On the Use of Second and Third Moments for the Comparison of Linear Gaussian and Simple Bilinear White Noise Processes

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Abstract
The linear Gaussian white noise process (LGWNP) is an independent and identically distributed (iid) sequence with zero mean and finite variance with distribution $N(0, \sigma^2)$. Some processes, such as the simple bilinear white noise process (SBWNP), have the same covariance structure like the LGWNP. How can these two processes be distinguished and/or compared? If $X_1, X_2, \cdots, X_n$ is a realization of the SBWNP. This paper studies in detail the covariance structure of $X_t, t \in Z; d = 1, 2, 3$. It is shown from this study that; 1) the covariance structure of $X_t, t \in Z$ is non-normal with distribution equivalent to the linear ARMA(2, 1) model; 2) the covariance structure of $X^1_t, t \in Z$ is iid; 3) the variance of $X^1_t, t \in Z$ can be used for comparison of SBWNP and LGWNP.

Keywords
White Noise Process, Normality, Stationarity, Invertibility, Covariance Structure

1. Introduction
A stochastic process $X_t, t \in Z$, where $Z = \{\cdots, -1, 0, 1, \cdots\}$ is called a white noise or purely random process, if with finite mean and finite variance, all the autocovariances are zero except at lag zero. In many applications, $X_t, t \in Z$ is assumed to be normally distributed with mean zero and variance, $\sigma^2 < \infty$, and
the series is called a linear Gaussian white noise process with the following properties [1]-[7].

\[ E(X_t) = \mu \]  
\[ R(0) = \text{var}(X_t) = E(X_t - \mu)^2 \]  
\[ R(k) = \text{cov}(X_t, X_{t+k}) = E[(X_t - \mu)(X_{t+k} - \mu)] = \begin{cases} \sigma^2, & k = 0 \\ 0, & \text{otherwise} \end{cases} \]  
\[ \rho(k) = \text{corr}(X_t, X_{t+k}) = \frac{R(k)}{R(0)} = \begin{cases} 1, & k = 0 \\ 0, & \text{otherwise} \end{cases} \]  
\[ \phi_k = \text{corr}(X_t, X_{t+k} / X_{t+1}, X_{t+2}, \cdots, X_{t+k-1}) = 0 \quad \forall k \]

where \( R(k) \) is the autocovariance function at lag \( k \), \( \rho_k \) is the autocorrelation function at lag \( k \) and \( \phi_k \) is the partial autocorrelation function at lag \( k \).

In other words, a stochastic process \( X_t, t \in Z \) is called a linear Gaussian white noise if \( X_t, t \in Z \) is a sequence of independent and identically distributed (iid) random variables with finite mean and finite variance. Under the assumption that the sample \( X_1, X_2, \cdots, X_n \) is an iid sequence, we compute the sample autocorrelations as

\[ \hat{\rho}_k(k) = \frac{\sum_{i=1}^{n}(X_i - \bar{X})(X_{i+k} - \bar{X})}{\sum_{i=1}^{n}(X_i - \bar{X})^2} \]  

where

\[ \bar{X} = \frac{1}{n} \sum_{i=1}^{n} X_i \]

The iid hypothesis is always tested with the Ljung and Box [8] statistic

\[ Q_{LB}(m) = n(n+2)\sum_{k=2}^{m} \left( \frac{\hat{\rho}_k(k)}{n-k} \right)^2 \]  

where \( Q_{LB}(m) \) is asymptotically a chi-squared random variable with \( m \) degree of freedom.

Several values of \( m \) are often used and simulation studies suggest that the choice of \( m \approx \ln(n) \) provides better power performance [9].

If the data are iid, the squared data \( X_1^2, X_2^2, \cdots, X_n^2 \) are also iid [10]. Another portmanteau test formulated by Mcleod and Li [10] is based on the same statistic used for the Ljung and Box [8]

\[ Q_{ML}(m) = n(n+2)\sum_{k=2}^{m} \left( \frac{\hat{\rho}_k^2(k)}{n-k} \right) \]  

where the sample autocorrelations of the data are replaced by the sample autocorrelations of the squared data, \( \hat{\rho}_k^2(k) \).
As noted by Iwueze et al. [11], a stochastic process \( X_t, t \in Z \) may have the covariance structure (1.1) through (1.5) even when it is not the linear Gaussian white noise process. Iwueze et al. [11] provided additional properties of the linear Gaussian white noise process for proper identification and characterization from other processes with similar covariance structure (1.1) through (1.5).

Let \( Y_t = X^d_t, d = 1, 2, 3, \ldots \) where \( X_t, t \in Z \), be the linear Gaussian white noise process, the mean \( E(Y_t) = E(X^d_t) \), the variance \( \text{var}(Y_t) = \text{var}(X^d_t) \), autocovariances \( R_y(k) = \text{cov}(Y_t Y_{t+k}) = \text{cov}(X^d_t X^d_{t+k}) \) were obtained to be [11]

\[
E(Y_t) = E(X^d_t) = \begin{cases} \sigma^{2m}(2m-1)!! & \text{if } d = 2m, m = 1, 2, \ldots \\ 0 & \text{if } d = 2m+1, m = 0, 1, 2, \ldots \end{cases} \tag{1.10}
\]

\[
\text{Var}(Y_t) = \text{Var}(X^d_t) = \begin{cases} \sigma^{4m} \left( \prod_{k=1}^{2m} (2k-1) - \left( \prod_{k=1}^{m} (2k-1) \right)^2 \right) & \text{if } d = 2m \\ \sigma^{4m+2} \left( \prod_{k=1}^{2m+1} (2k-1) \right) & \text{if } d = 2m+1 \end{cases} \tag{1.11}
\]

\[
R_y(k) = R_{X^d}(\ell) = \begin{cases} \sigma^{2m} \left( \prod_{k=1}^{2m} (2k-1) - \left( \prod_{k=1}^{m} (2k-1) \right)^2 \right) & \text{if } d = 2m, \ell = 0 \\ \sigma^{2m+2} \left( \prod_{k=1}^{2m+1} (2k-1) \right) & \text{if } d = 2m+1, \ell = 0 \\ 0 & \text{if } \ell \neq 0 \end{cases} \tag{1.12}
\]

where

\[
(2m-1)!! = \prod_{k=1}^{m} (2k-1) \tag{1.13}
\]

It is clear from (1.12) that when \( X_t, t \in Z \) are iid, the powers \( Y_t = X^d_t, d = 1, 2, 3, \ldots \) of \( X_t, t \in Z \) are also iid. Iwueze et al. [11] also showed the probability density function (pdf) of \( Y_t = X^2_t \) to be the pdf of a gamma distribution with parameters

\[
\alpha = \frac{1}{2}, \beta = 2\sigma^2. \quad \text{That is, } \quad Y_t = X^2_t \sim G(\alpha, \beta), \alpha = \frac{1}{2}, \beta = 2\sigma^2.
\]

when \( X_t \sim N(0, \sigma^2) \) and [11] concluded that all powers of a linear Gaussian white noise process are iid but not normally distributed.

Using the coefficient of symmetry and kurtosis, Iwueze et al. [11] confirmed the non-normality of \( Y_t = X^d_t, d = 2, 3, \ldots \). Table 1 gives the mean, variance, the coefficient of symmetry \( \beta_1 \) and kurtosis \( \beta_2 \) defined as follows

\[
\beta_1 = \frac{\mu_3(d)}{\left( \mu_2(d) \right)^{3/2}} \tag{1.14}
\]

\[
\beta_2 = \frac{\mu_4(d)}{\left( \mu_2(d) \right)^2} \tag{1.15}
\]

where

\[
\mu_2(d) = E\left[ \left( X^d_t - E\left( X^d_t \right) \right)^2 \right] = \text{var}(X^d_t) \tag{1.16}
\]
Table 1. Mean, Variance, Coefficient of symmetry ($\beta_1$) and kurtosis ($\beta_2$) for $Y = X^d_t$, $d = 1, 2, 3, \cdots, 6$, when $X_t \sim N(0, \sigma^2)$.

<table>
<thead>
<tr>
<th>$d$</th>
<th>$Y_t$</th>
<th>$E(Y_t)$ ($\mu_t$)</th>
<th>$\mu_t^2$ (d)</th>
<th>$\mu_t^4$ (d)</th>
<th>$\beta_1$</th>
<th>$\beta_2$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>$X_t$</td>
<td>0</td>
<td>$\sigma^2$</td>
<td>0</td>
<td>3$\sigma^4$</td>
<td>0</td>
</tr>
<tr>
<td>2</td>
<td>$X_t^2$</td>
<td>$\sigma^2$</td>
<td>2$\sigma^4$</td>
<td>8$\sigma^6$</td>
<td>60$\sigma^8$</td>
<td>2.828</td>
</tr>
<tr>
<td>3</td>
<td>$X_t^3$</td>
<td>0</td>
<td>15$\sigma^6$</td>
<td>0</td>
<td>10395$\sigma^{12}$</td>
<td>0</td>
</tr>
<tr>
<td>4</td>
<td>$X_t^4$</td>
<td>3$\sigma^4$</td>
<td>96$\sigma^6$</td>
<td>9504$\sigma^{10}$</td>
<td>1907712$\sigma^{12}$</td>
<td>10.104</td>
</tr>
<tr>
<td>5</td>
<td>$X_t^5$</td>
<td>0</td>
<td>945$\sigma^{10}$</td>
<td>0</td>
<td>654729075$\sigma^{12}$</td>
<td>0</td>
</tr>
<tr>
<td>6</td>
<td>$X_t^6$</td>
<td>15$\sigma^6$</td>
<td>10170$\sigma^8$</td>
<td>33998400$\sigma^{10}$</td>
<td>$3.142 \times 10^9 \sigma^{12}$</td>
<td>33.150</td>
</tr>
</tbody>
</table>

Source: Iwueze et al. (2017).

$$\mu_t^2 (d) = E\left[\left(X_t^d - E\left(X_t^d\right)\right)^2\right]$$ (1.17)

$$\mu_t^4 (d) = E\left[\left(X_t^d - E\left(X_t^d\right)\right)^4\right]$$ (1.18)

Using the standard deviations when $\sigma^2 = 1$ and the kurtosis of $Y_t = X_t^d, d = 1, 2, 3, \cdots$, Iwueze et al. [11] determined the optimal value of $d$ to be three ($d = 3$). Hence, for effective comparison of the linear Gaussian white noise process with any stochastic process with similar covariance structure, $Y_t = X_t^d, d = 1, 2, 3$ must be used.

The most commonly used white noise process is the linear Gaussian white noise process. The process is one of the major outcomes of any estimation procedure which is used in checking the adequacy of fitted models. The linear Gaussian white noise process also plays significant role as a basic building block in the construction of linear and non-linear time series models. However, the major problem is that there are many non-linear processes that exhibit the same covariance structure (Equation (1.1) through Equation (1.5)) as the linear Gaussian white noise process. One of such non-linear models is the bilinear models.

The study of bilinear models was introduced by Granger and Andersen [12] and Subba Rao [13]. Granger and Andersen [14] established that all series generated by the simple bilinear model

$$X_t = \beta X_{t-k} e_{t-j} + e_t, \quad k > j$$ (1.19)

appear to be second order white noise where $\beta$ is a constant and $e_t, t \in Z$ is an independent identically distributed random variable with $E(e_t) = 0$, $E(e_t^2) = \sigma^2 < \infty$. Guegan [15] studied the existence problem of a simple bilinear process $X_t, t \in Z$ satisfying

$$X_t = \beta X_{t-2} e_{t-1} + e_t$$ (1.20)

Martins [16] obtained the autocorrelation function of the process $X_t^2, t \in Z$ for the simple bilinear model defined by (1.19) when $e_t, t \in Z$ is iid with a Gaussian distribution. Again, Martins [16] studied the third order moment
structure of (1.19) with non-independent shocks. Recently, properties of the simple bilinear model (1.19) were addressed by Malinski and Bielinska [17], Malinski and Figwer [18] and Malinski [19]. Iwueze [20] studied the more general bilinear white noise model

\[ X_t = \left( \sum_{j=1}^{m} \beta_j X_{t-j} \right) e_{t-q} + e_t \]  

(1.21)

where \( e_t, t \in \mathbb{Z} \) is as defined in (1.19). Iwueze [20] was able to show the following.

1) The series \( X_t, t \in \mathbb{Z} \) satisfying (1.21) is strictly stationary, ergodic and unique.

2) The series \( X_t, t \in \mathbb{Z} \) satisfying (1.21) is invertible.

3) The series \( X_t, t \in \mathbb{Z} \) satisfying (1.21) has the same covariance structure as the linear Gaussian white noise processes.

4) Obtained the covariance structure of (1.21) to be

\[ \mu = E(X_t) = 0 \]  

(1.22)

\[ R(k) = \begin{cases} \frac{\sigma^2}{1 - \sum_{j=1}^{m} \sigma^2 \beta_j^2}, & k = 0 \\ 1 - \sum_{j=1}^{m} \sigma^2 \beta_j^2, & 0, \text{ otherwise} \end{cases} \]  

(1.23)

5) The series satisfying (1.21) is invertible if

\[ 2 \sum_{j=1}^{m} \beta_j^2 \sigma^2 < 1 \]  

(1.24)

For the simple bilinear model (1.19), it follows that

\[ R(k) = \begin{cases} \frac{1}{1 - \sigma^2 \beta^2}, & \sigma^2 \beta^2 < 1 \\ 0, & \text{otherwise} \end{cases} \]  

(1.25)

and the invertibility condition is

\[ \sigma^2 \beta^2 < \frac{1}{2} \]  

(1.26)

It is worthy to note that the stationarity condition

\[ \sigma^2 \beta^2 < 1 \]  

(1.27)

is structure \((k, n)\) independent [19] for model (1.19) and our study in this paper will concentrate on model (1.20). The purpose of this paper is to meet the following goals for the simple bilinear model satisfying (1.20).

1) Determine \( \text{Var}(X_t^d), d = 2, 3 \) for the simple bilinear model (1.20).

2) Determine the covariance structure of \( X_t^d, d = 2, 3 \), when \( X_t, t \in \mathbb{Z} \) satisfies (1.20).

3) Determine for what values of \( \beta \) the simple bilinear white noise process will be identified as a Linear Gaussian white noise process.
4) Determine for what values of $\beta$ the simple bilinear model will be normally distributed.

This paper is further divided into four sections in order to establish and achieve these goals. Section 2 discusses the covariance structure of $Y_t = X_t^d, d = 1, 2, 3$ when $X_t = \beta X_{t-2}e_{t-1} + e_t$, $e_t \sim iid N(0, \sigma^2)$, Section 3 presents the methodology, Section 4 is the results and discussion while, Section five is the conclusion.

2. Covariance Structure of $Y_t = X_t^d, d = 1, 2, 3$, When $X_t = \beta X_{t-2}e_{t-1} + e_t$, $e_t \sim iid N(0, \sigma^2)$

**Theorem 2.1.**

Let $e_t, t \in Z$ be the linear Gaussian white noise process with $E(e_t) = 0$ and $E(e_t^3) = \sigma^2 < \infty$. Suppose there exists a stationary and invertible process $X_t, t \in Z$ satisfying $X_t = \beta X_{t-2}e_{t-1} + e_t$ for every $t \in Z$ for some constant $\beta$, then $Y_t = X_t^2$ has the following properties:

$$E(Y_t) = \mu_t = \frac{\sigma^2}{1 - \beta^2}; \quad \sigma^2 \beta^2 < 1 \quad (2.1)$$

$$R_t(k) = \text{cov}(Y_t, Y_{t+k}) = \begin{cases} 
\frac{2\sigma^4}{(1 - \beta^2)^2(1 - 3\sigma^2 \beta^4)}, & \sigma^2 \beta^2 < \frac{1}{\sqrt{3}}, \ k = 0 \\
\frac{2\sigma^4 \beta^2}{(1 - \beta^2)^2}, & \sigma^2 \beta^2 < 1, \ k = 1 \\
\sigma^2 \beta^2 R_t(k - 2), & k = 2, 3, \ldots 
\end{cases} \quad (2.2)$$

$$\rho_t(k) = \frac{R_t(k)}{R_t(0)} = \begin{cases} 
1, & k = 0 \\
\sigma^2 \beta^2(1 - 3\sigma^2 \beta^4), & k = 1 \\
\sigma^2 \beta^2 \rho_t(k - 2), & k = 2, 3, \ldots 
\end{cases} \quad (2.3)$$

$Y_t = X_t^2, t \in Z$ has the same covariance structure as the linear ARMA(2, 1) process (2.4)

$$X_t = \lambda + \phi_1 X_{t-1} + \phi_2 X_{t-2} + \theta_t a_{t-1} + a_t, \quad \phi_i = 0 \quad (2.4)$$

where $a_t$ is the sequence of independent and identically distributed random variable with $E(a_t) = 0$ and $\text{Var}(a_t) = \sigma^2 < \infty$.

**Proof:**

Let $Y_t = X_t^2 = (\beta X_{t-2}e_{t-1} + e_t)^2 = \beta^2 X_{t-2}^2 e_{t-1}^2 + e_t^2 + 2\beta X_{t-2}e_{t-1}e_t$

$$E(Y_t) = E(X_t^2) = \beta^2 E(X_{t-2}^2) E(e_{t-1}^2) + E(e_t^2) + 2\beta E(X_{t-2}) E(e_{t-1}) E(e_t)$$

$$E(Y_t) = E(X_t^2) = \beta^2 E(X_{t-2}^2) E(e_t^2) + E(e_t^2) = \sigma^2 \beta^2 E(X_{t-2}^2) + \sigma^2$$

$$E(Y_t) = E(X_t^2) = \sigma^2$$
\[
\mu_t = E\left(X_t^2\right) = \frac{\sigma^2}{1-\sigma^2 \beta^2}; \quad \sigma^2 \beta^2 < 1 \quad (2.5)
\]

\[
Var(Y_t) = Var\left(X_t^2\right) = E\left(X_t^4\right) - \left[E\left(X_t^2\right)\right]^2
\]

\[
X_t^4 = \beta^4 X_t^2 e_t^4 + 4 \beta^3 X_t^2 e_t^2 e_t^4 e_t^2 + 6 \beta^2 X_t^2 e_t^2 e_t^2 e_t^2 + 4 \beta X_t^2 e_t^2 e_t^2 e_t^2 e_t^2 + e_t^4
\]

\[
E\left(X_t^4\right) = 3\sigma^4 \beta^4 E\left(X_t^4\right) + 6\sigma^4 \beta^2 E\left(X_t^2\right) + 3\sigma^4
\]

\[
\left(1 - 3\sigma^4 \beta^4\right) E\left(X_t^4\right) = \frac{6\sigma^4 \beta^2}{1-\sigma^2 \beta^2} + 3\sigma^4
\]

\[
\Rightarrow E\left(X_t^4\right) = \frac{3\sigma^4 \left(1 + \sigma^2 \beta^2\right)}{(1-\sigma^2 \beta^2)(1-3\sigma^4 \beta^4)}, \quad \sigma^2 \beta^2 < \frac{1}{\sqrt{3}} \quad (2.6)
\]

Now,

\[
Var(Y_t) = Var\left(X_t^2\right) = E\left(X_t^4\right) - \left[E\left(X_t^2\right)\right]^2
\]

\[
= \frac{3\sigma^4 \left(1 + \sigma^2 \beta^2\right)}{(1-\sigma^2 \beta^2)(1-3\sigma^4 \beta^4)} - \left(\frac{\sigma^2}{1-\sigma^2 \beta^2}\right)^2 \quad (2.7)
\]

\[
= \frac{3\sigma^4 \left(1 + \sigma^2 \beta^2\right) - \sigma^4 \left(1-3\sigma^4 \beta^4\right)}{(1-\sigma^2 \beta^2)^2 (1-3\sigma^4 \beta^4)}
\]

Hence,

\[
R_t(0) = Var(Y_t) = Var\left(X_t^2\right) = \frac{2\sigma^4}{(1-\sigma^2 \beta^2)^2 (1-3\sigma^4 \beta^4)}, \quad \sigma^2 \beta^2 < \frac{1}{\sqrt{3}} \quad (2.8)
\]

\[
R_t(k) = E[Y_t Y_{t-k}] - \mu_t^2 = E\left[X_t^2 X_{t-k}^2\right] - \mu_t^2, k = 0,1,2,\ldots
\]

\[
Y_{t-k} = X_t^2 X_{t-k}^2 = \beta^2 X_t^2 X_{t-k}^2 e_t^2 e_t^2 + 2\beta X_t^2 X_{t-k}^2 e_t^2 e_t^2 + X_t^2 e_t^2
\]

\[
E\left[Y_{t-k}\right] = \beta^2 E\left[X_t^2 X_{t-k}^2 e_t^2 e_t^2\right] + \sigma^2 E\left(X_t^2\right)
\]

\[
E\left[Y_{t-k}\right] = \beta^2 E\left[X_t^2 X_{t-k}^2 e_t^2 e_t^2\right] + \sigma^2 E\left(X_t^2\right)
\]

\[
X_t^2 X_{t-k}^2 e_t^2 e_t^2 = X_t^2 \left(\beta^2 X_t^2 e_t^2 e_t^2 + 2\beta X_t^2 e_t^2 e_t^2 + e_t^2\right) e_t^2
\]

\[
X_t^2 X_{t-k}^2 e_t^2 = \beta^2 X_t^2 X_{t-k}^2 e_t^2 e_t^2 + 2\beta X_t^2 X_{t-k}^2 e_t^2 e_t^2 + X_t^2 e_t^2
\]

By the assumption of stationarity,

\[
E\left[X_t^2 X_{t-k}^2 e_t^2\right] = \sigma^2 \beta^2 E\left[X_t^2 X_{t-k}^2 e_t^2\right] + 3\sigma^4 E\left(X_t^2\right)
\]

\[
\left(1 - \sigma^2 \beta^2\right) E\left[X_t^2 X_{t-k}^2 e_t^2\right] = 3\sigma^4 \left(\frac{\sigma^2}{1-\sigma^2 \beta^2}\right)
\]

\[
E\left[X_t^2 X_{t-k}^2 e_t^2\right] = \frac{3\sigma^6}{(1-\sigma^2 \beta^2)^2}, \quad \sigma^2 \beta^2 < 1 \quad (2.9)
\]

\[
E\left[Y_{t-k}\right] = \beta^2 \left[\frac{3\sigma^6}{(1-\sigma^2 \beta^2)^2}\right] + \sigma^2 \left(\frac{\sigma^2}{1-\sigma^2 \beta^2}\right) = \sigma^4 \left(1 + 2\sigma^2 \beta^2\right) \left(\frac{1}{1-\sigma^2 \beta^2}\right) \quad (2.10)
\]
Hence,

\[ R_y(1) = E(Y_{Y_{t-1}}) = E^2(Y_t) = \frac{\sigma^4(1 + 2\sigma^2\beta^2)}{(1 - \sigma^2\beta^2)^2} - \left(\frac{\sigma^2}{1 - \sigma^2\beta^2}\right)^2 = \frac{2\sigma^6\beta^2}{(1 - \sigma^2\beta^2)^2} \] (2.11)

\[ Y_{Y_{t-2}} = X_t^2 X_{t-2}^2 = (\beta^2 X_{t-2}^2 e_{t-1} + 2\beta X_{t-2} e_{t-1} e_t + e_t^2) X_{t-2}^2 \]

\[ Y_{Y_{t-2}} = \beta^2 X_{t-2}^2 e_{t-1} + 2\beta X_{t-2} e_{t-1} e_t + X_{t-2}^2 e_t^2 \]

\[ E[Y_{Y_{t-2}}] = \sigma^2 \beta^2 E[X_{t-2}^2] + \sigma^2 E(X_{t-2}^2) \]

\[ E[Y_{Y_{t-2}}] = \sigma^2 \beta^2 E(Y_{t-2}^2) + \sigma^2 E(Y_t) \]

\[ \Rightarrow E[Y_{Y_{t-2}}] = \sigma^2 \beta^2 E(Y_t^2) + \sigma^2 \mu_y \]

\[ R_y(2) + \mu_y^2 = \sigma^2 \beta^2 \left[ R_y(0) + \mu_y^2\right] + \sigma^2 \mu_y \] (2.12)

\[ R_y(2) = \sigma^2 \beta^2 R_y(0) + \sigma^2 \beta^2 \mu_y + \sigma^2 \mu_y - \mu_y^2 \]

\[ = \sigma^2 \beta^2 R_y(0) + \sigma^2 \mu_y - \mu_y (1 - \sigma^2 \beta^2) \]

Note that

\[ \mu_r = E(Y_r) = E(X_r^2) = \frac{\sigma^2}{1 - \sigma^2 \beta^2} \]

\[ \Rightarrow (1 - \sigma^2 \beta^2) \mu_r = \sigma^2 \]

\[ 1 - \sigma^2 \beta^2 = \frac{\sigma^2}{\mu_y} \] (2.13)

Hence

\[ R_y(2) = \sigma^2 \beta^2 R_y(0) + \sigma^2 \mu_y - \mu_y \left(\frac{\sigma^2}{\mu_y}\right) \] (2.14)

\[ = \sigma^2 \beta^2 R_y(0) + \sigma^2 \mu_y - \sigma^2 \mu_y = \sigma^2 \beta^2 R_y(0) \]

We have shown that

\[ \sigma^2 \beta^2 \mu_y^2 + \sigma^2 \mu_y - \mu_y^2 = 0 \] (2.15)

Similarly;

\[ Y_{Y_{t-3}} = X_t^2 X_{t-3}^2 = (\beta^2 X_{t-3}^2 e_{t-1} + 2\beta X_{t-3} e_{t-1} e_t + e_t^2) X_{t-3}^2 \]

\[ Y_{Y_{t-3}} = \beta^2 X_{t-3}^2 X_{t-2}^2 e_{t-1} + 2\beta X_{t-3}^2 e_{t-1} e_t + X_{t-3}^2 e_t^2 \]

\[ E[Y_{Y_{t-3}}] = \sigma^2 \beta^2 E[X_{t-3}^2 X_{t-1}^2] + \sigma^2 E(X_t^2) = \sigma^2 \beta^2 E(Y_{t-1}^2) + \sigma^2 E(Y_t) \]

\[ \Rightarrow R_y(3) + \mu_y^3 = \sigma^2 \beta^2 \left[R_y(1) + \mu_y^2\right] + \mu_y^3 \]

\[ = \sigma^2 \beta^2 R_y(1) + \sigma^2 \beta^2 \mu_y^2 + \sigma^2 \mu_y - \mu_y^3 \] (2.16)

Generally;

\[ R_y(k) = \sigma^2 \beta^2 R_y(k-2), \quad k = 2, 3, \ldots \] (2.17)

Hence,
\[
R_r(k) = \begin{cases}
\frac{2\sigma^4}{(1-\sigma^2)^2(1-\sigma^4)}, & \sigma^2\beta^2 < \frac{1}{\sqrt{3}}, \ k = 0 \\
\frac{2\sigma^4\beta^2}{(1-\sigma^2)^2}, & \sigma^2\beta^2 < 1, \ k = 1 \\
\sigma^2\beta^2R_r(k-2), & k = 2,3,\ldots
\end{cases}
\] (2.18)

and
\[
\rho_r(k) = \begin{cases}
1, & k = 0 \\
\sigma^2\beta^2(1-3\sigma^4\beta^4), & k = 1 \\
\sigma^2\beta^2\rho_r(k-2), & k = 2,3,\ldots
\end{cases}
\] (2.19)

With this result, it is clear that when \( X_t, t \in Z \) is defined by (1.20), \( Y_t = X_t^2 \) has the same covariance structure as the linear ARMA(2, 1) process. Its linear equivalence is
\[
Y_t = \lambda + \phi_1X_{t-1} + \phi_2Y_{t-2} + \theta a_{t-1} + a_t, \quad \phi_1 = 0
\] (2.20)

where \( a_t \) is the purely random process with \( E(a_t) = 0 \) and \( Var(a_t) = \sigma_a^2 < \infty \). Table 2 compares \( Y_t = X_t^2 \) with its linear ARMA(2, 1) equivalence.

**Theorem 2.2:**

Let \( e_t, t \in Z \) be the linear Gaussian white noise process with \( E(e_t) = 0 \) and \( E(e_t^2) = \sigma_e^2 < \infty \). Suppose there exists a stationary and invertible process \( X_t, t \in Z \) satisfying
\[
X_t = \beta X_{t-2} + e_t \quad \text{for every} \quad t \in Z \quad \text{and some constant} \quad \beta,
\]
then the mean and variance of \( Y_t = X_t^3, t \in Z \) are
\[
E(Y_t) = \mu_t = 0
\] (2.21)

<table>
<thead>
<tr>
<th>Properties</th>
<th>Process</th>
<th>Linear ARMA(2, 1)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Structure</td>
<td>( X_t = \beta X_{t-2}e_{t-2} + e_t ) ( e_t \sim N(0, \sigma_e^2) ) ( Y_t = X_t^2 \sim \text{ARMA}(2,1) )</td>
<td>( Y_t = \lambda + \phi_1X_{t-1} + \phi_2Y_{t-2} + \theta a_{t-1} + a_t, \quad \phi_1 = 0 ) ( E(a_t) = 0 ), ( Var(a_t) = \sigma_a^2 )</td>
</tr>
<tr>
<td>Mean</td>
<td>( \mu_t = E(Y_t) = E(X_t^2) = \sigma_e^2/1-\sigma_e^2 \sigma_t^2 &lt; 1 )</td>
<td>( \mu_t = E(Y_t) = \lambda \frac{1}{1-\phi_1} \left[ \frac{\lambda}{1-\phi_1} \mu_t \right] )</td>
</tr>
</tbody>
</table>
| Autocovariance | \( R_t(k) = \begin{cases}
\frac{2\sigma^4}{(1-\sigma^2)^2}, & \sigma^2\beta^2 < \frac{1}{\sqrt{3}}, \ k = 0 \\
\frac{2\sigma^4\beta^2}{(1-\sigma^2\beta^2)}, & \sigma^2\beta^2 < 1, \ k = 1 \\
\sigma^2\beta^2R_t(k-2), & k = 2,3,\ldots
\end{cases} \) | \( R_t(k) = \begin{cases}
\sigma_t^2 \left( 1 + \phi_t^2 \right) / \left( 1 - \phi_t \right), & \phi_t \neq 1, \ k = 1 \\
\phi_t^2 R_t(k-2), & k = 2,3,\ldots
\end{cases} \) |
| Autocorrelation | \( \rho_t(k) = \begin{cases}
1, & k = 0 \\
\sigma_t^2 \left( 1+\sigma_t^2\beta^2 \right), & k = 1 \\
\sigma_t^2 \beta^2 \rho_t(k-2), & k = 2,3,\ldots
\end{cases} \) | \( \rho_t(k) = \begin{cases}
1, & k = 0 \\
\phi_t^2 \left( 1+\phi_t^2 \right) / \left( 1 + \phi_t \right), & k = 1 \\
\phi_t \rho_t(k-2), & k = 2,3,\ldots
\end{cases} \) |
\[ R_t(k) = \begin{cases} 15\sigma^6 \left( 1 + 2\sigma^2 \beta^2 + 6\sigma^4 \beta^4 + 3\sigma^6 \beta^6 \right) \left( 1 - \sigma^2 \beta^2 \right) \left( 1 - 3\sigma^4 \beta^4 \right) \left( 1 - 15\sigma^6 \beta^6 \right), & \sigma^2 \beta^2 < \frac{1}{\sqrt{15}}, \ k = 0 \\ 0, & k \neq 0 \end{cases} \] (2.22)

\[ \rho_t(k) = \begin{cases} 1, & k = 0 \\ 0, & k \neq 0 \end{cases} \] (2.23)

The covariance structure of \( Y_t = X^3_t, t \in \mathbb{Z} \) is that of the linear white noise process.

**Proof:**

Let \( Y_t = X^3_t = (\beta X_{t-2}, \varepsilon_{t-1} + \varepsilon_t)^3 = \beta^3 X^3_{t-2}, \varepsilon_{t-1}^3 + 3\beta^2 X^3_{t-2}, \varepsilon^2_{t-1} \varepsilon_t + 3\beta X^3_{t-2}, \varepsilon^3_{t-1} + \varepsilon_t^3 \) (2.24)

\[ E(Y_t) = E(X^3_t) = \mu_t = \beta^3 E(X^3_{t-2}) + 3\beta^2 \sigma^2 E(X^3_{t-2}) + 3\beta \sigma^4 E(X^3_{t-2}) + \sigma^6 E(\varepsilon_t^3) \]

\[ Y_t = X^6_t = (\beta X_{t-2}, \varepsilon_{t-1} + \varepsilon_t)^6 = \beta^6 X^6_{t-2}, \varepsilon_{t-1}^6 + 6\beta^5 X^6_{t-2}, \varepsilon^5_{t-1} \varepsilon_t + 15\beta^4 X^6_{t-2}, \varepsilon^4_{t-1} \varepsilon_t^2 + 20\beta^3 X^6_{t-2}, \varepsilon^3_{t-1} \varepsilon_t^3 + 15\beta^2 X^6_{t-2}, \varepsilon^2_{t-1} \varepsilon_t^4 + 6\beta X^6_{t-2}, \varepsilon^1_{t-1} \varepsilon_t^5 + \varepsilon_t^6 \] (2.26)

\[ E(Y_t^2) = \beta^6 E(X^6_{t-2}, \varepsilon_{t-1}^6) + 6\beta^5 E(X^6_{t-2}, \varepsilon^5_{t-1} \varepsilon_t) + 15\beta^4 E(X^6_{t-2}, \varepsilon^4_{t-1} \varepsilon_t^2) + 20\beta^3 E(X^6_{t-2}, \varepsilon^3_{t-1} \varepsilon_t^3) + 15\beta^2 E(X^6_{t-2}, \varepsilon^2_{t-1} \varepsilon_t^4) + 6\beta E(X^6_{t-2}, \varepsilon^1_{t-1} \varepsilon_t^5) + E(\varepsilon_t^6) \]

\[ = \beta^6 E(X^6_{t-2}, \varepsilon_{t-1}^6) + 15\sigma^6 \beta^6 E(X^6_{t-2}, \varepsilon^6_{t-1}) + 45\sigma^6 \beta^4 E(X^6_{t-2}, \varepsilon^4_{t-1}) + 15\sigma^6 \beta^2 E(X^6_{t-2}, \varepsilon^2_{t-1}) + 15\sigma^6 E(Y_t^2) \]

\[ = 15\sigma^6 \beta^6 E(Y_t^2) + 45\sigma^6 \beta^4 \left( \frac{3\sigma^4 \left( 1 + \sigma^2 \beta^2 \right)}{(1 - \sigma^2 \beta^2)(1 - 3\sigma^4 \beta^4)} \right) + 15\sigma^6 \]

\[ (1 - 15\sigma^6 \beta^6) E(Y_t^2) = 45\sigma^6 \beta^4 \left( \frac{3\sigma^4 \left( 1 + \sigma^2 \beta^2 \right)}{(1 - \sigma^2 \beta^2)(1 - 3\sigma^4 \beta^4)} \right) + 45\sigma^6 \beta^2 \left( \frac{\sigma^2}{1 - \sigma^2 \beta^2} \right) + 15\sigma^6 \]

\[ = \frac{1}{(1 - \sigma^2 \beta^2)(1 - 3\sigma^4 \beta^4)} \left[ 45\sigma^6 \beta^4 \left( 3\sigma^4 \left( 1 + \sigma^2 \beta^2 \right) \right) \right] \]

\[ + 45\sigma^6 \beta^2 \left[ \sigma^2 \left( 1 - 3\sigma^4 \beta^4 \right) \right] + 15\sigma^6 \left( 1 - \sigma^2 \beta^2 \right) \left( 1 - 3\sigma^4 \beta^4 \right) \]

\[ = \frac{1}{(1 - \sigma^2 \beta^2)(1 - 3\sigma^4 \beta^4)} \left[ 135\sigma^{10} \beta^4 + 135\sigma^{12} \beta^6 + 45\sigma^{14} \beta^8 \right] \]

\[ - 135\sigma^{12} \beta^6 + 15\sigma^6 - 45\sigma^{10} \beta^4 - 15\sigma^6 - 45\sigma^{12} \beta^8 \]

\[ = \frac{1}{(1 - \sigma^2 \beta^2)(1 - 3\sigma^4 \beta^4)} \left[ 90\sigma^{10} \beta^4 + 30\sigma^8 \beta^2 + 15\sigma^6 + 45\sigma^{12} \beta^8 \right] \] (2.27)

\[ = \frac{15\sigma^4 \left( 1 + 2\sigma^2 \beta^2 + 6\sigma^4 \beta^4 + 3\sigma^6 \beta^6 \right)}{(1 - \sigma^2 \beta^2)(1 - 3\sigma^4 \beta^4)}, \quad \sigma^2 \beta^2 < \frac{1}{\sqrt{15}} \]
Some Results

\[ E\left(X_{i+1}X_i e_i\right) = \sigma^2 E\left(X_i\right) = 0 \]

**Proof:**

\[ X_{i+1}X_i e_i = X_{i+1}\left[\beta X_{i+2} e_{i+1} + e_i\right]e_i = \beta X_{i+2}X_i e_{i+1}e_i + X_{i+1} e_i^2 \]

\[ E\left(X_{i+1}X_i e_i\right) = \sigma^2 E\left(X_i\right) = 0 \]

**Proof:**

\[ X_{i+1}X_i e_i^2 = X_{i+1}\left[\beta X_{i+2} e_{i+1} + e_i\right]e_i^2 = \beta X_{i+2}X_i e_{i+1}e_i + X_{i+1} e_i^4 \]

\[ E\left(X_{i+1}X_i e_i^2\right) = \sigma^2 E\left(X_i\right) = 0 \]

**Proof:**

\[ X_{i+1}X_i e_i^3 = X_{i+1}\left[\beta X_{i+2} e_{i+1} + e_i\right]e_i^3 = \beta X_{i+2}X_i e_{i+1}e_i + X_{i+1} e_i^6 \]

\[ E\left(X_{i+1}X_i e_i^3\right) = \sigma^2 E\left(X_i\right) = 0 \]
\[ E(X_{t+1}^2 X_t^2 e_{t+1}^2) = 3\beta^2 (3\sigma^4) E(X_{t+2}^2 X_{t+1}^2 e_{t+1}^2) \]
\[ = 9\sigma^4 \beta^2 E(X_{t+2}^2 X_t^2 e_{t+1}^2) \]
\[ = 9\sigma^4 \beta^2 E(X_{t+1}^2 X_t^2 e_{t+1}^2) \]

Hence,
\[ E(Y_{t+1}) = \beta^2 \left[ 9\sigma^4 \beta^2 E(X_{t+1}^2 X_t^2 e_{t+1}^2) \right] = 9\sigma^4 \beta^2 E(X_{t+1}^2 X_t^2 e_{t+1}^2) \]

Now,
\[ X_{t+1}^2 X_t^2 e_{t+1}^2 = X_{t+1}^2 \left[ \beta^2 X_{t+2}^2 e_{t+1}^2 + 3\beta^2 X_{t+2}^2 e_{t+1}^2 + 3\beta X_{t+2}^2 e_{t+1}^2 + e_{t+1}^2 \right] e_{t+1}^2 \]
\[ = \beta^4 X_{t+2}^2 X_{t+1}^2 e_{t+1}^2 + 3\beta^3 X_{t+2}^2 X_{t+1}^2 e_{t+1}^2 + 3\beta X_{t+2}^2 X_{t+1}^2 e_{t+1}^2 + X_{t+1}^2 e_{t+1}^2 \]
\[ E(X_{t+1}^2 X_t^2 e_{t+1}^2) = \sigma^2 \beta^2 E(X_{t+2}^2 X_{t+1}^2 e_{t+1}^2) + 9\sigma^4 \beta^2 E(X_{t+1}^2 X_t^2 e_{t+1}^2) \]
\[ = \sigma^2 \beta^2 E(X_{t+2}^2 X_{t+1}^2 e_{t+1}^2) \]

But,
\[ E(Y_{t+1}) = 9\sigma^4 \beta^2 E(X_{t+1}^2 X_t^2 e_{t+1}^2) = 9\sigma^4 \beta^2 \left( \sigma^2 \beta^2 E(X_{t+2}^2 X_{t+1}^2 e_{t+1}^2) \right) \]
\[ = 9\sigma^4 \beta^2 E(X_{t+2}^2 X_{t+1}^2 e_{t+1}^2) \]

Now,
\[ X_{t+1}^2 X_t^2 e_{t+1}^2 = X_{t+2}^2 \left[ \beta^2 X_{t+2}^2 e_{t+1}^2 + 2\beta X_{t+2}^2 e_{t+1}^2 + e_{t+1}^2 \right] e_{t+1}^2 \]
\[ = \beta^2 X_{t+2}^2 X_{t+1}^2 e_{t+1}^2 + 2\beta X_{t+2}^2 X_{t+1}^2 e_{t+1}^2 + X_{t+1}^2 e_{t+1}^2 \]
\[ E(X_{t+1}^2 X_t^2 e_{t+1}^2) = 2\beta (3\sigma^4) E(X_{t+2}^2 X_{t+1}^2 e_{t+1}^2) = 6\sigma^4 \beta^2 E(X_{t+1}^2 X_t^2 e_{t+1}^2) = 0 \]

Hence,
\[ E(Y_{t+1}) = 9\sigma^4 \beta^2 \left[ 6\sigma^4 \beta E(X_{t+1}^2 X_t^2 e_{t+1}^2) \right] = 54\sigma^{10} \beta^3 E(X_{t+1}^2 X_t^2 e_{t+1}^2) = 0 \]
\[ \Rightarrow R_y(1) = 0 \text{, when } Y = X_t^2 \]
\[ Y_{t+2} = X_{t+2}^3 = X_{t+2}^2 \left[ \beta^3 X_{t+2}^2 e_{t+1}^2 + 2\beta^2 X_{t+2}^2 e_{t+1}^2 + 3\beta X_{t+2}^2 e_{t+1}^2 + e_{t+1}^2 \right] X_{t+2}^2 \]
\[ = \beta^3 X_{t+2}^2 e_{t+1}^2 + 3\beta^2 X_{t+2}^2 e_{t+1}^2 + 3\beta X_{t+2}^2 e_{t+1}^2 + X_{t+1}^2 e_{t+1}^2 \]
\[ E(Y_{t+2}) = 0 \]
\[ \Rightarrow R_y(2) = 0 \text{, when } Y = X_t^3 \]

Generally, \( R_y(k) = 0 \forall k \neq 0 \), when \( Y = X_t^k \).

Therefore, given \( X_t = \beta X_{t+2}^2 e_{t+1}^2 + e_t, \ e_t \sim N(0, \sigma^2) \) and \( Y = X_t^3 \), the following are true \( E(Y_t) = E(X_t^3) = 0 \).

\[ R_y(k) = \begin{cases} 15\sigma^6 \left( 1 + 2\sigma^2 \beta^2 + 6\sigma^4 \beta^4 + 3\sigma^6 \beta^6 \right) \left( 1 - \sigma^2 \beta^2 \right) \left( 1 - 3\sigma^4 \beta^4 \right) \left( 1 - 15\sigma^6 \beta^6 \right), & \sigma^2 \beta^2 < \frac{1}{\sqrt{15}} \ , k = 0 \\ 0, & k \neq 0 \end{cases} \]

\[ \rho_k(k) = \begin{cases} 1, & k = 0 \\ 0, & k \neq 0 \end{cases} \]
The covariance structure of $Y_t = X_t^2$, $t \in \mathbb{Z}$ identifies the process as linear white noise.

3. Methodology

3.1. Normality Checking

The Jarque-Bera (JB) test [21] [22] [23] will be used to determine for which values of $\beta$ a simple bilinear model (1.20) is normally distributed or not. The JB test statistic is

$$JB = n \left( \frac{{\hat{\gamma}_1^2}}{6} + \frac{({\hat{\gamma}_2 - 3})^2}{24} \right)$$  \hspace{1cm} (3.1)

where

$$\hat{\gamma}_1 = \frac{1}{n} \sum_{t=1}^{n} (X_t - \bar{X})^3 \left( \frac{1}{n} \sum_{t=1}^{n} (X_t - \bar{X})^2 \right)^{3/2}$$  \hspace{1cm} (3.2)

$$\hat{\gamma}_2 = \frac{1}{n} \sum_{t=1}^{n} (X_t - \bar{X})^4 \left( \frac{1}{n} \sum_{t=1}^{n} (X_t - \bar{X})^2 \right)^2$$  \hspace{1cm} (3.3)

$n$ is the sample size while, $\hat{\gamma}_1$ and $\hat{\gamma}_2$ are the sample skewness and kurtosis coefficients. The asymptotic null distribution of JB is $\chi^2$ with 2 degrees of freedom.

3.2. White Noise Test

The modified Ljung-Box test statistic [11] given by

$$Q^*(m) = n(n + 2) \sum_{k=1}^{m} \left( \frac{{\hat{\rho}_{x_d}(k)}}{n-k} \right)^2$$  \hspace{1cm} (3.4)

is used to test the iid hypothesis for $X_t^d$, $d = 1, 2, 3$ for the simple bilinear model (1.20). It is important to note from Theorem 2.1 that $X_t^2$ has ARMA(2, 1) structure while from Theorem 2.2, $X_t^3$ is iid. This test will look for $\beta$ values where both $X_t^2$ and $X_t^3$ are jointly iid. That will help determine the values of $\beta$ for which the simple bilinear model (1.20) is not distinguishable from the linear Gaussian white noise process (LGWNP). Then, the hypothesis of iid data is rejected at level $\alpha$ if the observed $Q^*(m)$ is larger than the $1 - \frac{\alpha}{2}$ quartile of the $\chi^2(m)$ distribution, where $m \approx \ln(n)$ [9].

3.3. Use of Chi-Square Test for Comparison of the Simple Bilinear White Noise Process and the Linear Gaussian White Noise Process

From Theorem 2.3, the third power of the simple bilinear process is iid. A test is
needed to confirm that the simple bilinear process (1.20) is not a linear Gaussian white noise process (LGWNP). For the LGWNP $X_t, t \in T$; $E(X_t) = \mu$, $\text{var}(X_t) = \sigma^2 < \infty$ and $\text{var}(X_t^3) = 15\sigma^6$. To show that the simple bilinear process (1.20) is not LGWNP, we need to test the hypothesis;

$$H_0 : \sigma^2_{x_t} = 15\sigma^6$$  \hspace{1cm} (3.5)

against the alternative hypothesis

$$H_0 : \sigma^2_{x_t} \neq 15\sigma^6$$  \hspace{1cm} (3.6)

The chi-square test [24] [25] can be used to perform the test. The chi-square test statistic is

$$\chi^2 = \frac{(n-1)S^2_{x_t}}{15\sigma^6}$$  \hspace{1cm} (3.7)

where $S^2_{x_t}$ is the sample variance of $X_t^3, X_t, t \in Z$ that follows (1.20), $\hat{\sigma}^2_{x_t}$ is an estimate of the true variance of the simple bilinear process (1.20) and $n$ is the number of observations of the series. The null hypothesis is rejected at level $\alpha$ if the observed value of $\chi^2$ is larger than $1 - \frac{\alpha}{2}$ quartile of the chi-square distribution with $n-1$ degree of freedom. It should be noted that this test works well when the underlying original population $X_t, t \in Z$ is normally distributed.

4. Results and Discussion

One thousand random digits $e_t, t \in Z$ that met the condition $e_t \sim N(0,1)$ were simulated using Minitab 16 series software. Only one random digit, shown in Appendix I, was used for demonstration in the study because of want of space. The estimates of the descriptive statistics (mean, variance, skewness ($\gamma_1$) and kurtosis ($\gamma_2$)) and other tests (Jarque Bera (JB) test, modified Ljung Box test ($Q^*$) and chi-square calculated test statistic) for the powers $e_t^d, d = 1, 2, 3$ of the random digit are shown in Table 3. The results obtained using the JB, $Q^*$ and the chi-square test indicated $e_t, t \in Z$ as a LGWNP at 5% level of significance.

The LGWNP were used to simulate the SBWNP $X_t = BX_{t-2}e_{t-1} + e_t$, $e_t \sim N(0,1)$ for $-0.60 \leq \beta \leq 0.60$ satisfying the existence of $E(X_t^3)$ using Fortran 77 program. The estimates of the descriptive statistic and that for the test statistic (JB, $Q^*$ and the chi-square calculated test statistic) are shown in Table 4. The values of the JB statistic show that the SBWNP are normally distributed for $-0.56 \leq \beta \leq 0.60$. Similarly, the values of $Q^*$ and the chi-square calculated test statistic ($H_0$) show that the SBWNP is iid and can be identified as a LGWNP for some $\beta$ values. The values of $\beta$ where the SBWNP will be identified as an LGWNP are summarized in Table 5.

5. Conclusion

We have been able to establish the covariance structure for $X_t^d, d = 1, 2, 3; t \in Z$
Table 3. Descriptive Statistics and estimate of the test statistic for rejecting the null hypothesis of equality of the variance of higher moment for the simulated series, \( X_t = \epsilon_t, \epsilon_t \sim N(0,1) \), as linear Gaussian white noise process.

<table>
<thead>
<tr>
<th>Statistic</th>
<th>Mean</th>
<th>Median</th>
<th>( S_j )</th>
<th>( \gamma_1 )</th>
<th>( \gamma_2 )</th>
<th>JB value</th>
<th>( Q^* )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( X_i )</td>
<td>0.0000</td>
<td>0.1261</td>
<td>1.0000</td>
<td>-0.28</td>
<td>-0.04</td>
<td>1.87</td>
<td>3.36</td>
</tr>
<tr>
<td>( X_i )</td>
<td>0.9931</td>
<td>0.4763</td>
<td>1.9074</td>
<td>1.90</td>
<td>2.79</td>
<td>133.19</td>
<td>0.04</td>
</tr>
<tr>
<td>( X_i )</td>
<td>-0.2728</td>
<td>0.0020</td>
<td>11.5236</td>
<td>-0.61</td>
<td>6.47</td>
<td>259.67</td>
<td>-0.14</td>
</tr>
</tbody>
</table>

Table 4. Descriptive statistics and estimate of the test statistic for simulated bilinear series \( X_t = \beta X_{t-2} e_{t-1} + \epsilon_t, \epsilon_t \sim N(0,1) \) and \(-0.60 \leq \beta \leq 0.60\).

<table>
<thead>
<tr>
<th>( \beta )</th>
<th>Statistic</th>
<th>Estimated Values</th>
<th>Estimate of Test Statistic</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Mean</td>
<td>Variance</td>
<td>( \gamma_1 )</td>
</tr>
<tr>
<td>-0.60</td>
<td>( X_i )</td>
<td>0.0418</td>
<td>1.9037</td>
</tr>
<tr>
<td></td>
<td>( X_i )</td>
<td>1.8923</td>
<td>11.3331</td>
</tr>
<tr>
<td>-0.59</td>
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$X_i$ | 0.0187 | 0.9618 | −0.34 | 0.01 | 2.85 | 1.56 | -
$X_i'$ | 0.9555 | 1.7813 | 1.94 | 3.08 | 147.18 | 7.72 | -
$X_i''$ | −0.2645 | 10.4608 | −1.62 | 6.82 | 342.26 | 3.83 | 112.09
$X_i'''$ | 0.0200 | 0.9620 | −0.35 | 0.01 | 2.91 | 1.52 | -

0.25
$X_i$ | 0.9557 | 1.7848 | 1.94 | 3.08 | 147.17 | 7.80 | -
$X_i'$ | −0.2643 | 10.4851 | −1.64 | 6.82 | 344.13 | 3.75 | 112.28
$X_i''$ | 0.0214 | 0.9623 | −0.35 | 0.02 | 2.98 | 1.49 | -
$X_i'''$ | 0.9565 | 1.7935 | 1.94 | 3.08 | 147.16 | 7.88 | -

0.26
$X_i$ | 0.9561 | 1.7888 | 1.94 | 3.08 | 147.16 | 7.88 | -
$X_i'$ | −0.2641 | 10.5152 | −1.66 | 6.83 | 345.65 | 3.66 | 112.49
$X_i''$ | 0.0229 | 0.9628 | −0.36 | 0.02 | 3.05 | 1.46 | -
$X_i'''$ | 0.9573 | 1.7989 | 1.94 | 3.08 | 147.23 | 8.05 | -

0.27
$X_i$ | 0.9566 | 1.7935 | 1.94 | 3.08 | 147.18 | 7.96 | -
$X_i'$ | −0.2638 | 10.5514 | −1.67 | 6.83 | 346.83 | 3.57 | 112.71
$X_i''$ | 0.0244 | 0.9634 | −0.36 | 0.03 | 3.12 | 1.43 | -
$X_i'''$ | 0.9578 | 1.8048 | 1.94 | 3.08 | 147.33 | 8.14 | -

0.28
$X_i$ | 0.9573 | 1.7989 | 1.94 | 3.08 | 147.23 | 8.05 | -
$X_i'$ | −0.2636 | 10.5939 | −1.69 | 6.83 | 347.68 | 3.49 | 112.95
$X_i''$ | 0.0259 | 0.9641 | −0.36 | 0.03 | 3.19 | 1.41 | -
$X_i'''$ | 0.9581 | 1.8115 | 1.94 | 3.08 | 147.52 | 8.24 | -

0.29
$X_i$ | 0.9581 | 1.8048 | 1.94 | 3.08 | 147.33 | 8.14 | -
$X_i'$ | −0.2633 | 10.6429 | −1.69 | 6.82 | 348.21 | 3.41 | 113.22
$X_i''$ | 0.0275 | 0.9651 | −0.37 | 0.04 | 3.27 | 1.40 | -
$X_i'''$ | 0.9591 | 1.8115 | 1.94 | 3.08 | 147.52 | 8.24 | -

0.30
$X_i$ | 0.9591 | 1.8115 | 1.94 | 3.08 | 147.52 | 8.24 | -
$X_i'$ | −0.2630 | 10.6987 | −1.70 | 6.82 | 348.45 | 3.33 | 113.46
$X_i''$ | 0.0291 | 0.9662 | −0.37 | 0.05 | 3.34 | 1.40 | -
$X_i'''$ | 0.9603 | 1.8187 | 1.94 | 3.09 | 147.79 | 8.35 | -

0.31
$X_i$ | 0.9603 | 1.8187 | 1.94 | 3.09 | 147.79 | 8.35 | -
$X_i'$ | −0.2626 | 10.7616 | −1.70 | 6.82 | 348.44 | 3.26 | 113.74
$X_i''$ | 0.0308 | 0.9675 | −0.38 | 0.05 | 3.42 | 1.40 | -
$X_i'''$ | 0.9617 | 1.8266 | 1.95 | 3.09 | 148.18 | 8.46 | -

0.32
$X_i$ | 0.9617 | 1.8266 | 1.95 | 3.09 | 148.18 | 8.46 | -
$X_i'$ | −0.2622 | 10.8318 | −1.69 | 6.82 | 348.25 | 3.19 | 114.02
$X_i''$ | 0.0326 | 0.9689 | −0.38 | 0.06 | 3.50 | 1.41 | -
$X_i'''$ | 0.9633 | 1.8352 | 1.95 | 3.10 | 148.72 | 8.59 | -

0.33
$X_i$ | 0.9633 | 1.8352 | 1.95 | 3.10 | 148.72 | 8.59 | -
$X_i'$ | −0.2617 | 10.9098 | −1.68 | 6.83 | 347.93 | 3.12 | 114.35
$X_i''$ | 0.0343 | 0.9706 | −0.38 | 0.06 | 3.58 | 1.43 | -
$X_i'''$ | 0.9650 | 1.8445 | 1.95 | 3.11 | 149.41 | 8.73 | -

0.34
$X_i$ | 0.9650 | 1.8445 | 1.95 | 3.11 | 149.41 | 8.73 | -
$X_i'$ | −0.2611 | 10.9958 | −1.67 | 6.84 | 347.58 | 3.06 | 114.64
$X_i''$ | 0.0362 | 0.9724 | −0.39 | 0.07 | 3.66 | 1.45 | -
$X_i'''$ | 0.9670 | 1.8544 | 1.96 | 3.12 | 150.28 | 8.87 | -

0.35
$X_i$ | 0.9670 | 1.8544 | 1.96 | 3.12 | 150.28 | 8.87 | -
$X_i'$ | −0.2603 | 11.0904 | −1.66 | 6.85 | 347.31 | 3.01 | 114.99
$X_i''$ | 0.0381 | 0.9744 | −0.39 | 0.08 | 3.74 | 1.49 | -
$X_i'''$ | 0.9691 | 1.8651 | 1.96 | 3.14 | 151.36 | 9.03 | -

0.36
$X_i$ | 0.9691 | 1.8651 | 1.96 | 3.14 | 151.36 | 9.03 | -
$X_i'$ | −0.2594 | 11.1938 | −1.63 | 6.87 | 347.25 | 2.96 | 115.35
$X_i''$ | 0.0400 | 0.9767 | −0.40 | 0.08 | 3.81 | 1.53 | -
$X_i'''$ | 0.9715 | 1.8765 | 1.97 | 3.16 | 152.67 | 9.21 | -

0.37
$X_i$ | 0.9715 | 1.8765 | 1.97 | 3.16 | 152.67 | 9.21 | -
$X_i'$ | −0.2583 | 11.3067 | −1.61 | 6.90 | 347.55 | 2.91 | 115.69
Continued

| tX | 0.0420 | 0.9791 | −0.40 | 0.09 | 3.88 | 1.59 | -  
|----|--------|--------|-------|------|------|------|-------|
| 0.38 | 0.9741 | 1.8888 | 1.97  | 3.19 | 154.22 | 9.40 | -  
| X'  | −0.2569 | 11.4295 | −1.58 | 6.94 | 348.40 | 2.87 | 116.09 |  
| X'  | 0.0440 | 0.9818 | −0.40 | 0.09 | 3.95 | 1.65 | -  
| 0.39 | 0.9769 | 1.9019 | 1.98  | 3.19 | 156.05 | 9.61 | -  
| X'  | −0.2553 | 11.5629 | −1.58 | 6.94 | 349.99 | 2.84 | 116.48 |  
| X'  | 0.0461 | 0.9847 | −0.41 | 0.10 | 4.02 | 1.73 | -  
| 0.40 | 0.9800 | 1.9159 | 1.99  | 3.26 | 158.16 | 9.84 | -  
| X'  | −0.2534 | 11.7074 | −1.50 | 7.06 | 352.57 | 2.81 | 116.89 |  
| X'  | 0.0482 | 0.9879 | −0.41 | 0.11 | 4.08 | 1.82 | -  
| 0.41 | 0.9834 | 1.9309 | 1.99  | 3.30 | 160.59 | 10.09 | -  
| X'  | −0.2511 | 11.8638 | −1.45 | 7.14 | 356.40 | 2.79 | 117.31 |  
| X'  | 0.0504 | 0.9913 | −0.41 | 0.12 | 4.13 | 1.92 | -  
| 0.42 | 0.9870 | 1.9470 | 2.00  | 3.35 | 163.34 | 10.36 | -  
| X'  | −0.2484 | 12.0330 | −1.39 | 7.25 | 361.79 | 2.77 | 117.75 |  
| X'  | 0.0526 | 0.9950 | −0.41 | 0.12 | 4.18 | 2.03 | -  
| 0.43 | 0.9909 | 1.9643 | 2.01  | 3.40 | 166.42 | 10.65 | -  
| X'  | −0.2452 | 12.2158 | −1.33 | 7.38 | 369.08 | 2.76 | 118.22 |  
| X'  | 0.0549 | 0.9990 | −0.41 | 0.12 | 4.21 | 2.15 | -  
| 0.44 | 0.9951 | 1.9829 | 2.02  | 3.46 | 169.86 | 10.97 | -  
| X'  | −0.2416 | 12.4134 | −1.27 | 7.53 | 378.64 | 2.75 | 118.70 |  
| X'  | 0.0572 | 1.0033 | −0.41 | 0.13 | 4.24 | 2.29 | -  
| 0.45 | 0.9996 | 2.0030 | 2.03  | 3.53 | 173.66 | 11.32 | -  
| X'  | −0.2373 | 12.6270 | −1.19 | 7.71 | 390.89 | 2.75 | 119.20 |  
| X'  | 0.0595 | 1.0079 | −0.42 | 0.14 | 4.25 | 2.44 | -  
| 0.46 | 1.0044 | 2.0247 | 2.04  | 3.61 | 177.82 | 11.70 | -  
| X'  | −0.2323 | 12.8582 | −1.11 | 7.92 | 406.27 | 2.75 | 119.73 |  
| X'  | 0.0620 | 1.0128 | −0.41 | 0.14 | 4.25 | 2.60 | -  
| 0.47 | 1.0096 | 2.0482 | 2.05  | 3.69 | 182.34 | 12.10 | -  
| X'  | −0.2267 | 13.1088 | −1.03 | 8.16 | 425.24 | 2.75 | 120.29 |  
| X'  | 0.0644 | 1.0181 | −0.41 | 0.15 | 4.24 | 2.78 | -  
| 0.48 | 1.0152 | 2.0737 | 2.06  | 3.78 | 187.25 | 12.54 | -  
| X'  | −0.2202 | 13.3809 | −0.93 | 8.44 | 448.27 | 2.76 | 120.88 |  
| X'  | 0.0670 | 1.0238 | −0.41 | 0.16 | 4.21 | 2.97 | -  
| 0.49 | 1.0211 | 2.1016 | 2.07  | 3.87 | 192.53 | 13.02 | -  
| X'  | −0.2127 | 13.6772 | −0.83 | 8.75 | 475.84 | 2.78 | 121.52 |  
| X'  | 0.0695 | 1.0298 | −0.41 | 0.16 | 4.16 | 3.18 | -  
| 0.50 | 1.0275 | 2.1319 | 2.08  | 3.97 | 198.22 | 13.53 | -  
| X'  | −0.2043 | 14.0009 | −0.73 | 9.09 | 508.36 | 2.81 | 122.22 |  
| X'  | 0.0980 | 1.1188 | −0.32 | 0.28 | 2.89 | 6.02 | -  
| 0.60 | 1.1207 | 2.6703 | 2.27  | 5.60 | 312.17 | 20.67 | -  
| X'  | −0.0402 | 20.1794 | 0.86  | 13.91 | 1178.64 | 5.41 | 137.37 |  

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Table 5. Values of $\beta$ for comparison of SBWNP as a LGWNP at 0.05 and 0.10 $\alpha$ levels.

<table>
<thead>
<tr>
<th>$\alpha$ % level</th>
<th>Values of $\beta$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$Q^*$</td>
<td>$H_x$</td>
</tr>
<tr>
<td>5</td>
<td>$[-0.23,0.44]$</td>
</tr>
<tr>
<td>10</td>
<td>$[-0.19,0.37]$</td>
</tr>
</tbody>
</table>

We have also determined the values of $\beta$ for which the simple bilinear model (1.20) is normally distributed and in which the process can be determined as a LGWNP or not. We recommend that for proper comparison of SBWNP with LGWNP, the SBWNP should be considered for normality, white noise test and test of equality of variance of its third moment being equivalent to the theoretical values of the LGWNP.

References


Appendix I

Simulated Random Digits; \( e_i, e_i \sim N(0,1) \) (Read Across).

| -0.57532 | -0.17491 | 0.35244 | 0.30620 | -0.76520 | -0.10381 | -0.78604 | 0.19891 | 0.48466 | -1.04050 | 0.25694 | 2.13936 |
| 0.81740 | -1.61037 | 2.38415 | 0.74182 | -1.83436 | -0.97443 | 0.06649 | -0.80814 | -2.14835 | -1.39147 | -1.19600 | 0.16246 |
| 1.10204 | -0.75625 | 1.43986 | 0.41147 | 0.34040 | -0.27339 | -0.66471 | 0.72426 | -0.24697 | -0.73065 | 1.22347 | 1.89188 |
| -0.78388 | 0.99457 | -0.94385 | 1.99912 | 0.00884 | 0.10762 | -2.23041 | -0.20387 | 1.20197 | -0.12003 | 1.83635 | -0.06882 |
| -2.38069 | 0.01037 | 0.55983 | -1.86577 | 0.75661 | -0.83977 | -0.06520 | -0.25303 | 0.57397 | -0.10694 | -1.87199 | -0.61338 |
| -0.96019 | -0.69799 | 0.41226 | -0.13727 | 0.73620 | -0.25448 | 0.27995 | 0.82692 | 1.07422 | 0.72309 | 0.44146 | 0.76731 |
| 0.72838 | 0.39809 | 0.18794 | 0.06831 | 0.45853 | -0.79068 | -1.97602 | -1.55625 | 0.98349 | 2.09313 | -1.26609 | 0.50341 |
| -0.98639 | 0.78335 | 0.56394 | -0.00389 | -0.60469 | 0.68956 | 0.09199 | -0.84437 | 0.28016 | -0.36120 | 0.16969 | -0.32149 |
| -1.97702 | -0.98212 | -1.26901 | 0.93133 | 0.63846 | -0.83151 | 0.68592 | 0.18103 | -0.69071 | 0.35337 | 0.67619 | 0.82779 |
| 1.25023 | 0.50671 | 1.39091 | -0.27367 | -0.09697 | 1.01271 | 1.21921 | 0.67856 | 0.37606 | 1.16306 | -0.11180 | -2.39334 |
| 1.13787 | -0.46900 | -1.07178 | 0.09855 | 1.96154 | -0.45406 | -1.57186 | 0.93940 | -0.00755 | 0.32726 | 0.57558 | 0.48859 |
| 0.45601 | 0.14352 | -2.13818 | 0.23375 | -1.82588 | 0.13979 | -0.25057 | 1.17289 | 0.12739 | 0.35428 | 0.12472 | -0.92299 |