Chapter 1. Forward and Inverse Problem in Geophysics

1.1. Introduction

Geophysical methods are based on the study of the propagation of different physical fields inside the Earth. The most important geophysical fields are gravity, magnetic, electromagnetic and seismic wave fields. The observations of these fields first depend on the physical properties of the rock. Traditional geophysical data analysis methods include the establishment of different geological models and observational data. Numerical modeling of geophysical data for a given model parameter is often referred to as a **forward problem** (see **Figure 1**). The solution to the forward problem enables us to predict geophysical data from specific geological structures.

The ultimate goal of geophysical observations is to determine geological structures from geophysical data. Due to the complex internal structure of the earth, this is a very difficult problem. Usually we use a more or less simple model to approximate the real geology and try to determine the model parameters from the data. We call this problem an **inverse problem** (see **Figure 1**). The success of geophysical interpretation depends on whether or not we can approximate the true geological structure through a reasonable model and effectively solve the corresponding inverse problems.

1.2. Formulation of Forward and Inverse Problems

When studying geophysical methods, we should also consider that certain sources can generate geophysical fields. So the general forward and inverse problems can be described as follows:

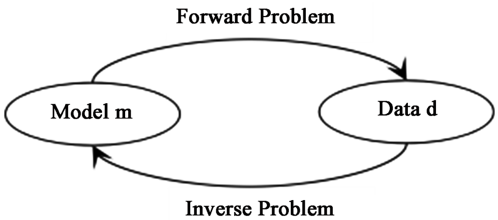

Figure 1. The traditional definition of the forward and inverse problems.

FORWARD PROBLEM:

Model {model parameters $\mathbf{m}$, sources s} $\rightarrow$ data $\mathbf{d}$:

$$\mathbf{d} = A_s(\mathbf{m}), \tag{1.1}$$

where A_s is the forward problem operator depending on a source s.

INVERSE PROBLEM:

{data $\mathbf{d}$, sources s}$\rightarrow$model {model parameters $\mathbf{m}$}

$$\mathbf{m} = A_s^{-1}(\mathbf{d}) \tag{1.2}$$

or {data $\mathbf{d}$} $\rightarrow$ model and sources {model parameters $\mathbf{m}$, sources s}:

$$(\mathbf{m},\mathbf{s}) = A^{-1}(\mathbf{d}) \tag{1.3}$$

where A_s^{-1} and A^{-1} are inverse problem operators.

We will call the question (1.2) and the inverse model problem. Note that this problem (1.2) applies to electromagnetic field or acoustic field propagation and is often referred to as inverse scattering problem.

In some geophysical applications, the inverse problem is only formulated with respect to the sources of the observed field:

{data $\mathbf{d}$} $\rightarrow$ {sources s}

$$\mathbf{s} = A^{-1}(\mathbf{d}) \tag{1.4}$$

The problem (1.4) is called the inverse source problem. In this case, it is assumed that the model parameters (physical properties of the medium) are known. Typical examples of this problem are the gravity inverse problem and seismological inverse problem. In the first case, the density distribution of the rock is the source of the gravity field. In the second case, the goal is to estimate the location and type of seismic source from the observed seismic field.

In the solution of any inverse problem, there are three important questions arising:

1) Does the solution exist?

2) Is it unique?

3) Is it stable?

The problem of the existence of the solution is related to the mathematical formula of the inverse problem. From the physical point of view, there should be some solution because we study the real geological structure inside the Earth. However, from a mathematical point of view, there may be no suitable numerical model to fit our observed data on a given set of models.

The uniqueness of the solution can be illustrated by the following formula. Suppose we have two different models, $\mathbf{m}_1$ and $\mathbf{m}_2$, and two different sources, s_1 and s_2, that produce the same data $\mathbf{d}_0$:

$$A(\mathbf{m}_1, s_1) = \mathbf{d}_0,\ A(\mathbf{m}_2, s_2) = \mathbf{d}_0.$$

In this case, it is impossible to distinguish two models, $\mathbf{m}_1, \mathbf{m}_2$, from the given data. That is why uniqueness issues are important for inversion.

The last problem of solution stability is also a key issue in inversion theory. In fact, geophysical data always contaminated by some noise $\delta\mathbf{d}$. The problem is whether the difference in the response of different modes is greater than noise level. For example, let two different models $\mathbf{m}_1, \mathbf{m}_2$, and two different sources, s_1, s_2, generate two different data sets, $\mathbf{d}_1, \mathbf{d}_2$, which can be represented as follows:

$$A(\mathbf{m}_1, s_1) = \mathbf{d}_1,\ A(\mathbf{m}_2, s_2) = \mathbf{d}_2.$$

Assume that the two models and the source are different obviously, while the data difference is within the noise level ε:

$$\|\delta\mathbf{m}\| = \|\mathbf{m}_1 - \mathbf{m}_2\| > C,\ \|\delta s\| = \|s_1 - s_2\| > C,$$

$$\|\delta\mathbf{d}\| = \|\mathbf{d}_1 - \mathbf{d}_2\| < \varepsilon,\ C \gg \varepsilon,$$

where symbol $\|\cdot\|$ denotes some norm, or measure of difference between two models, sources and data sets.

In this case, it is also impossible to distinguish the two models from the observed data.

Taking into account the importance of these three issues to the inversion problem solution, Hadamard, a renowned French mathematician, said that if all three issues mentioned above have a positive answers, then a mathematical problem can be expressed correctly. In other words, the mathematical problem is considered to be well-posed, if its solution does exist, is unique and is stable.

One problem is according to Hadamard (1902) ill-posed if the solution did not exist, or it was not unique, or it was not a continuous function of the data (that is, if the small perturbation of the data corresponds to any large perturbation of the solution). Hadamard considered that ill-posed mathematical problems were not physically and/or mathematically meaningful.

However, it turns out that most mathematical physics and geophysics problems (actually most natural science problems) are ill-posed. Fortunately, it was found later that Hadamard's view was wrong: the ill-posed problems are both physically and mathematically meaningful and could be solved.

In the middle of last Century Andrei N. Tikhonov developed the foundations of the theory of ill-posed problem solutions. He introduced a regularization method to approximate an ill-posed problem by a number of well-posed problems.

1.3. Existence and Uniqueness of Inversion Problem Solutions

1.3.1. Existence of the Solution

We can write an operator equation in a general form for the forward modeling:

$$\mathbf{d} = A(\mathbf{m}), \tag{1.5}$$

where denotes different kind of geophysical data, and $\mathbf{m}$ denotes corresponding model parameters.

There are two aspects to the existence of the inverse problem solution. One is the physical existence of observed data generated by the distribution of certain physical parameters, and the other is the existence of the mathematical solution of operator Equation (1.5). There is no doubt about the physical existence of the inverse problem solution. However, the existence of mathematical existence is questionable. To better understand this phenomenon, please note that the measured geophysical data $\mathbf{d}$ always contains the error $\delta\mathbf{d}$:

$$\mathbf{d}_\delta = \mathbf{d} + \delta\mathbf{d} \tag{1.6}$$

The question is whether it is possible to find a model $\mathbf{m}$ that can accurately generates the observed noisy data $\mathbf{d}_\delta$:

$$\mathbf{d}_\delta = A(\mathbf{m}_\delta) \tag{1.7}$$

The answer is that sometimes we cannot find such a model and it is easy to understand why. Indeed, one should keep in mind that noise has nothing to do with true model parameters, and it can produce reasons that have nothing in common with the geophysical field equations described above. Therefore, noise cannot be described by the same operator Equation (1.5) as theoretical geophysical data. For this reason, we should not expect that we can always find a physical model that matches actual observational data exactly. But why should we try to find such a model?

The solution to an inverse problem is usually a class that simplifies the model. Therefore, the problem must be a quasi-solution to the inverse problem, that is, a solution that best fits the observations from a selected model class. Therefore, we get a practical existence: the solution to the inverse problem exists if there is $\mathbf{m}_\delta$ such that

$$\left\|\mathbf{d}_\delta - A(\mathbf{m}_\delta)\right\| \le \delta$$

where δ is the measurement error, and $\|\cdot\|$ denotes some measure of difference between the theoretical (predicted) data $A(\mathbf{m}_\delta)$ and observed noisy data $\mathbf{d}_\delta$. It is important to understand that we should not even try to get an exact solution of the inverse problem. It has no practical meaning for noisy data. So we should think about some approximate method to the inversion based on searching for a model that could fit the observed data within the given accuracy δ. This simple idea is the cornerstone of regularization theory.

1.3.2. Uniqueness of the Solution

Another important issue we mentioned earlier is the uniqueness of the solution. Again, natural mothers do not seem to like simple solutions: In general, the solutions to the geophysical inverse problems are not unique.

When analyzing non-uniqueness problems, the above two types of inverse problems should be distinguished: the inverse model (or inverse scattering) problem and the inverse source problem. The advantage of the inverse scattering problem is that many different locations of the source can be selected to illuminate the medium being inspected. Each source location will generate a different data set as a rule, reducing the ambiguity of inverse solution (Blok and Oristaglio, 1995).

The inverse source problem is more ambiguous because there is usually a source distribution that produces zero external fields. This type of source is called a nonradiating source. A detailed analysis of the analytical performance of nonradiating source was given in a classic paper by Bleistein and Cohen (1976).

1.3.3. Instability of the Solution

Another key issue of inversion theory is instability. This problem reflects the difference in the noise level of the two observed data sets, and the distribution of the corresponding model parameters may be completely different.

A demonstration of instability can be provided for practically all geophysical inversion problems (see, for example, Lavrent'ev et al., 1986). That is why any reasonable algorithm for an inverse problem solution must be able to take this effect into account.

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