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The Application of GPS Technique in Determining the Earth's Potential Field

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Abstract. Two approaches to determining the Earth's external potential field by using GPS technique are proposed. The first one is that the relation between the geopotential difference and the light signal's frequency shift, between two separated points, is applied. The second one is that the spherical harmonic expansion series and a new technique dealing with the "downward continuation" problem are applied. Given the boundary value provided by GPS "geopotential frequency shift" on the Earth's surface, the Earth's external field could be determined based on the "fictitious compress recovery" method. Given the boundary value derived by on-board GPS technique on the satellite surface, the Earth's external field could be determined by using a new technique for solving the "downward continuation" problem, which is also based on the "fictitious compress recovery" method. The main idea of the "fictitious compress recovery" is that an iterative procedure of "compress " and "recovery" between the given boundary (the Earth's surface or the satellite surface) and the surface of Bjerhammar sphere is executed and a fictitious field is created, which coincides with the real field in the domain outside the Earth. Simulation tests support the new approach.

Key words: GPS observation, fictitious compress recovery, geopotential frequency shift, potential field determination

1 Introduction

The GPS technique plays an important role in geoscience and has very broad applications in various fields, especially in determining the coordinates of the interested points. One possible application of the GPS technique, which might not be taken good attention in geodesy, is that the geopotential difference between two (even far away) separated points on the Earth's surface might be directly determined by using GPS signals (Shen et al, 1993; Brumberg and Groten, 2002). Conventionally, the potential difference between two points are determined by gravimetry and levelling, the drawback of which is that it is almost impossible to connect two points which are located on two continents, because it is well known that the potential surface of the mean sea level is not an equi-potential surface. Hence, if GPS "geopotential frequency shift" approach could be applied in practice in the future, a unified world datum might be established. Meanwhile, once the geopotential on the Earth's surface is given, the Earth's external geopotential field could be strictly determined by using the "fictitious compress recovery" method (Shen, 2004), which can be also applied for determining the Earth's external geopotential field (and consequently solving the "downward continuation" problem), if the gravitational potential on the satellite surface (which is defined by the flying satellite) is given, for which the GPS technique provides a good opportunity.

Satellite on-board GPS receiver provides the position $x_i(t)$ of the satellite (e.g., CHAMP or GRACE satellite), and consequently its velocity $v_i(t)$ by using differential approach. Suppose an accelerometer is equipped with the flying satellite, one could determine the Earth's gravitational potential along the satellite orbit based on the energy conservation law, e.g., using the well known energy integral approach (e.g., Gerlach et al, 2003; Visser et al, 2003). Then, one can determine the potential field based on the truncated spherical harmonic expansion (in this case there are finite harmonic coefficients to be determined) and by using, e.g., the least squares adjustment (e.g., Rummel et al, 1993), which is referred to as the conventional approach for convenience.

However, the problem is that so determined field might not be valid in the domain between the Earth's surface and the surface of Brilloun sphere, the smallest sphere that encloses the whole Earth, because it could not be guaranteed that the spherical harmonic expansion series is convergent in that domain (e.g., Moritz, 1978; Sjöberg, 1980; Rummel et al, 1993). To solve this problem, two approaches could be applied. If the provided (discrete) potential values are uniformly distributed on the satellite surface, one could determine the fictitious field that coincides with the real field in the whole domain outside the Earth based on the "fictitious compress recovery" method, naturally solving the "downward continuation" problem, which is greatly interested by geodesists. Otherwise, using the conventional approach one first determines the potential field, which is valid in the domain outside Brillouin sphere; then, choose a simply closed surface (it is recommended that an ellipsoidal or spherical surface is chosen, Cf. Remark 2 in Sec.6) that encloses Brillouin sphere, and applying the "fictitious compress recovery" method one could determine the real field between the Earth's surface and the surface of Brillouin sphere, also solving the "downward continuation" problem.

Since the "fictitious compress recovery" method (Shen, 2004) is essential, it will be summarized in Sec.2. In Sec.3, the GPS "geopotential frequency shift" approach is presented, and in Sec.4, the "downward continuation" approach is presented. In Sec.5, preliminary simulation results are provided, and finally, the conclusions and discussions end this paper.

2 The "fictitious compress recovery" method

Choose an inner sphere or Bjerhammar sphere K (Bjerhammar, 1964), which is an open set (excluding the boundary ∂K of the sphere K) and entirely located inside the Earth, with its centre coinciding with the Earth's mass centre (Shen, 2004). In the domain $\overline{K}$, which denotes the domain outside the sphere K , including the boundary ∂K , there exists such a potential field $V^*(P)$ satisfying

$$\begin{aligned} &\Delta V^*(P) = 0, \quad P \in \overline{K} \\ &V^*(P)|_{\partial K} = V^*{}_{\partial K} \\ &\lim_{P\to\infty} V^*(P) = 0 \end{aligned} \tag{1}$$

and on the boundary of the Earth, $V^*(P)$ has the same value as the Earth's real potential field $V(P)$ has. Then, the fictitious potential field $V^*(P)$ coincides exactly with the Earth's real potential field $V(P)$ in the domain $\overline{\Omega}$ (Shen, 2004), where Ω denotes the domain (open set) occupied by the Earth (excluding the Earth's boundary), $\overline{\Omega}$ denotes the domain outside the Earth (including the boundary of the Earth), and let $\partial\Omega$ denote the boundary of the Earth. To realize this idea, the "compress" and "recovery" technique (Shen, 2004) could be used.

Set the observed gravitational potential value $V(P)|_{\partial\Omega} \equiv V_{\partial\Omega}$ (note that $V_{\partial\Omega}$ is obtained by the observed geopotential W subtracting the centrifugal potential Q) on the surface of K along the radial direction. Then, based on Poisson integral a regular harmonic solution $V^{*(1)}(P)$ could be determined in the domain $\overline{K}$:

$$V^{*(1)}(P) = \frac{r^2 - R^2}{4\pi R}\int_{\partial K} \frac{V_{\partial\Omega}}{l^3} d\sigma, \quad P \in \overline{K} \tag{2}$$

which can be taken as the first approximation of the Earth's real potential field $V(P)$ in the domain outside the Earth. The first-order residual field $\delta^{(1)}(P)$ is defined as follows (Shen, 2004):

$$\delta^{(1)}(P) = \delta^{(0)}(P) - V^{*(1)}(P) \equiv V(P) - V^{*(1)}(P), \;\; P \in \overline{\Omega} \tag{3}$$

where $\delta^{(0)}(P) \equiv V(P)$ is the Earth's real potential field. It should be noted that $\delta^{(1)}(P)$ is defined only in the domain $\overline{\Omega}$. With $V^{*(1)}(P)$ the boundary value $V^{*(1)}\big|_{\partial\Omega} \equiv V^{*(1)}_{\partial\Omega}$ can be calculated. Based on Eq.(3), set the first-order residual boundary value

$$\delta^{(1)}\big|_{\partial\Omega} = V_{\partial\Omega} - V^{*(1)}_{\partial\Omega} \tag{4}$$

again on the surface of Bjerhammar sphere K (note that the boundary value $\delta^{(1)}\big|_{\partial\Omega}$ is identically compressed on ∂K along the radial direction), a second-order regular harmonic solution $V^{*(2)}(P)$ can be determined in $\bar{K}$:

$$V^{*(2)}(P) = \frac{r^2 - R^2}{4\pi R}\int_{\partial\Omega} \frac{\delta^{(1)}\big|_{\partial\Omega}}{l^3} d\sigma, \quad P \in \bar{K} \tag{5}$$

$V_1^* + V_2^*$ can be taken as the second approximation of the Earth's real potential field $V(P)$ in the domain outside the Earth.

Similarly, the second-order residual field $\delta^{(2)}(P)$ is defined as follows:

$$\delta^{(2)}(P) = \delta^{(1)}(P) - V^{*(2)}(P), \quad P \in \bar{\Omega} \tag{6}$$

and set the second-order residual boundary value

$$\delta^{(2)}|_{\partial\Omega} \equiv \delta^{(1)}|_{\partial\Omega} - V^{*(2)}|_{\partial\Omega} \tag{7}$$

again on the surface of the sphere K (along the radial direction), a third-order regular harmonic solution $V^{*(3)}(P)$ in $\bar{K}$ is determined. This procedure can be repeated until a series solution $V^*(P)$ is provided in the domain outside the sphere K (Shen, 2004):

$$V^*(P) = \sum_{n=1}^{\infty} V^{*(n)}(P), \quad P \in \bar{K} \tag{8}$$

which is a regular harmonic function in $\bar{K}$, and coincides exactly with the Earth's real potential field $V(P)$ in the domain $\bar{\Omega}$, the domain outside the Earth:

$$V^*(P) = \sum_{n=1}^{\infty} V^{*(n)}(P) \equiv V(P), \quad P \in \bar{\Omega} \tag{9}$$

Hence, once the geopotential or gravitational potential on the Earth's physical surface $\partial\Omega$ is given, the Earth's external field can be exactly determined. It should be pointed out that the "fictitious compress recovery" method has wide applications in geophysics (Shen *et al*., 2004).

3 The "geopotential frequency shift" approach

The geopotential frequency shift approach by using GPS signals was briefly proposed in Shen *et al*. (1993), and the technical details could be found in Shen (1998).

Suppose a light signal with frequency f is emitted from point P by an emitter, and the signal is received at point Q by a receiver. Because of the geopotential difference between these two points, the frequency of the received light signal is not f but f'. Using f_P and f_Q to denote f and f' respectively, the following equation holds (Pound and Snider, 1965; Shen *et al*., 1993):

$$\Delta f \equiv f_Q - f_P = -\frac{f}{c^2}\Delta W_{PQ} \equiv -\frac{f}{c^2}(W_Q - W_P) \tag{10}$$

where c is the velocity of light in vacuum, W_P and W_Q are the geopotentials at points P and Q, respectively. Expression (10) is called in literature the gravity frequency shift equation (Pound and Snider, 1965), or properly called the geopotential frequency shift equation (due to the fact that the frequency shift is caused by the geopotential difference). Katila and Riski (1981) confirmed Eq.(10) with the accuracy level 10^{-2}. Vessot *et al*. (1980) proved that Eq.(10) is correct to the accuracy of 10^{-4}. Scientists believe that Eq.(10) is correct, because it is a result derived from the theory of general relativity. In fact, Eq.(10) can be also derived out based on quantum theory and energy conservation law (Shen, 1998).

Suppose the geopotential at point P is given, then, from Eq.(10) the geopotential at an arbitrary point Q can be determined by measuring the geopotential frequency shift between P and Q:

$$W_Q = W_P - \frac{c^2 \Delta f}{f} \tag{11}$$

If the point P is chosen on the geoid, it holds that

$$W_Q = C_0 - \frac{c^2 \Delta f}{f} \tag{12}$$

where C_0 is the geoid geopotential constant. It should be noted that C_0 might not be correct because of a constant shift δW, which will give rise to W_Q the same shift at an arbitrary point Q. This is a systematic error, and it could be filtered out by using the"fictitious compress recovery" method (the determination of C_0 or δW in details is beyond the scope of present paper and will be explored in a separated paper). Then, the geopotential at an arbitrary point Q on the Earth's physical surface $\partial\Omega$ can be determined based on the geopotential frequency shift equation. Then, the main problem is how to measure the frequency shift between two points. The basic principle of measuring the frequency shift can be stated as follows.

Set at point P an emitter which emits a light signal with frequency f and a receiver at point Q, which receives the light signal emitted by the emitter at point P. Suppose the received signal's frequency is f'. Then, it could be compared the frequency f' of the received light signal with it-self's standard frequency f (this is not only the emitting frequency at point P but also the standard innate frequency of the receiver at point Q), and the frequency shift $\Delta f = f' - f$ can be determined. Consequently, according to Eq.(11) the geopotential difference ΔW_{PQ} between P and Q can be determined. Applying the same principle it will be found the geopotential difference $\Delta W_{OP} = W_P - W_O$ between the geoid and the point P, where $W_O = C_0$ is the geopotential at point O located on the geoid. If C_0 is a known constant, W_P as well as W_Q can be found.

Now, suppose the light signal emitter E is located in a satellite. Two GPS receivers at P and Q receive the light signals coming from E corresponding to an emitting time t (seeing Fig.1), and suppose the received signals' frequencies corresponding to time t are recorded by receivers at P and Q in some way, respectively, i.e., f_P and f_Q corresponding to time t are recorded by receivers at P and Q, respectively. Note that the time at which the signal is received by P is generally different from that by Q. By comparing the received frequencies f_P and f_Q it could be determined the geopotential difference $\Delta W_{PQ} = W_Q - W_P$ (Shen *et al.*, 1993), which is just given by Eq.(10).

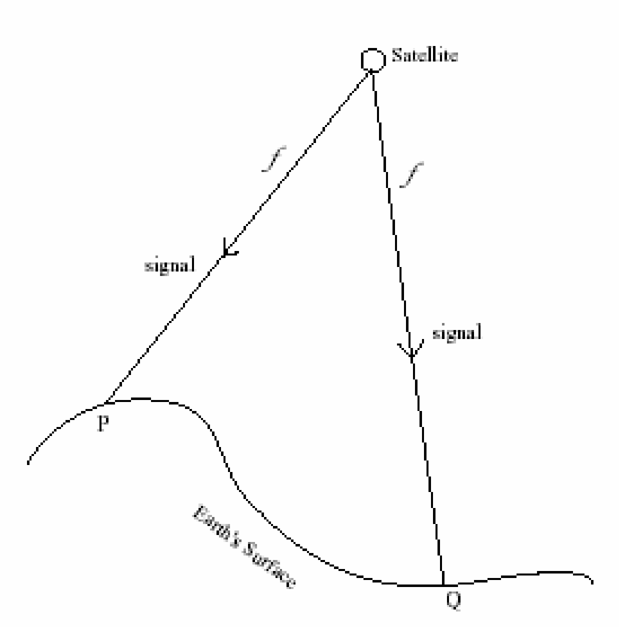

Fig. 1 Two receivers at points *P* and *Q* receive simultaneously the satellite-emitted light signal with frequency f

By this way, theoretically, the geopotential on the Earth's whole surface could be determined based on the geopotential frequency shift approach by using GPS technique. Then, the "fictitious compress recovery" method could be applied for determining the Earth's external potential field $V(P)$. Note that by GPS technique the coordinates x^i at any point on the Earth's physical surface can be determined, where x^i denote $x^1 \equiv x,\ x^2 \equiv y,\ x^3 \equiv z$. Consequently, once the geopotential W on the Earth's surface is determined, the gravitational potential V on the Earth's physical surface is determined.

4 The "downward continuation" approach

Suppose there are quite a few satellites flying around the Earth and many observation stations distributed at various points on the Earth's physical surface. Generally, the Earth's gravitational potential V could be expanded into spherical harmonic series (Heiskanen and Moritz,1967):

$$V(p) = GM\sum_{n=0}^{\infty}\sum_{m=0}^{n}\frac{a^n}{r^{n+1}}(a_{nm}\cos m\lambda + b_{nm}\sin m\lambda)P_{nm}(\cos\theta) \quad (13)$$

where G is the gravitational constant, M the Earth's mass, a the Earth's semi-major axis, r the distance between the coordinate origin o and the field point $P(x^i)$, a_{nm} and b_{nm} are constant coefficients to be determined based on various observations (GPS observations in our case), $P_{nm}(\cos\theta)$ are the associated Legendre functions, λ and θ are longitude and co-latitude, respectively. The expression (13) is at least correct in the domain outside a satellite surface ∂S. In the domain near the Earth's surface, the series (13) might be divergent (Moritz, 1978; Sjöberg, 1980; Rummel et al, 1993; Shen, 1995).

Now, suppose Eq.(13) holds in $\bar{S}$, the domain outside the satellite surface ∂S. To determine the field $V(P)$, the truncation technique should be applied (otherwise the infinite harmonic coefficients a_{nm} and b_{nm} can't be determined), i.e., only the first terms until degree N are left:

$$V_1(P) = GM\sum_{n=0}^{N}\sum_{m=0}^{n}\frac{a^n}{r^{n+1}}(a_{nm}\cos m\lambda + b_{nm}\sin m\lambda)P_{nm}(\cos\theta) \quad (14)$$

Hence, only the finite harmonic coefficients a_{nm} and b_{nm} are left as the unknown parameters. Then, by establishing the relation between the potential (or the gravitation $\partial_i V$ either the gravitational gradient $\partial_i\partial_j V$) and the GPS observations $x^i(t)$ (note that once $x^i(t)$ are determined as GPS observation values, $dx^i(t)/dt$ and $d^2x^i(t)/dt^2$ are also determined as GPS "observation values"), the finite harmonic coefficients a_{nm} and b_{nm} could be determined by using the least-squares method. That means the potential field $V_1(P)$ outside the satellite surface ∂S is determined. However, as mentioned before, it can't be guaranteed that Eq.(14) holds also in the domain near the Earth's surface due to the divergence problem caused by Eq.(13). Hence, it occurs the "downward continuation" problem, which has attracted many geodesists' interests and attention. The "downward continuation" problem was not solved satisfactorily by using conventional methods due to the well-known "ill-posed" problem. Recently, this problem was solved satisfactorily (Shen and Ning, 2004; Shen and Wang *et al.*, 2005; Cf. Remark 1) by using the "fictitious compress recovery" method (Shen, 2004).

By GPS observations, suppose it has been established a model of the Earth's potential field, noted as EGM1, which is correct in the domain $\bar{S}$ (the domain outside the satellite surface ∂S). Then, on the boundary ∂S it has been known the boundary value $V_{\partial S} = V_1|_{\partial S}$, which can be assumed very accurate without loss of generality. Choose Bjerhammar sphere K, using the boundary value $V_{\partial S}$ (which is calculated by EGM1) and by applying the "fictitious compress recovery" method it could be determined a regular harmonic function $V^*(P)(P \in \overline{K})$, which coincides exactly with the real field $V(P)$ in the domain $\overline{S}$ under the assumption that the boundary value $V_{\partial S}$ is error-free (Shen, 2004). Furthermore, it has been proved that the determined fictitious field $V^*(P)(P \in \overline{K})$ coincides also with the real field $V(P)$ in the domain $\overline{\Omega} - \overline{S}$, the domain between the satellite surface and the Earth's physical surface (Shen and Ning, 2004; Shen and Wang *et al*., 2005; Cf. Remark 1). For convenience, the determined field constrained in $\overline{\Omega}$ based on the "fictitious compress recovery" method is referred to as FEGM (or FEGM1).

Remark 1: Suppose the boundary value $V_{\partial S}$ on the satellite surface ∂S (it can be also a spherical surface ∂K_Γ) is given. It will be briefly proved that the real field can be determined based on the "fictitious compress recovery" method, referred to Shen and Ning (2004). Set

$$\phi(P) = V^*(P) - V(P), \quad P \in \overline{\Omega} \tag{15}$$

where $V^*(P)$ is a fictitious regular harmonic solution (based on $V_{\partial S}$ and the "fictitious compress recovery" method) in the domain $\overline{K}$, the domain outside Bjerhammar sphere K. Hence, $\phi(P)$ is a regular harmonic function in the domain $\overline{\Omega}$, and satisfies the following equation:

$$\phi(P) = V^*(P) - V(P) = 0, \quad P \in \overline{S} \tag{16}$$

Based on Eq.(15), on the boundary $\partial\Omega$ it holds:

$$\phi(P)|_{\partial\Omega} = V^*(P)|_{\partial\Omega} - V(P)|_{\partial\Omega} \tag{17}$$

Applying the "fictitious compress recovery" method (Shen, 2004), it can be determined a regular harmonic function $\phi^*(P)$ in the domain $\overline{K}$, and it holds

$$\phi^*(P) = \phi(P), \quad P \in \overline{\Omega} \tag{18}$$

Since $\phi^*(P)$ is regular and harmonic in $\overline{K}$, it can be expanded into a (uniformly convergent) spherical harmonic series (the mathematical expression form looks like Eq.(13)). Using Eqs.(16) and (18) it must hold that $\phi^*(P) \equiv 0$ $(P \in \overline{K})$. Then, from Eqs.(15) and (18) it holds that $V^*(P) = V(P)$ $(P \in \overline{\Omega})$. The proof is completed.

Hence, after applying the new "downward continuation" approach, referred to as the "fictitious downward continuation" (Shen and Wang et al, 2005; Shen and Yan et al, 2005), the Earth's external potential field can be determined based the established model EGM1 (the Earth's potential field in the domain $\overline{S}$). If more precise EGM1 (in the domain $\overline{S}$) is established by various satellite observations (e.g. GPS, CHAMP, GRACE, GOCE, etc), a more precise field $V(P)$ in $\overline{\Omega}$ could be determined.

If given the gravitational potential values distributed uniformly on the satellite surface, a fictitious potential field in the domain outside Bjerhammar sphere can be directly determined based on the "fictitious compress recovery" method, and the determined fictitious field coincides with the real field in the whole domain outside the Earth. In this way, FEGM is directly established.

5 Preliminary simulation results

In this section a simulation test is provided, which is referred to (Li, 2005; Shen and Li, 2005). Choose a spherical coordinate system (r, θ, λ) and a gravitational potential model, a 4-sphere anomaly model: three small spheres O_i ($i = 1, 2, 3$) are located inside a large sphere O_0, with the parameters listed in Tab.1.

Tab.1 Parameters of the four spheres

Sphere	O_0	O_1	O_2	O_3
Centre [km, deg,deg]	(0, 0, 0)	(4000, 0, 0)	(2000, 90, 120)	(3000, 120, 240)
Radius [km]	R_0 6371	R_1 300	R_2 500	R_3 600
Potential at surface [m^2s^{-2}]	V_0 1000	V_1 -100	V_2 400	V_3 200

Based on the above model, the real potential field $V(P)$ outside the large sphere (which is assumed as the "Earth") is known, expressed as

$$V(P) = \frac{R}{r}V_0 + \sum_{i=1}^{3}\frac{R_i}{r_i}V_i, \quad P \in \overline{\Omega} \tag{19}$$

where r_i ($i = 1, 2, 3$) is the distance from O_i to the field point P.

Now, it is supposed that only the boundary value $V_{\partial S}$ on a satellite surface is known, calculated from Eq.(19), and the aim is to determine the real field $V(P)$ in the domain outside the "Earth" (i.e., in the domain outside the large sphere), based only on the given boundary value $V_{\partial S}$. The boundary value $V_{\partial S}$ on the satellite surface is supposed to be obtained based on a polar satellite (equipped with a GPS receiver and an accelerometer), using the well known energy integral approach (e.g., Gerlach et al, 2003; Visser et al, 2003). The satellite surface is supposed to be a rotation-ellipsoidal surface, with its geometric centre coinciding with the coordinate origin, the major-axis $a = 6371 + 250$ km, and the eccentricity $e = 0.01$. The radius of the inner sphere (i.e., Bjerhammar sphere) is taken as 6000 km. The simulation calculation (especially the calculation of Poisson integral) is executed based on grid approach, with $1^o \times 1^o$ grid defined by parallel latitude line and longitude line on the surface of the inner sphere. The grids on the satellite surface are 1-1 corresponding to the grids on the spherical surface. Consequently there are 64800 discrete values, which are uniformly distributed on the satellite surface.

Now, with the given (discrete) boundary values $V_{\partial S}$, the fictitious distribution on the surface of the inner sphere is determined based on the "fictitious compress recovery" method (Cf. Sec.2), and consequently the fictitious field $V^*(P)$ in the domain outside the inner sphere is determined. In theory, the fictitious field $V^*(P)$ coincides with the real field in the domain outside the "Earth". Hence, we need only to compare the calculated values with the real values on the surface of the "Earth", because, if two regular harmonic fields (in the domain outside the "Earth") coincide on the boundary of the "Earth", they must coincide in the whole domain outside the "Earth". The calculated results are summarized in Tab.2, where $n = 8$ and $n = 15$ express the iterative procedure times, respectively, $\partial\Omega$ and ∂S express the satellite surface and the surface of the "Earth", respectively, and $\Delta V = V - V^*$ expresses the residual potential value between the real value and the calculated (fictitious) value. Fig.2 shows the residuals between the real values and the calculated corresponding values on the "Earth's surface".

Tab.2 Results based on the boundary values on the satellite surface

n	$\max(\lvert \Delta V \rvert)$ [m^{2}s^{-2}]		$mean(\Delta V)$ [m^{2}s^{-2}]		RMS [m^{2}s^{-2}]	
	On ∂S	**On $\partial\Omega$**	On ∂S	**On $\partial\Omega$**	On ∂S	**On $\partial\Omega$**
8	0.006	**0.01**	-1.7× e-4	**-1.7× e-4**	1.0× e-3	**0.002**
15	0.002	**0.008**	2.0× e-5	**9.2× e-5**	2.4× e-4	**0.001**

From Tab.2, we can draw the following conclusions: suppose the given boundary value $V_{\partial S}$ is error-free, then, based on the boundary value $V_{\partial S}$ and the "fictitious compress recovery" method, after 15-times iterative procedures, one gets a fictitious (regular harmonic) field $V^*(P)(P \in \overline{K})$, which coincides with the real field $V(P)$ on the "Earth's surface" (seeing the black numbers in Tab.2) under the accuracy (RMS) level 0.1 mm (note that 0.001m^{2}s^{-2} corresponds to the height 0.1 mm), and based on the extreme value principle (e.g., Kellogg, 1929) we can conclude that the fictitious field $V^*(P)(P \in \overline{K})$ coincides with the real field $V(P)$ in the whole domain outside the "Earth" at least under the accuracy (RMS) level 0.1 mm, which is confirmed by further experiments, seeing Tabs.3 and 4 (note that in theory, the experimental tests summarized in Tabs.3 and 4 are not necessary).

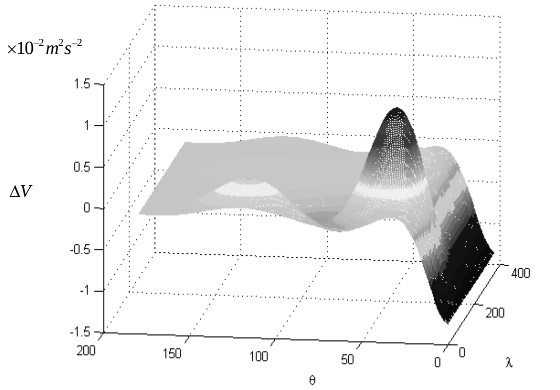

Fig. 2 Residual potential values on the surface of the "Earth" (From Shen and Li, 2005)

Quite arbitrarily, 12 test field points, which are located in the domain between the satellite surface and the surface of the "Earth", are chosen, and then, the residual values (i.e., the differences between the calculated fictitious values and the corresponding real values) on those points are calculated. The results are listed in Tab.3, from which it can be seen that the fictitious field coincides with the real field (at least at the chosen points) under a high accuracy (RMS) level, around 0.08 mm.

Tab. 3 Results at test points in the domain between two boundaries ($\Delta V = V - V^*$; unit: m^{2}s^{-2})

(r, θ, λ) [km, deg, deg]	ΔV	(r, θ, λ) [km, deg, deg]	ΔV
(6380, 0, 0)	-0.0026	(6380, 90, 120)	-8.15e-7
(6600, 0, 0)	-0.0014	(6420, 100, 250)	-9.65e-8
(6400, 15, 0)	-2.62e-4	(6610, 120, 60)	5.01e-7
(6500, 10, 60)	-1.07e-4	(6500, 125, 200)	-2.72e-10
(6550,2 5, 180)	-2.55e-5	(6480, 135, 60)	6.81e-7
(6600, 20, 270)	-5.18e-5	(6570, 150, 210)	-9.68e-8
$mean(\Delta V)$	-3.70e-4		
RMS	7.67e-4		

Further, 5 test field points, which are located in the domain outside the satellite surface, are chosen, and then the residual values on those points are calculated. The results are listed in Tab.4, from which it can be seen that the fictitious field coincides with the real field (at least at the chosen points) under the accuracy (RMS) level around 0.03 mm.

Tab.4 Results at test points in the domain out side the satellite surface ($\Delta V = V - V^*$)

(r,θ,λ) [km, º,º]	(6700, 0, 0)	(6750, 30, 90)	(6800, 60, 120)	(6720, 120, 240)	(6850, 180, 0)
ΔV [m^{2}s^{-2}]	-6.8 × e-4	1.6 × e-5	-2.2 × e-6	-4.8 × e-11	-5.4 × e-4
$mean(\Delta V)$	-2.4e-4 m^{2}s^{-2}				
RMS	3.0e-4 m^{2}s^{-2}				

In summary, the "fictitious compress recovery" is valid and reliable, based on which the "downward continuation" problem is satisfactorily solved.

Previous to the above mentioned simulation test, Shen and Wang et al (2005) completed an experimental test, which is summarized as follows. Two spherical surfaces ∂K_1 and ∂K_2 with radii $R_1 = 6378$ km and $R_2 = 6680$ km were chosen (Cf. Fig.3), where ∂K_1 and ∂K_2 simulate the Earth's surface (exactly saying the surface of Brillouin sphere) and the satellite surface, respectively.

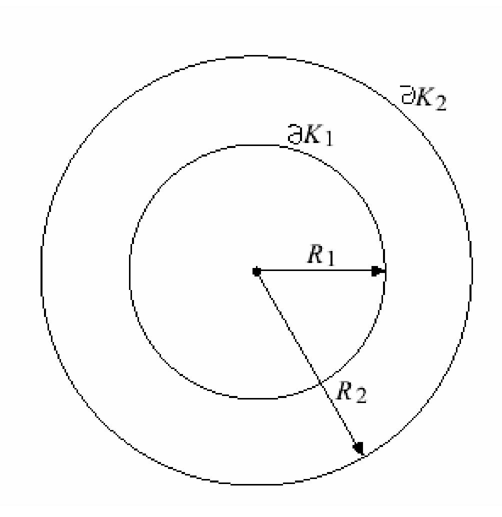

Fig.3 ∂K_1 and ∂K_2 simulate the surface of the Earth or Brillouin sphere and the satellite surface, respectively

With EGM96 model the potential value $V|_{\partial K_1}$ on ∂K_1 is known, and using Poisson integral the potential value $V|_{\partial K_2}$ on the "satellite" surface was calculated, which was assumed to be the "observations", that means the potential values $V|_{\partial K_2}$ on the satellite surface were taken as the initial boundary values. Then, with $10^o \times 10^o$ grid and based on the "fictitious compress recovery" method the fictitious field $V^*(P)$ was calculated, which is compared with the "real" value $V|_{\partial K_1}$ on ∂K_1. The largest difference between the real value $V_1|_{\partial K_1}$ and the calculated fictitious value $V^*|_{\partial K_1}$ is 0.04 m^{2}s^{-2}, which corresponds to a height variation 0.4 cm. Hence, the experimental test (Shen and Wang et al, 2005) supports the "fictitious compress recovery" method and the "fictitious downward continuation".

It is noted that the satellite surface ∂S can be also replaced by a spherical surface, which completely encloses the Earth (Cf. Remark 2). The simulation test in details as well as various other simulation tests will be provided in a separated paper.

6 Conclusions and discussions

If the geopotential on the Earth's boundary $\partial\Omega$ is determined, the Earth's external potential field $V(P)$ can be determined based on the "fictitious compress recovery" method. To realize this, the GPS "geopotential frequency shift" approach (Shen et al, 1993) is proposed. However, it is most likely that in the very near future it is very difficult to realize this approach in practice, due to the fact that the accuracy of the determined potential difference by using the GPS "geopotential frequency shift" approach is too low, which depends on the frequency stability of the signal receiver. At present, the frequency stability is around $10^{-15} - 10^{-16}$ (HMC Project, 2005), which corresponds to the height variation about 1m.

If the gravitational potential $V(P)$ on the satellite surface ∂S or the surface ∂K_Γ of a sphere K_Γ (which completely encloses the whole Earth) is determined, e.g., using the energy integral approach (Cf. Gerlach et al, 2003; Visser et al, 2003), the Earth's external potential field $V(P)$ can be also determined based on the "fictitious compress recovery" method (Cf. Remark 2). This is a new approach for solving the "downward continuation" problem, referred to as the "fictitious downward continuation" (Cf. Sec.4). To realize this, it can be first determined the field $V_1(P)$ outside the satellite surface (or outside the sphere K_Γ) based on the spherical harmonic expansion (14) and by using GPS observations, and then the whole field outside the Earth could be determined, or the potential field could be directly determined if the determined boundary values are uniformly distributed on the satellite surface. Hence, in any case, the real field $V(P)$ in $\overline{\Omega}$ can be exactly determined based on the boundary value $V_{\partial S}$ (or $V_{\partial K_\Gamma}$) and by using the "fictitious compress recovery" method.

At present, the determined position deviation (from the real position) by on-board GPS receiver is around 5 cm, which gives rise to the velocity deviation about 10 cm/s, if the time keeping error is neglected. Consequently, the deviation of the "observed gravitational potential" on the satellite surface due to the position deviation is around 0.01 m^2/s^2 (of course this is not the real case, because the accuracy of the accelerometer is relatively low). Hence, neglecting other error sources, the determined potential field has the deviation around 0.01 m^2/s^2.

Generally, suppose the accuracy of the given value on the satellite surface (several hundred kilometres above the Earth's surface) is $\sigma_{\partial S}$, then, the accuracy $\sigma(r)$ of the determined fictitious field (based on the "fictitious compress recovery" method) is on the same accuracy level as σ in the domain between the Earth's surface and the satellite surface, expressed by the following relation (Shen and Tao, 2004):

$$\sigma(r) = \frac{r}{R}\sigma_{\partial S} \equiv (1+\frac{h}{R})\sigma_{\partial S} \tag{19}$$

where R is the average radius of the Earth, h is the height of the field point above the Earth's surface.

Remark 2: The satellite surface ∂S can be replaced by the surface ∂K_Γ of a sphere K_Γ that encloses the whole Earth. In this case, Eqs.(13) and (14) hold also in the domain $\overline{K}_\Gamma$, the domain outside the sphere K_Γ (there does not exist divergence problem any more in the domain $\overline{K}_\Gamma$). Based on Eq.(14) and satellite observations, the field $V_1(P)$ in the domain $\overline{K}_\Gamma$ could be determined. Then, after the "fictitious compress recovery" method is applied, the real field $V(P)$ in the whole domain $\overline{\Omega}$ outside the Earth could be determined, under the assumption that $V_1(P)$ coincides with the real field $V(P)$ in the domain $\overline{K}_\Gamma$

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