Cost Edge-Coloring of a Cactus

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Abstract

Let $C$ be a set of colors, and let $\omega(c)$ be an integer cost assigned to a color $c$ in $C$. An edge-coloring of a graph $G = (V, E)$ is assigning a color in $C$ to each edge $e \in E$ so that any two edges having end-vertex in common have different colors. The cost $\omega(f)$ of an edge-coloring $f$ of $G$ is the sum of costs $\omega(f(e))$ of colors $f(e)$ assigned to all edges $e$ in $G$. An edge-coloring $f$ of $G$ is optimal if $\omega(f)$ is minimum among all edge-colorings of $G$. A cactus is a connected graph in which every block is either an edge or a cycle. In this paper, we give an algorithm to find an optimal edge-coloring of a cactus in polynomial time. In our best knowledge, this is the first polynomial-time algorithm to find an optimal edge-coloring of a cactus.

Keywords

Cactus, Cost Edge-Coloring, Minimum Cost Maximum Flow Problem

1. Introduction

Let $G = (V, E)$ be a graph with vertex set $V$ and edge set $E$, and let $C$ be a set of colors. An edge-coloring of $G$ is to color all the edges in $E$ so that any two adjacent edges are colored with different colors in $C$. The minimum number of colors required for edge-colorings of $G$ is called the chromatic index, and is denoted by $\chi'(G)$. It is well-known that $\Delta(G) \leq \chi'(G) \leq \Delta(G) + 1$ for every simple graph $G$ and that $\chi'(G) = \Delta(G)$ for every bipartite (multi)graph $G$, where $\Delta(G)$ is the maximum degree of $G$ [1]. The ordinary edge-coloring problem is to compute the chromatic index $\chi'(G)$ of a given graph $G$ and find an edge-coloring of $G$ using $\chi'(G)$ colors. Let $\omega$ be a cost function which assigns an integer $\omega(c)$ to each color $c \in C$, then the cost edge-coloring problem is to find an optimal edge-coloring of $G$, that is, an edge-coloring $f$ such that $\sum_{e \in E} \omega(f(e))$ is minimum among all edge-colorings of $G$. An optimal edge-coloring does not always use the minimum number $\chi'(G)$ of colors. For example, suppose that $\omega(c_i) = 1$ and $\omega(c_i) = 2$ for each index $i \geq 2$, then the graph $G$ with $\chi'(G) = 3$ in Figure 1(a) can be uniquely colored with the three cheapest colors $c_1$, $c_2$, and $c_3$ as in Figure 1(a), but this edge-coloring is not optimal; an optimal edge-coloring of $G$ uses the four cheapest colors $c_1$, $c_2$, $c_3$, and $c_4$, as illustrated in Figure 1(b). However, every simple graph $G$ has an edge-coloring

using $\Delta(G)$ or $\Delta(G)+1$ colors \cite{2} \cite{3}. The edge-chromatic sum problem, introduced by Giaro and Kubale \cite{4}, is merely the cost edge-coloring problem for the special case where $\omega(c_i) = i$ for each integer $i \geq 1$.

The cost edge-coloring problem has a natural application for scheduling \cite{5}. Consider the scheduling of biprocessor tasks of unit execution time on dedicated machines. An example of such tasks is the file transfer problem in a computer network in which each file engages two corresponding nodes, sender and receiver, simultaneously \cite{6}. Another example is the biprocessor diagnostic problem in which links execute concurrently the same test for a fault tolerant multiprocessor system \cite{7}. These problems can be modeled by a graph $G$ in which machines correspond to the vertices and tasks correspond to the edges. An edge-coloring of $G$ corresponds to a schedule, where the edges colored with color $c_i \in C$ represent the collection of tasks that are executed in the $i$th time slot. Suppose that a task executed in the $i$th time slot takes the cost $\omega(c_i)$. Then the goal is to find a schedule that minimizes the total cost, and hence this corresponds to the cost edge-coloring problem.

The cost edge-coloring problem is APX-hard even for bipartite graphs \cite{8}, and hence there is no polynomial-time approximation scheme (PTAS) for the problem unless $P = NP$. On the other hand, Zhou and Nishizaki gave an algorithm to solve the cost edge-coloring problem for trees $T$ in time $O(n\Delta^2 \log(n\Delta))$, where $n$ is the number of vertices in $T$, $\Delta$ is the maximum degree of $T$, and $N_{\omega}$ is the maximum absolute cost $|\omega(c)|$ of colors $c$ in $C$ \cite{5}. The algorithm is based on a dynamic programming (DP) approach, and computes a DP table for each vertex of a given tree $T$ from the leaves to the root of $T$. In this paper, we give a polynomial-time algorithm to solve the cost edge-coloring problem for cacti. In our best knowledge, this is the first polynomial-time algorithm to find an optimal edge-coloring of a cactus.

2. Preliminaries

In this section, we define some basic terms.

Let $G = (V, E)$ be a graph with a set $V$ of vertices and a set $E$ of edges. We sometimes denote by $V(G)$ and $E(G)$ the vertex set and the edge set of $G$, respectively. We denote by $n(G)$ and $m(G)$, respectively, or simply by $n$ and $m$, the number of vertices and edges in $G$, that is, $n(G) = |V|$ and $m(G) = |E|$. The degree $d(v)$ of a vertex $v$ is the number of edges in $E$ incident to $v$. We denote the maximum degree of $G$ by $\Delta(G)$ or simply by $\Delta$. A cactus $G$ can be represented by an under tree $T$, which is a rooted tree. In the underlay tree $T$ of $G$, each node represents a block which is either a bridge (edge) of $G$ or an elementary cycle of $G$. If there is an edge between nodes $b_1$ and $b_2$ of $T$, then bridges or cycles of $G$ represented by $b_1$ and $b_2$ share exactly one vertex in $G$. Each node $b$ of $T$ corresponds to a subgraph $G_b$ of $G$ induced by all bridges and cycles represented by the nodes that are descendants of $b$ in $T$. Figure 2(a) depicts the subgraph $G_{b_1}$ for the child $b_1$ of the root of $T$. Clearly $G = G_{r}$ and $G_{b_1}$ is a cactus for each node $b$ of $T$. One can easily find an underlay tree $T$ of a given cactus $G$ in linear time, and hence one may assume that an underlay tree of $G$ is given. We denote by $ch(b)$ the number of edges joining a node $b$ and its children in $T$. Then, $ch(r) = d(r)$, and $ch(b) = d(b) - 1$ for every vertex $b \in V \setminus \{r\}$.

Let $C$ be a set of colors. An edge-coloring $f : E \rightarrow C$ of a graph $G$ is to color all edges of $G$ by colors in $C$ so that any two adjacent edges are colored with different colors. Let $\omega : C \rightarrow \mathbb{R}^+$, where $\mathbb{R}^+$ is the set of real numbers. One may assume with loss of generality that $\omega$ is non-decreasing, that is, $\omega(c_i) \leq \omega(c_{i+1})$ for any
index $i$, $1 \leq i \leq |C|$. Since trivially any graph $G$ has an optimal edge-coloring using colors at most $2\Delta(G) - 1$, we assume for the sake of convenience that $|C| = 2\Delta(G) - 1$, and we write $C = \{c_1, c_2, \ldots, c_{2\Delta-1}\}$. The cost $\omega(f)$ of an edge-coloring $f$ of a graph $G = (V, E)$ is defined as follows:

$$\omega(f) = \sum_{e \in E} \omega(f(e)).$$

An edge-coloring $f$ of $G$ is called an optimal one if $\omega(f)$ is minimum among all edge-colorings of $G$. The cost edge-coloring problem is to find an optimal edge-coloring of a given graph $G$. The cost of an optimal edge-coloring of $G$ is called the minimum cost of $G$, and is denoted by $\omega(G)$.

Let $f$ be an edge-coloring of a graph $G$. For each vertex $v$ of $G$, let $C_f(G, v)$ be the set of all colors that are assigned to edges incident to $v$, that is,

$$C_f(G, v) = \{f(e) \mid e \text{ is an edge incident to } v \text{ in } G\}.$$ 

We say that a color $c \in C$ is missing at $v$ if $c \not\in C_f(G, v)$. Let $\text{Miss}(f, v)$ be the set of all colors missing at $v$, that is, $\text{Miss}(f, v) = C \setminus C_f(G, v)$.

3. Algorithm

In this section we prove the following theorem.

**Theorem 1.** An optimal edge-coloring of a cactus can be found in polynomial time.

As a proof of Theorem 1, we give a dynamic programming algorithm in the remainder of this section to compute the minimum cost $\omega(G)$ of a given cactus $G$. Our algorithm can be easily modified so that it actually finds an optimal edge-coloring $f$ of $G$ with $\omega(f) = \omega(G)$.

A dynamic programming method is a standard one to solve a combinatorial problem on graphs with tree-construction. We also use it, and compute the minimum cost $\omega(G)$ of a cactus $G$ with an under tree $T$ by the bottom-up tree computation.

3.1. Ideas and Definitions

Let $b$ be a node of $T$ with its parent $b'$, and let $v$ be the vertex on both two blocks $b$ and $b'$. Let $b_1, b_2, \ldots, b_{\text{ch}(b)}$ be the children of $b$ in $T$. Then one can observe that the minimum cost $\omega(G_b)$ of the subgraph $G_b$ rooted at $b$ cannot be computed directly from the minimum costs $\omega(G_{b_j})$ of all the subgraphs $G_{b_j}$, $1 \leq j \leq \text{ch}(b)$. Our idea is to introduce a new parameter $\omega(G_b, i_1, i_2)$ defined for each node $b$ of $T$ and each pair of colors $c_{i_1}, c_{i_2} \in C$ as follows:

$$\omega(G_b, i_1, i_2) = \min \{\omega(f) \mid f \text{ is an edge-coloring of } G_b \text{ and } c_{i_1}, c_{i_2} \in C(f, v)\}.$$ 

If $G_b$ has no such edge-coloring we define $\omega(G_b, i_1, i_2) = +\infty$. Note that $\omega(G_b, i_1, i_2) = +\infty$ if either the block $b$ is an edge and $i_1 \neq i_2$ or the block $b$ is a cycle and $i_1 = i_2$. Clearly,

$$\omega(G_b) = \min_{1 \leq i_1, i_2 \leq 2\Delta-1} \omega(G_b, i_1, i_2).$$
We compute the values \( \omega(G_b, i_1, i_2) \) for all indices \( i_1, i_2, 1 \leq i_1, i_2 \leq 2\Delta - 1 \), from leaves to root \( r \). Thus the DP table for each node \( b \) consists of the \( O(\Delta^2) \) values \( \omega(G_b, i_1, i_2) \), \( 1 \leq i_1, i_2 \leq 2\Delta - 1 \).

Our algorithm computes \( \omega(G_b, i_1, i_2) \) for all pairs of colors \( c_i, c_j \in C \) from the leaves to the root \( r \) of \( T \), by means of dynamic programming. Then \( \omega(G) \) can be computed at the root \( r \) from all the values \( \omega(G_b, i_1, i_2) \) as follows:

\[
\omega(G) = \begin{cases} 
\min \{ \omega(G_b, i_1, i_2) | c_i \in C \} & \text{if the block } b \text{ is an edge;} \\
\min \{ \omega(G_b, i_1, i_2) | c_i, c_j \in C \text{ and } i_1 \neq i_2 \} & \text{if the block } b \text{ is a cycle}
\end{cases}
\]

and it can be computed in polynomial time. Thus the remainder problem is how to compute all the values \( \omega(G_b, i_1, i_2) \) for each node \( b \in V(T) \) of \( T \) and all pairs of colors \( c_i, c_j \in C \).

3.2. Algorithm

In this subsection, we explain how to compute all the values \( \omega(G_b, i_1, i_2) \) for each node \( b \in V(T) \) of \( T \) and all pairs of colors \( c_i, c_j \in C \).

3.2.1. The Node \( b \) Is a Leaf in \( T \)

In this case, the block \( b \) is either an edge or a cycle. Therefore we have the following two cases to consider.

**Case 1:** the block \( b \) is an edge.

In this case, clearly

\[
\omega(G_b, i_1, i_2) = \begin{cases} 
\omega(c_i) & \text{if } i_1 = i_2; \\
+\infty & \text{if } i_1 \neq i_2,
\end{cases}
\]

and all the values \( \omega(G_b, i_1, i_2), c_i, c_j \in C \), can be computed in time polynomial in \(|C|\).

**Case 2:** the block \( b \) is a cycle.

In this case, we describe the following algorithm to compute \( \omega(G_b, i_1, i_2) \) in time polynomial in the size of \( G_b \) and \(|C|\).

Algorithm AlgLeaf(\( G_b, i_1, i_2 \)):

1: let \( C = \{c_1, c_2, \cdots, C_{2\Delta-1}\} \);
2: let \( v_1, v_2, \cdots, v_x \) be the vertices lied on the cycle of \( G_b \) in the clockwise order;
3: assume that \( v_1 \) is also on other blocks, that is, \( d(G, v_1) \geq 2 \) and \( d(G, v_j) = 2 \) for all \( j, 2 \leq j \leq x \);
4: if \( i_1 = i_2 \) then
5: return \( \omega(G_b, i_1, i_2) = +\infty \);
6: else
7: if \( i_1 \) or \( i_2 = 1 \) then
8: assume without loss of generality that \( i_1 = 1 \);
9: if \( i_2 \neq 2 \) then
10: return \( \omega(G_b, i_1, i_2) = \omega(c_2) + \omega(c_1) \times \lfloor (x - 1) / 2 \rfloor + \omega(c_2) \times \lfloor (x - 1) / 2 \rfloor \);
11: else
12: if \( x \) is even then
13: return \( \omega(G_b, i_1, i_2) = \omega(c_1) \times x / 2 + \omega(c_2) \times x / 2 \);
14: else
15: return \( \omega(G_b, i_1, i_2) = \omega(c_1) \times (x - 1) / 2 + \omega(c_2) \times (x - 1) / 2 + \omega(c_3) \);
16: end if
17: end if
18: else
19: if \( i_1 \) or \( i_2 = 2 \) then
20: assume without loss of generality that \( i_1 = 2 \) and \( i_2 \geq 3 \);
21: return \( \omega(G_b, i_1, i_2) = \omega(c_2) + \omega(c_1) \times \lfloor (x - 1) / 2 \rfloor + \omega(c_2) \times \lfloor (x - 1) / 2 \rfloor \);
22: else
23: return \( \omega(G_b, i_1, i_2) = \omega(c_3) + \omega(c_2) + \omega(c_1) \times \lfloor (x - 2) / 2 \rfloor + \omega(c_2) \times \lfloor (x - 2) / 2 \rfloor \);
24: end if
25: end if
26: end if
3.2.2. The Node \( b \) Is an Internal Node

In order to compute \( \omega(G_b, i_1, i_2) \) for each pair of indices \( i_1 \) and \( i_2, \ 1 \leq i_1, i_2 \leq |C| \), we introduce a new parameter \( \omega^*(B, v, i_1, i_2) \) defined as follows.

Let \( B = \{h_1, h_2, \ldots\} \) be a set of blocks of \( T \) such that all these blocks share exactly one vertex \( v \) in \( G \). For each pair of colors \( c_1, c_2 \in C \) we define

\[
\omega^*(B, v, i_1, i_2) = \min \{ \omega(f) | f \text{ is an edge-coloring of } G, \text{ and } c_1, c_2 \in \text{Miss}(f, v) \}.
\]

We show how to compute the all the values \( \omega^*(B, v, i_1, i_2) \) from the \( |B| \times |C|^2 \) values \( \omega(G_b, i_1, i_2) \), \( 1 \leq j \leq |B| \) and \( 1 \leq i_1, i_2 \leq |C| \). The problem of computing \( \omega^*(B, v, i_1, i_2) \) can be reduced to the minimum cost flow problem on a bipartite graph \( K(i_1, i_2) \) as follows.

We first introduce \( |B| \times |C|^2 \) isolated vertices \( v_{i,j}^l, 1 \leq j \leq |B| \) and \( 1 \leq i_1, i_2 \leq |C| \). Then add \( |C| \) vertices \( v_1, 1 \leq l \leq |C| \), corresponding to colors \( c_1 \), and add a source \( s \) and a sink \( t \). Connect the source \( s \) to all the \( |C| \) vertices \( v_1, 1 \leq l \leq |C| \), with capacity 1 and cost 0. For each vertex \( v_j \), \( 1 \leq l \leq |C| \) and \( l \neq i_1, i_2 \), connect \( v_j \) to all the vertices \( v_{j,i}^l, 1 \leq j \leq |B| \) and \( 1 \leq i_1, i_2 \leq |C| \), satisfying \( l_1 = l \) or \( l_2 = l \) with capacity 1 and cost 0. Finally, for each vertex \( v_{j,i}^l \), \( 1 \leq j \leq |B| \) and \( 1 \leq i_1, i_2 \leq |C| \), connect \( v_{j,i}^l \) to the sink \( t \) with capacity 2 and cost \( \omega(G_b, l_1, l_2) \). The minimum cost flow problem is to find a maximum flow from \( s \) to \( t \) with the sum of costs of edges on the flow. Clearly \( \omega^*(B, v, i_1, i_2) \) is equal to the cost of the minimum cost maximum flow in \( K(i_1, i_2) \).

The minimum cost maximum flow problem can be solved in time polynomial in the size of the graph \([9][10]\), and hence the value \( \omega^*(B, v, i_1, i_2) \) for a pair of indices \( i_1 \) and \( i_2, \ 1 \leq i_1, i_2 \leq |C| \), can be computed in time polynomial in \( |B| \) and \( |C| \) since \( K(i_1, i_2) \) has at most \( O(|B| \times |C|^2) \) vertices and edges. Therefore the \( |C|^2 \) values \( \omega^*(B, v, i_1, i_2) \) for all pairs of indices \( i_1 \) and \( i_2, \ 1 \leq i_1, i_2 \leq |C| \), can be computed total in time polynomial in \( |B| \) and \( |C| \).

We are now ready to compute \( \omega(G_b, i_1, i_2) \). Since the block \( b \) is either an edge or a cycle, we have the following two cases to consider.

**Case 1:** the block \( b \) is an edge \( e = (u,v) \).

Let \( B = \{h_1, h_2, \ldots, h_{ch(b)}\} \) be the set of blocks of the children of \( b \) in \( T \). Then all the blocks \( h_1, h_2, \ldots, h_{ch(b)} \) share exactly one vertex \( v \) in \( G \). In this case, clearly

\[
\omega(G_b, i_1, i_2) = \begin{cases} \omega^*(B, v, i_1, i_2) & \text{if } i_1 = i_2, \\ +\infty & \text{if } i_1 \neq i_2; \end{cases}
\]

and it can be computed in time polynomial in the size of \( G_b \) and \( |C| \).

**Case 2:** the block \( b \) is a cycle.

In this case, let \( v_1, v_2, \ldots, v_s \) be the vertices lied on the cycle of \( G_b \) in the clockwise order. Assume that \( v_1 \) is the vertex shared by the block \( b \) and its parent block, and let \( B(v_j), \ 2 \leq j \leq x \), be the set of blocks which shares \( v_j \); \( B(v_j) = \emptyset \) if no such blocks exist. In order to compute \( \omega(G_b, i_1, i_2) \) we define

\[
\omega_{k,j}(i_1, i_j) = \min \left\{ \sum_{l \in \omega_j} \omega^*(B(v_p), v_p, l_{p-1}, l_p) + \sum_{l \in \omega_j} \omega(c_p) \right\}
\]

for each \( j, \ 2 \leq j \leq x \), where \( l_1 = i_1 \). Then clearly

\[
\omega(G_b, i_1, i_2) = \omega_{i,j}(i_1, i_2).
\]

Therefore it suffices to show how to compute \( \omega_{i,j}(i_1, i_2) \) in polynomial time for each \( j, \ 2 \leq j \leq x \), as follows.

By Equation (1) we have

\[
\omega_{i,j+1}(i_1, i_{j+1}) = \min \left\{ \sum_{l \in \omega_{j+1}} \omega^*(B(v_p), v_p, l_{p-1}, l_p) + \sum_{l \in \omega_{j+1}} \omega(c_p) \right\}
\]

\[
= \min \left\{ \omega_{i,j}(i_1, i_j) + \omega^*(B(v_{j+1}), v_{j+1}, l_j, l_{j+1}) + \sum_{l \in \omega_{j+1}} \omega(c_p) \right\}.
\]
and hence $\omega^*(i,j,l)$ for all $j, 2 \leq j \leq x$, can be recursively computed total in time $O(x | C |)$ if all the values $\omega^*(B(v_j),v_j,l_1,l_2)$, $1 \leq l_1,l_2 \leq |C|$, are given. Since we have mentioned before that all the values $\omega^*(B(v_j),v_j,l_1,l_2)$ can be computed in time polynomial in $|B(v_j)|$ and $|C|$, one can compute all $\omega^*(i,j,l)$ and hence $\omega(G_i,i,l)$ total in time polynomial in $n(G_i)$ and $|C|$.

4. Conclusion

In this paper, we show that the cost edge-coloring problem for a cactus $G$ can be solved in polynomial time. It is still open to solve the problem in polynomial time for outerplanar graphs.

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References