Open Journal of Statistics, 2014, 4, 776-788
Published Online October 2014 in SciRes. http://www.scirp.org/journal/ojs
http://dx.doi.org/10.4236/ojs.2014.49073

Mixture Ratio Estimators Using Multi-Auxiliary Variables and Attributes for Two-Phase Sampling

Paul Mwangi Waweru, John Kung'u, James Kahiri

Department of Statistics and Actuarial Science, Kenyatta University, Nairobi, Kenya
Email: Waweru.paul12@yahoo.com, johnkungu08@yahoo.com

Received 25 July 2014; revised 27 August 2014; accepted 12 September 2014

Abstract

In this paper, we have proposed three classes of mixture ratio estimators for estimating population mean by using information on auxiliary variables and attributes simultaneously in two-phase sampling under full, partial and no information cases and analyzed the properties of the estimators. A simulated study was carried out to compare the performance of the proposed estimators with the existing estimators of finite population mean. It has been found that the mixture ratio estimator in full information case using multiple auxiliary variables and attributes is more efficient than mean per unit, ratio estimator using one auxiliary variable and one attribute, ratio estimator using multiple auxiliary variable and multiple auxiliary attributes and mixture ratio estimators in both partial and no information case in two-phase sampling. A mixture ratio estimator in partial information case is more efficient than mixture ratio estimators in no information case.

Keywords

Ratio Estimator, Multiple Auxiliary Variables, Multiple Auxiliary Attributes, Two-Phase Sampling, Bi-Serial Correlation Coefficient

1. Introduction

The history of using auxiliary information in survey sampling is as old as history of the survey sampling. The work of Neyman [1] may be referred to as the initial works where auxiliary information has been used. Cochran [2] used auxiliary information in single-phase sampling to develop the ratio estimator for estimation of population mean. In the ratio estimator, the study variable and the auxiliary variable had a high positive correlation and the regression line was passing through the origin. Hansen and Hurwitz [3] also suggested the use of auxiliary

How to cite this paper: Waweru, P.M., Kung'u, J. and Kahiri, J. (2014) Mixture Ratio Estimators Using Multi-Auxiliary Variables and Attributes for Two-Phase Sampling. *Open Journal of Statistics*, **4**, 776-788.
http://dx.doi.org/10.4236/ojs.2014.49073

information in selecting the sample with varying probabilities. If regression line is still linear but does not pass through the origin the regression estimator is used. Watson [4] used the regression estimator of leaf area on leaf weight to estimate the average area of the leaves on a plant.

Olkin [5] was the first person to use information on more than one auxiliary variable, which was positively correlated with the variable under study, using a linear combination of ratio estimator based on each auxiliary variable. Raj [6] suggested a method of using multi-auxiliary information in sample survey. Singh [7] proposed a ratio-cum-product estimator and its multi-variable expression.

The concept of double sampling was first proposed by Neyman [1] in sampling human populations when the mean of auxiliary variable was unknown. It was later extended to multi-phase by Robson [8]. Ahmad [9] proposed a generalized multivariate ratio and regression estimators for multi-phase sampling while Zahoor, Muhhamad and Munir [10] suggested a generalized regression-cum-ratio estimator for two-phase sampling using multiple auxiliary variables.

Jhajj, Sharma and Grover [11] proposed a family of estimators using information on auxiliary attribute. They used known information of population proportion possessing an attribute (highly correlated with study variable *Y*). The attribute are normally used when the auxiliary variables are not available e.g. amount of milk produced and a particular breed of cow or amount of yield of wheat and a particular variety of wheat. Jhajj, Sharma and Grover [11] used the information on auxiliary attributes in ratio estimator in estimating population mean of the variable of interest using known attributes such as coefficient of variation, coefficient kurtosis and point bi-serial correlation coefficient. The estimator performed better than the usual sample mean and Naik and Gupta [12] estimator. Jhajj, Sharma and Grover [11] also used the auxiliary attribute in regression, product and ratio type exponential estimator following the work of Bahl and Tuteja [13].

Hanif, Haq and Shahbaz [14] proposed a general family of estimators using multiple auxiliary attribute in single- and double-phase sampling. The estimator had a smaller MSE compared to that of Jhajj, Sharma and Grover [11]. They also extended their work to ratio estimator which was generalization of Naik and Gupta [12] estimator in single- and double-phase samplings with full information, partial information and no information. Kung'u and Odongo [15] and [16] proposed ratio-cum-product estimators using multiple auxiliary attributes in single- and two-phase sampling. Moeen, Shahbaz and Hanif [17] proposed a class of mixture ratio and regression estimators for single-phase sampling for estimating population mean by using information on auxiliary variables and attributes simultaneously.

In this paper, we will extend the mixture ratio estimator proposed by Moeen, Shahbaz and Hanif [17] in single-phase sampling to two-phase sampling under full, partial and no information case strategies introduced by Samiuddin and Hanif [18] and also incorporate Arora and Bansi [19] approach in writing down the mean squared error.

2. Preliminaries

2.1. Notation and Assumption

Consider a population of *N* units. Let *Y* be the variable for which we want to estimate the population mean and $X_1, X_2, \cdots, X_k$ are k auxiliary variables. For two-phase sampling design let n_1 and n_2 $(n_2 < n_1)$ are sample sizes for first and second phase respectively. $x_{j(1)}$ and $x_{j(2)}$ denote the j^{th} auxiliary variables form first and second phase samples respectively and y_2 denote the variable of interest from second phase. X_j and C_{x_j} denote the population means and coefficient of variation of j^{th} auxiliary variables respectively and ρ_{yx_j} denotes the population correlation coefficient of Y and X_j.

Further, let $\theta_1 = \left(\frac{1}{n_1} - \frac{1}{N}\right)$ $\theta_2 = \left(\frac{1}{n_2} - \frac{1}{N}\right)$ $(\theta_1 < \theta_2)$

$$\begin{aligned} &y_{(2)} = Y + e_{y_{(2)}}, \quad x_{j(1)} = X_j + e_{x_{j(1)}} \\ &\text{and} \quad x_{j(2)} = X_j + e_{x_{j(2)}} \quad \left(j = (1, 2, \cdots, k)\right) \end{aligned} \tag{1.0}$$

where $\overline{e}_{y_2}$, $\overline{e}_{x_{j(1)}}$ and $\overline{e}_{x_{j(2)}}$ are sampling error and are very small. We assume that

$$E\left(\overline{e}_{y_2}\right) = E\left(\overline{e}_{x_{j(1)}}\right) = E\left(\overline{e}_{x_{j(2)}}\right) = 0. \tag{1.1}$$

Consider a sample of size *n* drawn by simple random sampling without replacement from a population of size N. Let y_j and τ_j denotes the observations on variable y and r respectively for the j^{th} unit where $j = k+1,\ k+2,\ \cdots,\ q$.

In defining the attributes we assume complete dichotomy so that,

$$\tau_{ij} = \begin{cases} 1, & \text{if } i^{th} \text{ unit of population possess } j^{th} \text{ auxiliary attribute,} \\ 0, & \text{otherwise.} \end{cases} \tag{1.2}$$

Let $A_j = \sum_{j=1}^{N} \tau_{ij}$ and $a_j = \sum_{i=1}^{n} \tau_{ij}$ be the total number of units in the population and sample respectively possessing attribute τ_j. Let $P_j = \frac{A_j}{N}$ and $p_{j(2)} = \frac{a_j}{n}$ be the corresponding proportion of units possessing a specific attributes τ_j and $\bar{y}$ is the mean of the main variable at second phase. Let $p_{j(1)}$ and $p_{j(2)}$ denote the j^{th} auxiliary attribute form first and second phase samples respectively and y_2 denote the variable of interest from second phase. The mean of main variable of interest at second phase will be denoted by $\bar{y}_2$. Also let us define

$$\bar{e}_{\tau_{j(1)}} = p_{j(1)} - P_j, \quad \bar{e}_{\tau_{j(2)}} = p_{j(2)} - P_j \tag{1.3}$$

where $\bar{e}_{y_2}$, $\bar{e}_{\tau_{j(1)}}$ and $\bar{e}_{\tau_{j(2)}}$ are sampling error and are very small. We assume that

$$E\left(\bar{e}_{y_2}\right) = E\left(\bar{e}_{\tau_{j(1)}}\right) = E\left(\bar{e}_{\tau_{j(2)}}\right) = 0.$$

Let $v_j = \frac{\bar{x}_{(2)j}}{\bar{X}_j}$, then $v_j - 1 = \frac{\bar{x}_{(2)j}}{\bar{X}_j} - 1 = \frac{\bar{x}_{(2)j} - \bar{X}_j}{\bar{X}_j} = \frac{\bar{e}_{x_{(2)j}}}{\bar{X}_j}$. Therefore $v_j = 1 + \frac{\bar{e}_{x_{(2)j}}}{\bar{X}_j}$. Similarly, $w_j = 1 + \frac{\bar{e}_{\tau_{(2)j}}}{P_j}$.

Also $z_j = \frac{\bar{x}_{(2)j}}{\bar{x}_{(1)j}}$, then

$$v_j - 1 = \frac{\bar{x}_{(2)j} - \bar{x}_{(1)j}}{\bar{x}_{(1)j}} = \frac{\bar{e}_{x_{(2)j}} - \bar{e}_{x_{(1)j}}}{\bar{x}_{(1)j}} = \frac{\bar{e}_{x_{(2)j}} - \bar{e}_{x_{(1)j}}}{\bar{e}_{x_{(1)j}} + \bar{X}_j} = \frac{\bar{e}_{x_{(2)j}} - \bar{e}_{x_{(1)j}}}{\bar{X}_j\left(1 + \frac{\bar{e}_{x_{(1)j}}}{\bar{X}_j}\right)} = \frac{\bar{e}_{x_{(2)j}} - \bar{e}_{x_{(1)j}}}{\bar{X}_j}\left(1 + \frac{\bar{e}_{x_{(1)j}}}{\bar{X}_j}\right)^{-1}.$$

We shall take $v_i - 1$ to term of order $\frac{1}{n}$ as $v_j - 1 = \frac{\bar{e}_{x_{(2)j}} - \bar{e}_{x_{i(1)j}}}{\bar{X}_j}$ hence,

$$z_j = 1 + \frac{\bar{e}_{x_{(2)j}} - \bar{e}_{x_{(1)j}}}{\bar{X}_j} = 1 - \frac{\bar{e}_{x_{(1)j}} - \bar{e}_{x_{(2)j}}}{\bar{X}_j}.$$

Similarly,

$$u_j = 1 - \frac{\bar{e}_{\tau_{(1)j}} - \bar{e}_{\tau_{(2)j}}}{P_j}. \tag{1.4}$$

The coefficient of variation and correlation coefficient are given by

$$\begin{gathered} C_y^2 = \frac{S_{y_2}^2}{\bar{Y}^2}, \quad C_{x_1}^2 = \frac{S_{x_1}^2}{\bar{X}_1^2}, \quad C_{\tau_1}^2 = \frac{S_{x_1}^2}{\bar{P}_1^2}, \quad \rho_{yx} = \frac{S_{yx}}{S_y S_x}, \\ \rho_{yz} = \frac{S_{yz}}{S_y S_z}, \quad \rho_{Pb_i} = \frac{S_{y\tau}}{S_y S_\tau}, \quad \text{and} \quad \rho_{Pb_i} = \frac{S_{y\tau}}{S_y S_\tau}. \end{gathered} \tag{1.5}$$

Then for simple random sampling without replacement for both first and second phases we write by using phase wise operation of expectations as:

$$E\left(\bar{e}_{y_2}\right)^2=\theta_2\bar{Y}^2C_{y_2}^2,\quad E\left(\bar{e}_{\tau_{j(1)}}-\bar{e}_{\tau_{j(2)}}\right)^2=\left(\theta_2-\theta_1\right)P_j^2C_{\tau_j}^2,\quad E\left(\bar{e}_{x_{j(1)}}-\bar{e}_{x_{j(2)}}\right)^2=\left(\theta_2-\theta_1\right)\bar{X}_j^2C_{x_j}^2,$$

$$E\left(\bar{e}_{y_2}\bar{e}_{x_{j(2)}}\right)^2=\theta_2\overline{YX}_jC_{y_2}C_{x_j}\rho_{yx_j},\quad E\left(\bar{e}_{y_2}\bar{e}_{\tau_{j(2)}}\right)^2=\theta_2\bar{Y}P_jC_{y_2}C_{\tau_j}\rho_{Pb\tau_j},$$

$$E\left(\bar{e}_{y(2)}\left(\bar{e}_{x_{j(1)}}-\bar{e}_{x_{j(2)}}\right)\right)=\left(\theta_1-\theta_2\right)\overline{YX}_jC_yC_{x_j}\rho_{yx_j},\quad E\left(\bar{e}_{x_{j(2)}}\left(\bar{e}_{x_{j(1)}}-\bar{e}_{x_{j(2)}}\right)\right)=\left(\theta_1-\theta_2\right)\bar{X}_j^2C_{x_j}^2,$$

$$E\left(\bar{e}_{\tau_{j(2)}}\left(\bar{e}_{\tau_{j(1)}}-\bar{e}_{\tau_{j(2)}}\right)\right)=\left(\theta_1-\theta_2\right)P_j^2C_{\tau_j}^2,\quad E\left(\bar{e}_{x_{i(2)}}\bar{e}_{x_{j(2)}}\right)=\theta_2\bar{X}_i\bar{X}_jC_{x_i}C_{x_j}\rho_{x_ix_j}\ (i\neq j),$$

$$E\left(\bar{e}_{\tau_{i(2)}}\bar{e}_{\tau_{j(2)}}\right)=\theta_2P_iP_jC_{\tau_i}C_{\tau_j}\rho_{Pb\tau_i\tau_j}\ (i\neq j),\quad E\left(\bar{e}_{y(2)}\left(\bar{e}_{x_{j(1)}}-\bar{e}_{x_{j(2)}}\right)\right)=\left(\theta_1-\theta_2\right)\overline{YX}_jC_yC_{x_j}\rho_{yx_j}, \tag{1.6}$$

$$E\left(\left(\bar{e}_{\tau_{i(1)}}-\bar{e}_{\tau_{i(2)}}\right)\left(\bar{e}_{\tau_{j(1)}}-\bar{e}_{\tau_{j(2)}}\right)\right)=\left(\theta_2-\theta_1\right)P_iP_jC_{\tau_j}C_{\tau_i}\rho_{\tau_j\tau_i};(i\neq j)$$

$$E\left(\left(\bar{e}_{x_{i(1)}}-\bar{e}_{x_{i(2)}}\right)\left(\bar{e}_{x_{j(1)}}-\bar{e}_{x_{j(2)}}\right)\right)=\left(\theta_2-\theta_1\right)\bar{X}_i\bar{X}_jC_{x_j}C_{x_i}\rho_{x_jx_i};(i\neq j)$$

$$E\left(\bar{e}_{x_{j(2)}}\left(\bar{e}_{x_{i(1)}}-\bar{e}_{x_{i(2)}}\right)\right)=\left(\theta_1-\theta_2\right)\bar{X}_i\bar{X}_jC_{x_j}C_{x_i}\rho_{x_jx_i};(i\neq j)$$

$$E\left(\bar{e}_{\tau_{i(2)}}\left(\bar{e}_{\tau_{i(1)}}-\bar{e}_{\tau_{i(2)}}\right)\right)=\left(\theta_1-\theta_2\right)P_iP_jC_{\tau_j}C_{\tau_i}\rho_{\tau_j\tau_i};(i\neq j)$$

$$A^{-1}=\frac{1}{|A|}\left(C^{\mathrm{T}}\right)_{ij}=\frac{Adj(A)}{|A|} \tag{1.7}$$

$$\frac{|R|_{y\underline{x}_q}}{|R|_{\underline{x}_q}}=\left(1-\rho_{y.\underline{x}_q}^2\right)\ \text{Arora and Lai [19]} \tag{1.8}$$

The following notations will be used in deriving the mean square errors of proposed estimators.

$|R|_{y\underline{x}_q}$: Determinant of population correlation matrix of variables $y,\ x_1,\ x_2,\ \cdots,\ x_{q-1}$ and x_q.

$\left|R_{yx_i}\right|_{y\underline{x}_q}$: Determinant of i^{th} minor of $|R|_{y\underline{x}_p}$ corresponding to the i^{th} element of ρ_{yx_i}.

$\rho_{y\underline{x}_r}^2$: Denotes the multiple coefficient of determination of y on $x_1,\ x_2,\ \cdots,\ x_{r-1}$ and x_r.

$\rho_{y\underline{x}_q}^2$: Denotes the multiple coefficient of determination of y on $y,\ x_1,\ x_2,\ \cdots,\ x_{q-1}$ and x_q.

$|R|_{\underline{x}_r}$: Determinant of population correlation matrix of variables $x_1,\ x_2,\ \cdots,\ x_{r-1}$ and x_r.

$|R|_{\underline{x}_q}$: Determinant of population correlation matrix of variables $x_1,\ x_2,\ \cdots,\ x_{q-1}$ and x_q.

$|R|_{y_i\underline{x}_r}$: Determinant of the correlation matrix of $y_i,\ x_1,\ x_2,\ \cdots,\ x_{r-1}$ and x_r.

$|R|_{y_i\cdot\underline{x}_q}$: Determinant of the correlation matrix of $y_i,\ x_1,\ x_2,\ \cdots,\ x_{q-1}$ and x_q.

$|R|_{y_i\cdot y_j\cdot\underline{x}_r}$: Determinant of the minor corresponding to $\rho_{y_iy_j}$ of the correlation matrix of $y_i,\ y_j,\ x_1,\ x_2,\ \cdots,\ x_{r-1}$ and x_r $(i\neq j)$.

$|R|_{y_i\cdot y_j\cdot\underline{x}_q}$: Determinant of the minor corresponding to $\rho_{y_iy_j}$ of the correlation matrix of

$$y_i,\ y_j,\ x_1,\ x_2,\ \cdots,\ x_{q-1}\ \text{and}\ x_q\ (i\neq j). \tag{1.9}$$

2.2. Mean per Unit in Two-Phase Sampling

The sample mean $\bar{y}_2$ using simple random sampling without replacement in two-phase sampling is given by is given by,

$$\bar{y}_2=\frac{1}{n_2}\sum_{i=1}^{n_2}y_i \tag{2.0}$$

while its variance is given,

$$\operatorname{Var}\left(\bar{y}_2\right)=\theta_2\bar{Y}^2C_y^2. \tag{2.1}$$

2.3. Ratio Estimator Using Auxiliary Variable in Two-Phase Sampling

Let $\bar{x}_2=\frac{1}{n_2}\sum_{i=1}^{n_2}x_i$ be the sample mean of the auxiliary variable in two-phase sampling. The ratio estimator when information on one auxiliary variables is available for population (*full information case*) is:

$$t_{\mathrm{RV}}=\bar{y}_2\left(\frac{\bar{X}_1}{\bar{x}_2}\right)^{\alpha_1}. \tag{2.2}$$

The mean square error of t_{RV} can be written as:

$$\operatorname{MSE}\left(t_{\mathrm{RV}}\right)=\theta_2\bar{Y}^2\left(C_{y_2}^2+\alpha_1^2C_x^2-2\alpha_1C_xC_y\rho_{yx}\right) \tag{2.3}$$

where $\alpha_1=\frac{C_y\rho_{yx}}{C_x}$ and ρ_{yx} are the optimum value and the correlation coefficient respectively.

2.4. Ratio Estimator Using Multiple Auxiliary Variables in Two-Phase Sampling

The ratio estimator by Haq [9] when information on k auxiliary variables is available for population (full information case) is:

$$t_{\mathrm{RMV}}=\bar{y}_2\left(\frac{\bar{X}_1}{\bar{x}_{(2)1}}\right)^{\beta_1}\left(\frac{\bar{X}_2}{\bar{x}_{(2)2}}\right)^{\beta_2}\cdots\left(\frac{\bar{X}_k}{\bar{x}_{(2)k}}\right)^{\beta_k}. \tag{2.4}$$

The optimum values of unknown constants are,

$$\beta_j=(-1)^{j+1}\frac{C_y}{C_{x_j}}\frac{\left|R_{yx_j}\right|_{y\underset{\sim}{x}_k}}{\left|R\right|_{\underset{\sim}{x}_k}}. \tag{2.5}$$

The mean square error of t_{RMV} can be written as:

$$\operatorname{MSE}\left(t_{\mathrm{RMV}}\right)=\theta_2\bar{Y}^2C_y^2\left(1-\rho_{y\underset{\sim}{x}_k}^2\right). \tag{2.6}$$

2.5. Ratio Estimator Using Auxiliary Attribute in Two-Phase Sampling

In order to have an estimate of the population mean $\bar{Y}$ of the study variable *y*, assuming the knowledge of the population proportion *P*, Naik and Gupta [12] defined ratio estimator of population mean when the prior information of population proportion of units possessing the same attribute is variable. Naik and Gupta [12] proposed the following estimator:

$$t_{\mathrm{RA}}=\bar{y}_2\left(\frac{P}{p_2}\right)^{\alpha_1}. \tag{2.7}$$

The MSE of t_{RA} up to the first order of approximation are given respectively by,

$$\operatorname{MSE}\left(t_{\mathrm{RA}}\right)=\theta_2\bar{Y}^2\left(C_y^2+\alpha_1^2C_P^2-2\alpha_1C_PC_y\rho_{Pb\tau_j}\right) \tag{2.8}$$

where $\alpha_1=\frac{C_y\rho_{yx}}{C_x}$ and $\rho_{Pb\tau}$ are the optimum value and the bi-serial correlation coefficient respectively.

2.6. Ratio Estimator Using Multiple Auxiliary Attributes in Two-Phase Sampling

The ratio estimators by Hanif, Haq and Shahbaz [14] for two-phase sampling using information on multiple auxiliary attributes is given by,

$$t_{\text{MRA}} = \bar{y}_2 \left(\frac{P_1}{p_{(2)1}}\right)^{\gamma_1} \left(\frac{P_2}{p_{(2)2}}\right)^{\gamma_2} \cdots \left(\frac{P_q}{p_{(2)q}}\right)^{\gamma_q}. \tag{2.9}$$

The optimum values of unknown constants are,

$$\gamma_j = (-1)^{j+1} \frac{C_y}{C_{\tau_j}} \frac{\left|R_{y\tau_j}\right|_{y\underline{\tau}_q}}{|R|_{\underline{\tau}_q}}. \tag{2.10}$$

The MSE of the t_{MRA} up to the first order of approximation is given by,

$$\text{MSE}\left(t_{\text{MRA}}\right) = \theta_2 \bar{Y}^2 C_{y_2}^2 \left(1 - \rho_{Pb\underline{\tau}_q}^2\right). \tag{2.11}$$

In general these estimators have a bias of order $\frac{1}{n}$. Since the standard error of the estimates is of order $\frac{1}{\sqrt{n}}$, the quantity $\text{bias}/s.e$ is of order $\frac{1}{\sqrt{n}}$ and becomes negligible as n becomes large.

3. Methodology

3.1. Mixture Ratio Estimators Using Multi-Auxiliary Variable and Attributes for Two-Phase Sampling (Full Information Case)

If we estimate a study variable when information on all auxiliary variables is available from population, it is utilized in the form of their means. By taking the advantage of mixture ratio estimators technique for two-phase sampling, a generalized estimator for estimating population mean of study variable Y with the use of multi auxiliary variables and attributes are suggested as:

$$t_{\text{MRVAF}(3.0)} = \bar{y}_2 \left(\frac{\bar{X}_1}{\bar{x}_{(2)1}}\right)^{\alpha_1} \left(\frac{\bar{X}_2}{\bar{x}_{(2)2}}\right)^{\alpha_2} \cdots \left(\frac{\bar{X}_k}{\bar{x}_{(2)k}}\right)^{\alpha_k} \left(\frac{P_{k+1}}{p_{(2)k+1}}\right)^{\beta_{k+1}} \left(\frac{P_{k+2}}{p_{(2)k+2}}\right)^{\beta_{k+2}} \cdots \left(\frac{P_t}{p_{(2)t}}\right)^{\beta_t}. \tag{3.0}$$

Using (1.0), (1.3) and (1.4) in (3.0) and ignoring the second and higher terms for each expansion of product and after simplification, we write,

$$t_{\text{MRVAF}(3.0)} = \bar{e}_{y_2} + \bar{Y} - \sum_{j=1}^{k} \alpha_i \bar{Y} \frac{\bar{e}_{x_{(2)j}}}{\bar{X}_j} - \sum_{j=k+1}^{h+k=t} \beta_j \bar{Y} \frac{\bar{e}_{\tau(2)j}}{P_j}. \tag{3.1}$$

The mean squared error of $t_{\text{MRVAF}(3.0)}$ is given by,

$$\text{MSE}\left(t_{\text{MRVAF}(3.0)}\right) = E\left(t_{\text{MRVAF}(3.0)} - \bar{Y}\right)^2 = E\left(\bar{e}_{y_2} - \sum_{j=1}^{k} \alpha_j \bar{Y} \frac{\bar{e}_{x_{(2)j}}}{\bar{X}_j} - \sum_{j=k+1}^{h+k=t} \beta_j \bar{Y} \frac{\bar{e}_{\tau(2)j}}{P_j}\right)^2. \tag{3.2}$$

We differentiate the Equation (3.2) partially with respect to $\alpha_j \ (i = 1, 2, \cdots, k)$ and $\beta_j \ (i = k+1, k+2, \cdots, t)$ then equate to zero, using (1.6), (1.7) and (1.9), we get

$$\alpha_j = (-1)^{j+1} \frac{C_y}{C_{x_j}} \frac{\left|R_{yx_j}\right|_{\underline{x}_t}}{|R|_{\underline{x}_t}} \tag{3.3}$$

$$\beta_j = (-1)^{j+1} \frac{C_y}{C_{\tau_j}} \frac{\left|R_{y\tau_j}\right|_{\underline{\tau}_t}}{|R|_{\underline{\tau}_t}}. \tag{3.4}$$

Using normal equation that is used to find the optimum values given (3.2) we can write,

$$\mathrm{MSE}\left(t_{\mathrm{MRVAF}(3.0)}\right)=E\left(\bar{e}_{y_2}\left(\bar{e}_{y_2}-\sum_{j=1}^{k}\alpha_i\bar{Y}\frac{\bar{e}_{x(2)j}}{\bar{X}_j}-\sum_{j=k+1}^{h+k=t}\beta_j\bar{Y}\frac{\bar{e}_{\tau(2)j}}{P_j}\right)\right). \tag{3.5}$$

Taking expectation and using (1.6) in (3.5), we get,

$$\mathrm{MSE}\left(t_{\mathrm{MRVAF}(3.0)}\right)=\theta_2\bar{Y}^2\left(C_y^2-\sum_{j=1}^{k}\alpha_i\frac{\bar{Y}}{\bar{X}_j}\bar{X}_jC_yC_{x_j}\rho_{yx_j}-\sum_{j=k+1}^{k+h=t}\frac{\bar{Y}}{P_j}P_j\beta_jC_yC_{\tau_j}\rho_{Pb_j}\right). \tag{3.6}$$

Substituting the optimum (3.3) and (3.4) in (3.6), we get,

$$\mathrm{MSE}\left(t_{\mathrm{MRVAF}(3.0)}\right)=\theta_2\bar{Y}^2\left(C_y^2+\sum_{j=1}^{k}(-1)^j\frac{C_y}{C_{x_j}}\frac{\left|R_{yx_j}\right|_{x_j}}{\left|R_{x_j}\right|}C_yC_{x_j}\rho_{yx_j}+\sum_{j=1}^{k+h=t}(-1)^j\frac{C_y}{C_{\tau_j}}\frac{\left|R_{y\tau_j}\right|_{\tau_j}}{\left|R_{\tau_j}\right|}C_yC_{\tau_j}\rho_{Pb_j}\right). \tag{3.7}$$

Or

$$\mathrm{MSE}\left(t_{\mathrm{MRVAF}(3.0)}\right)=\theta_2\bar{Y}^2C_y^2\left(1+\left(\sum_{j=1}^{k}(-1)^j\frac{\left|R_{yx_j}\right|_{x_j}}{\left|R_{x_j}\right|}\rho_{yx_j}+\sum_{j=k+1}^{k+h=t}(-1)^j\frac{\left|R_{y\tau_i}\right|_{\tau_j}}{\left|R_{\tau_j}\right|}\rho_{Pb_j}\right)\right). \tag{3.8}$$

Or

$$\mathrm{MSE}\left(t_{\mathrm{MRVAF}(3.0)}\right)=\theta_2\bar{Y}^2C_y^2\frac{|R|_{y(\underline{x},\underline{\tau})_t}}{|R|_{(\underline{x},\underline{\tau})_t}}. \tag{3.9}$$

Using (1.8), we get,

$$\mathrm{MSE}\left(t_{\mathrm{MRVA}(3.0)}\right)=\theta_2\bar{Y}^2C_y^2\left(1-\rho^2_{y.(\underline{x},\underline{\tau})_t}\right). \tag{3.10}$$

3.2. Mixture Ratio Estimator Using Multi-Auxiliary Variable and Attributes in Two-Phase Sampling (Partial Information Case)

In this case suppose we have no information on all s and t auxiliary variables but only for r and g auxiliary variables from population. Considering mixture ratio technique of estimating technique, the population mean of study variable Y can be estimated for two-phase sampling using multi-auxiliary variables and attributes as:

$$t_{\mathrm{MRVAP}(3.1)}=\bar{y}_2\left(\frac{\bar{x}_{(1)1}}{\bar{x}_{(2)1}}\right)^{\alpha_1}\left(\frac{\bar{X}_1}{\bar{x}_{(2)1}}\right)^{\beta_1}\left(\frac{\bar{x}_{(1)2}}{\bar{x}_{(2)2}}\right)^{\alpha_2}\left(\frac{\bar{X}_2}{\bar{x}_{(2)2}}\right)^{\beta_2}\cdots\left(\frac{\bar{x}_{(1)r}}{\bar{x}_{(2)r}}\right)^{\alpha_s}\left(\frac{\bar{X}_r}{\bar{x}_{(2)r}}\right)^{\beta_s}\left(\frac{\bar{x}_{(1)s+1}}{\bar{x}_{(2)s+1}}\right)^{\alpha_{r+1}}\left(\frac{\bar{x}_{(1)s+2}}{\bar{x}_{(2)s+2}}\right)^{\alpha_{r+2}}\cdots\left(\frac{\bar{x}_{(1)k}}{\bar{x}_{(2)k}}\right)^{\alpha_k}$$
$$\left(\frac{p_{(1)k+1}}{p_{(2)k+1}}\right)^{\gamma_{k+1}}\left(\frac{P_{k+1}}{p_{(2)k+1}}\right)^{\lambda_{k+1}}\left(\frac{p_{(1)k+2}}{p_{(2)k+2}}\right)^{\gamma_{k+2}}\left(\frac{P_{k+2}}{p_{(2)k+2}}\right)^{\lambda_{k+2}}\cdots\left(\frac{p_{(1)h}}{p_{(2)h}}\right)^{\gamma_h}\left(\frac{P_h}{p_{(2)h}}\right)^{\lambda_h}\left(\frac{p_{(1)h+1}}{p_{(2)h+1}}\right)^{\gamma_{h+1}}\left(\frac{p_{(1)h+2}}{p_{(2)h+2}}\right)^{\gamma_{h+2}}\cdots\left(\frac{p_{(1)t}}{p_{(2)t}}\right)^{\gamma_q}. \tag{3.11}$$

Using (1.0), (1.3) and (1.4) in (3.1) and ignoring the second and higher terms for each expansion of product and after simplification, we write,

$$\begin{aligned}t_{\mathrm{MRVAP}(3.1)}=\bar{y}_2&+\sum_{j=1}^{r}\bar{Y}\alpha_j\frac{\bar{e}_{x(1)j}-\bar{e}_{x(2)j}}{\bar{X}_j}-\sum_{j=1}^{r}\bar{Y}\beta_j\frac{\bar{e}_{x(2)j}}{\bar{X}_j}+\sum_{j=1}^{s+r=k}\bar{Y}\alpha_j\frac{\bar{e}_{x(1)j}-\bar{e}_{x(2)j}}{\bar{X}_j}+\sum_{i=1}^{k+t=h}\bar{Y}\gamma_i\frac{\bar{e}_{\tau(1)j}-\bar{e}_{\tau(2)j}}{P_i}\\&-\sum_{j=1}^{k+t=h}\bar{Y}\lambda_j\frac{\bar{e}_{\tau(2)j}}{P_j}+\sum_{j=h+1}^{h+q=q}\bar{Y}\gamma_j\frac{\bar{e}_{\tau(1)j}-\bar{e}_{\tau(2)j}}{P_j}.\end{aligned} \tag{3.12}$$

Mean squared error of $t_{\mathrm{MRVAP}(3.1)}$ estimator is given by,

$$t_{\mathrm{MRVAP}(3.1)} = E_1E_{2/1}\left(\bar{e}_{y_2} + \sum_{j=1}^{r}\bar{Y}\alpha_j \frac{\bar{e}_{x_{(1)j}} - \bar{e}_{x_{(2)j}}}{\bar{X}_j} - \sum_{j=1}^{r}\bar{Y}\beta_j \frac{\bar{e}_{x_{(2)j}}}{\bar{X}_j} + \sum_{j=1}^{s+r=k}\bar{Y}\alpha_j \frac{\bar{e}_{x_{(1)j}} - \bar{e}_{x_{(2)j}}}{\bar{X}_j} + \sum_{i=1}^{k+t=h}\bar{Y}\gamma_i \frac{\bar{e}_{\tau_{(1)j}} - \bar{e}_{\tau_{(2)j}}}{P_i} - \sum_{j=1}^{k+t=h}\bar{Y}\lambda_j \frac{\bar{e}_{\tau_{(2)j}}}{P_j} + \sum_{j=h+1}^{h+q=q}\bar{Y}\gamma_j \frac{\bar{e}_{\tau_{(1)j}} - \bar{e}_{\tau_{(2)j}}}{P_j}\right)^2. \tag{3.13}$$

We differentiate the Equation (3.13) with respect to $\alpha_i(i=1,2,\cdots,r)$, $\beta_i(i=1,2,\cdots,r)$, $\alpha_i(i=r+1,r+2,\cdots,k)$, $\gamma_i(i=k+1,k+2,\cdots,h)$, $\lambda_i(i=k+1,k+2,\cdots,h)$, $\gamma_i(i=h+1,h+2,\cdots,q)$ and equate to zero and use (1.6), (1.7) and (1.9). The optimum values are as follows,

$$\begin{aligned}
\alpha_j &= (-1)^{j+1}\frac{C_y}{C_{x_j}}\left(\frac{\left|R_{yx_j}\right|_{\underline{x}_m}}{|R|_{\underline{x}_m}} - \frac{\left|R_{yx_j}\right|_{\underline{x}_w}}{\left|R_{yx_j}\right|_{\underline{x}_w}}\right) && (j=1,2,\cdots,r);\\
\gamma_j &= (-1)^{j+1}\frac{C_y}{C_{\tau_j}}\left(\frac{\left|R_{y\tau_j}\right|_{\underline{x}_p}}{|R|_{\tau_m}} - \frac{\left|R_{y\tau_j}\right|_{\tau_w}}{\left|R_{\tau_w}\right|}\right) && j=k+1,k+2,\cdots,h;\\
\beta_j &= (-1)^{j+1}\frac{C_y}{C_{x_j}}\frac{\left|R_{yx_j}\right|_{\underline{x}_r}}{\left|R_{\underline{x}_w}\right|} && (j=1,2,\cdots,r);\\
\lambda_j &= \frac{C_y}{\left|R_{\tau_w}\right|}\frac{1}{C_{\tau_j}}\left((-1)^{j+1}\left|R_{y\tau_j}\right|_{\tau_w}\right) && j=k+1,k+2,\cdots,h;\\
\alpha_j &= (-1)^{j+1}\frac{C_y}{C_{x_j}}\frac{\left|R_{yx_{ji}}\right|_{\underline{x}_s}}{|R|_{\underline{x}_s}} && j=r+1,r+2,\cdots,k;\\
\gamma_j &= (-1)^{j+1}\frac{C_y}{|R|_{\underline{x}_p}C_{\tau_j}}\left|R_{y\psi_j}\right|_{\tau_w} && j= h+1,h+2,\cdots,q.
\end{aligned} \tag{3.14}$$

Using normal equation that are used to find the optimum values given (3.13) we can write

$$\mathrm{MSE}\left(t_{\mathrm{MRVAP}(3.1)}\right) = E_1E_{2/1}\left(\bar{e}_y\left(\bar{e}_{y_2} + \sum_{j=1}^{r}\bar{Y}\alpha_j \frac{\bar{e}_{x_{(1)j}} - \bar{e}_{x_{(2)j}}}{\bar{X}_j} - \sum_{j=1}^{r}\bar{Y}\beta_j \frac{\bar{e}_{x_{(2)j}}}{\bar{X}_j} + \sum_{j=1}^{s+r=k}\bar{Y}\alpha_j \frac{\bar{e}_{x_{(1)j}} - \bar{e}_{x_{(2)j}}}{\bar{X}_j} + \sum_{i=1}^{k+t=h}\bar{Y}\gamma_i \frac{\bar{e}_{\tau_{(1)j}} - \bar{e}_{\tau_{(2)j}}}{P_i} - \sum_{j=1}^{k+t=h}\bar{Y}\lambda_j \frac{\bar{e}_{\tau_{(2)j}}}{P_j} + \sum_{j=h+1}^{h+q=m}\bar{Y}\gamma_j \frac{\bar{e}_{\tau_{(1)j}} - \bar{e}_{\tau_{(2)j}}}{P_j}\right)\right). \tag{3.15}$$

Taking expectation and using (1.6) in (3.15), we get,

$$\begin{aligned}
MSE\left(t_{MRVAP(3.1)}\right) &= \theta_2\bar{Y}^2C_y^2 + (\theta_1-\theta_2)\sum_{j=1}^{r}\frac{\bar{Y}^2}{\bar{X}_j}\alpha_j\bar{X}_jC_yC_{x_j}\rho_{yx_j} - \theta_2\sum_{j=1}^{r}\frac{\bar{Y}^2}{\bar{X}_j}\beta_j\bar{X}_jC_yC_{x_j}\rho_{yx_j} + (\theta_1-\theta_2)\sum_{j=r+1}^{r+s=k}\frac{\bar{Y}^2}{\bar{X}_j}\alpha_j\bar{X}_jC_yC_{x_j}\rho_{yx_j}\\
&\quad + (\theta_1-\theta_2)\sum_{i=k+1}^{t+k=q}\frac{\bar{Y}^2}{P_j}\gamma_jP_jC_yC_{\tau_j}\rho_{Pb_j} - \theta_2\sum_{i=k+1}^{t+k=h}\frac{\bar{Y}^2}{P_j}\lambda_jP_jC_yC_{\tau_j}\rho_{Pb_j} + (\theta_1-\theta_2)\sum_{i=h+1}^{q}\frac{\bar{Y}^2}{P_j}\gamma_jP_jC_yC_{\tau_j}\rho_{Pb_j}.
\end{aligned} \tag{3.16}$$

Using the optimum value (3.14) in (3.16), we get,

$$
\begin{aligned}
&\mathrm{MSE}\left(t_{\mathrm{MRVAP}(3.1)}\right) \\
&=\bar{Y}^2C_y^2\left(\theta_2+\left(\theta_1-\theta_2\right)\sum_{j=1}^{r}(-1)^{j+1}\left(\frac{\left|R_{yx_j}\right|_{y.\underline{x}_q}}{|R|_{\underline{x}_q}}-\frac{\left|R_{yx_j}\right|_{y.\underline{x}_w}}{|R|_{\underline{\tau}_w}}\right)\rho_{yx_j}-\theta_2\sum_{j=1}^{r}(-1)^{j+1}\frac{\left|R_{yx_j}\right|_{y.\underline{x}_w}}{|R|_{\underline{\tau}_w}}\rho_{yx_j}\right.\\
&\quad+\left(\theta_1-\theta_2\right)\sum_{j=r+1}^{r+s=k}(-1)^{j+1}\frac{\left|R_{yx_i}\right|_{y.\underline{x}_q}}{|R|_{\underline{x}_q}}\rho_{yx_i}+\left(\theta_1-\theta_2\right)\sum_{j=k+1}^{t+k=h}(-1)^{j+1}\left(\frac{\left|R_{y\tau_j}\right|_{y.\underline{x}_q}}{|R|_{\underline{x}_q}}-\frac{\left|R_{y\tau_j}\right|_{y.\underline{\tau}_w}}{|R|_{\underline{\tau}_w}}\right)\rho_{Pb_j}\\
&\quad\left.-\theta_2\sum_{j=k+1}^{t+k=h}(-1)^{j+1}\frac{\left|R_{y\tau_j}\right|_{y.\underline{\tau}_w}}{|R|_{\underline{\tau}_w}}\rho_{Pb_j}+\left(\theta_1-\theta_2\right)\sum_{j=h+1}^{q}(-1)^{j+1}\frac{\left|R_{y\tau_j}\right|_{y.\underline{\tau}_q}}{|R|_{\underline{\tau}_q}}\gamma_j\rho_{Pb_j}\right).
\end{aligned}
\tag{3.17}
$$

Or

$$
\begin{aligned}
\mathrm{MSE}\left(t_{\mathrm{MRVAP}(3.1)}\right)=\bar{Y}^2C_y^2&\left(\theta_2+\left(\theta_1-\theta_2\right)\sum_{j=1}^{r}(-1)^{j+1}\frac{\left|R_{yx_j}\right|_{y.\underline{x}_q}}{|R|_{\underline{x}_q}}\rho_{yx_j}-\theta_1\sum_{j=1}^{r}(-1)^{j+1}\frac{\left|R_{yx_j}\right|_{y.\underline{x}_w}}{|R|_{\underline{x}_w}}\rho_{yx_j}\right.\\
&+\left(\theta_1-\theta_2\right)\sum_{j=r+1}^{r+s=k}(-1)^{j+1}\frac{\left|R_{yx_i}\right|_{y.\underline{x}_q}}{|R|_{\underline{x}_q}}\rho_{yx_i}+\left(\theta_1-\theta_2\right)\sum_{j=k+1}^{t+k=h}(-1)^{j+1}\frac{\left|R_{y\tau_j}\right|_{y.\underline{\tau}_q}}{|R|_{\underline{\tau}_q}}\rho_{Pb_j}\\
&\left.-\theta_1\sum_{j=k+1}^{t+k=h}(-1)^{j+1}\frac{\left|R_{y\tau_j}\right|_{y.\underline{\tau}_w}}{|R|_{\underline{x}_w}}\rho_{Pb_j}+\left(\theta_1-\theta_2\right)\sum_{j=h+1}^{q}(-1)^{j+1}\frac{\left|R_{y\tau_j}\right|_{y.\underline{\tau}_q}}{|R|_{\underline{\tau}_q}}\gamma_j\rho_{Pb_j}\right).
\end{aligned}
\tag{3.18}
$$

Or

$$
\begin{aligned}
&\mathrm{MSE}\left(t_{\mathrm{MRVAP}(3.1)}\right)\\
&=\bar{Y}^2C_y^2\left(\left(\theta_2-\theta_1\right)\left(1+\left(\sum_{j=1}^{r}(-1)^{j}\frac{\left|R_{yx_i}\right|_{\underline{x}_q}}{|R|_{\underline{x}_q}}\rho_{yx_j}+\sum_{j=r+1}^{r+s=k}(-1)^{j}\frac{\left|R_{yx_j}\right|_{y.\underline{x}_q}}{|R|_{\underline{x}_q}}\rho_{yx_j}+\sum_{j=k+1}^{t+k=h}(-1)^{j}\frac{\left|R_{y\tau_j}\right|_{y.\underline{\tau}_q}}{|R|_{\underline{\tau}_q}}\rho_{Pb_j}\right.\right.\right.\\
&\quad\left.\left.+\sum_{j=h+1}^{q}(-1)^{j}\frac{\left|R_{yx_j}\right|_{y.\underline{\tau}_q}}{|R|_{\underline{\tau}_q}}\gamma_i\rho_{Pb_j}\right)+\theta_1\left(1+\left(\sum_{j=1}^{r}(-1)^{j}\frac{\left|R_{yx_i}\right|_{y.\underline{x}_w}}{|R|_{\underline{x}_w}}\rho_{yx_i}+\sum_{j=k+1}^{t+k=h}(-1)^{j}\frac{\left|R_{yx_j}\right|_{y.\underline{x}_w}}{|R|_{\underline{x}_w}}\rho_{Pb_j}\right)\right)\right).
\end{aligned}
\tag{3.19}
$$

Or

$$
\begin{aligned}
\mathrm{MSE}\left(t_{\mathrm{MRVAP}(3.1)}\right)=\bar{Y}^2C_y^2&\left(\frac{\left(\theta_2-\theta_1\right)}{|R|_{\left(\underline{x}\,\underline{\tau}\right)_q}}\left(|R|_{\left(\underline{x}\,\underline{\tau}\right)_q}+\sum_{j=1}^{q}(-1)^{j}\frac{\left|R_{y.x_j.\tau_j}\right|_{\left(\underline{x}\,\underline{\tau}\right)_q}}{|R|_{\left(\underline{x}\,\underline{\tau}\right)_q}}\rho_{y\left(x_j,\tau_j\right)}\right)\right.\\
&\left.+\frac{\theta_1}{|R|_{\left(\underline{x}\,\underline{\tau}\right)_w}}\left(1+\left(\sum_{j=1}^{r}(-1)^{j}\frac{\left|R_{yx_i}\right|_{\underline{x}_r}}{|R|_{\left(\underline{x}\,\underline{\tau}\right)_w}}\rho_{yx_j}+\sum_{j=k+1}^{t+k=h}(-1)^{j}\frac{\left|R_{yx_j}\right|_{\underline{\tau}_w}}{|R|_{\left(\underline{x}\,\underline{\tau}\right)_w}}\rho_{Pb_j}\right)\right)\right).
\end{aligned}
\tag{3.20}
$$

Or

$$\mathrm{MSE}\left(t_{\mathrm{MRVAP}(3.1)}\right)=\bar{Y}^2C_y^2\left((\theta_2-\theta_1)\frac{|R|_{y.(\underset{\sim}{x},\underset{\sim}{\tau})_q}}{|R|_{(\underset{\sim}{x},\underset{\sim}{\tau})_q}}+\theta_1\frac{|R|_{y.(\underset{\sim}{x},\underset{\sim}{\tau})_w}}{|R|_{(\underset{\sim}{x},\underset{\sim}{\tau})_w}}\right). \tag{3.21}$$

Using (1.8) in (3.21), we get,

$$\mathrm{MSE}\left(t_{\mathrm{MRVAP}(3.1)}\right)=\bar{Y}^2C_y^2\left((\theta_2-\theta_1)\left(1-\rho^2_{y(\underset{\sim}{x},\underset{\sim}{\tau})_q}\right)+\theta_1\left(1-\rho^2_{y(\underset{\sim}{x},\underset{\sim}{\tau})_w}\right)\right). \tag{3.22}$$

Simplifying (3.22) we get,

$$\mathrm{MSE}\left(t_{\mathrm{MRVAP}(3.1)}\right)=\bar{Y}^2C_y^2\left(\theta_2\left(1-\rho^2_{y(\underset{\sim}{x},\underset{\sim}{\tau})_q}\right)+\theta_1\left(\rho^2_{y(\underset{\sim}{x},\underset{\sim}{\tau})_q}-\rho^2_{y(\underset{\sim}{x},\underset{\sim}{\tau})_w}\right)\right). \tag{3.23}$$

3.3. Mixture Ratio Estimator Using Multi-Auxiliary Variable and Attributes in Two-Phase Sampling (No Information Case)

If we estimate a study variable when information on all auxiliary variables is unavailable from population, it is utilized in the form of their means. By taking the advantage of mixture ratio technique for two-phase sampling, a generalized estimator for estimating population mean of study variable Y with the use of multi auxiliary variables and attributes are suggested as:

$$t_{\mathrm{MRVAN}(3.2)}=\bar{y}_2\left(\frac{\bar{x}_{(1)1}}{\bar{x}_{(2)1}}\right)^{\alpha_1}\left(\frac{\bar{x}_{(1)2}}{\bar{x}_{(2)2}}\right)^{\alpha_2}\cdots\left(\frac{\bar{x}_{(1)k}}{\bar{x}_{(2)k}}\right)^{\alpha_s}\left(\frac{p_{(1)k+1}}{p_{(2)k+1}}\right)^{\gamma_{k+1}}\left(\frac{p_{(1)k+2}}{p_{(2)k+2}}\right)^{\gamma_{k+2}}\cdots\left(\frac{p_{(1)t}}{p_{(2)t}}\right)^{\gamma_t}. \tag{3.24}$$

Using (1.0), (1.3) and (1.4) in (3.24) and ignoring the second and higher terms for each expansion of product and after simplification, we write,

$$t_{\mathrm{MRVAN}(3.2)}=\bar{y}_2+\sum_{j=1}^{k}\bar{Y}\alpha_i\frac{\bar{e}_{x_{j(1)}}-\bar{e}_{x_{j(2)}}}{\bar{X}_j}+\sum_{j=1}^{k+h=t}\bar{Y}\gamma_j\frac{\bar{e}_{\tau_{j(1)}}-\bar{e}\tau_{x_{j(2)}}}{P_j}. \tag{3.25}$$

Mean squared error of $t_{\mathrm{MRVAN}(3.2)}$ estimator is given by

$$\mathrm{MSE}\left(t_{\mathrm{MRVAN}(3.2)}\right)=E\left(\bar{e}_{y_2}+\sum_{j=1}^{k}\bar{Y}\alpha_j\frac{\bar{e}_{x_{j(1)}}-\bar{e}_{x_{j(2)}}}{\bar{X}_j}+\sum_{j=1}^{k+h=t}\bar{Y}\gamma_j\frac{\bar{e}_{\tau_{j(1)}}-\bar{e}_{\tau_{j(2)}}}{P_j}\right)^2. \tag{3.26}$$

We differentiate the Equation (3.27) partially with respect to $\alpha_i\ (i=1,2,\cdots,k)$ and $\beta_i\ (i=k+1,k+2,\cdots,t)$ then equate to zero, using (1.6), (1.7) and (1.9), we get

$$\alpha_j=(-1)^{j+1}\frac{C_y}{C_{x_j}}\frac{|R_{yx_j}|_{x_k}}{|R_{x_j}|} \tag{3.27}$$

$$\beta_j=(-1)^{j+1}\frac{C_y}{C_{\tau_j}}\frac{|R_{y\tau_i}|_{\tau_j}}{|R_{\tau_j}|}. \tag{3.28}$$

Using normal equation that are used to find the optimum values given (3.26) we can write,

$$\mathrm{MSE}\left(t_{\mathrm{MRVAN}(3.2)}\right)=E\left(\bar{e}_{y_2}\left(\bar{e}_{y_2}+\sum_{j=1}^{k}\bar{Y}\alpha_j\frac{\bar{e}_{x_{(1)j}}-\bar{e}_{x_{(2)j}}}{\bar{X}_i}+\sum_{j=k+1}^{k+h=t}\bar{Y}\gamma_j\frac{\bar{e}_{x_{(1)j}}-\bar{e}_{x_{(2)j}}}{P_j}\right)\right). \tag{3.29}$$

Taking expectation and using (1.6) in (3.29), we get

$$\mathrm{MSE}\left(t_{\mathrm{MRVAN}(3.2)}\right)=\bar{Y}^2C_y^2\left(\theta_2+(\theta_1-\theta_2)\sum_{j=1}^{k}\alpha_jC_yC_{x_j}\rho_{yx_j}+(\theta_1-\theta_2)\sum_{j=k+1}^{k+h=t}\beta_jC_yC_{\tau_j}\rho_{Pb_j}\right). \tag{3.30}$$

Or

$$\mathrm{MSE}\left(t_{\mathrm{MRVAN}(3.2)}\right)=\theta_2\bar{Y}^2C_y^2\left(\theta_2+\left(\theta_2-\theta_1\right)\sum_{j=1}^{k}(-1)^j\frac{\left|R_{yx_j}\right|_{y.\underline{x}_q}}{|R|_{(\underline{x},\underline{\tau})_q}}\rho_{yx_j}+\left(\theta_2-\theta_1\right)\sum_{j=1}^{k+h=t}(-1)^j\frac{\left|R_{y\tau_j}\right|_{y.\underline{x}_q}}{|R|_{(\underline{x},\underline{\tau})_q}}\rho_{Pb_j}\right). \quad (3.31)$$

Or

$$\mathrm{MSE}\left(t_{\mathrm{MRVAN}(3.2)}\right)=\theta_2\bar{Y}^2C_y^2\left(\left(\theta_2-\theta_1\right)\left(1+\sum_{j=1}^{q}(-1)^j\frac{\left|R_{y.x_i.\tau_j}\right|_{y.(\underline{x}.\underline{\tau})_q}}{|R|_{(\underline{x}.\underline{\tau})_q}}\rho_{y.(x_i.\tau_j)}+\theta_1\right)+\theta_1\right). \quad (3.32)$$

Or

$$\mathrm{MSE}\left(t_{\mathrm{MRVAN}(3.2)}\right)=\bar{Y}^2C_y^2\left(\frac{\left(\theta_2-\theta_1\right)}{|R|_{(\underline{x}.\underline{\tau})_q}}\left(|R|_{(\underline{x}.\underline{\tau})_q}+\sum_{i=1}^{t}(-1)^j\left|R_{y.x_i.\tau_j}\right|_{y.(\underline{x}.\underline{\tau})_q}\rho_{y.(x_j.\tau_j)}+\theta_1\right)\right). \quad (3.33)$$

Or

$$\mathrm{MSE}\left(t_{\mathrm{MRVAN}(3.2)}\right)=\bar{Y}^2C_y^2\left(\left(\theta_2-\theta_1\right)\frac{|R|_{y.(\underline{x},\underline{\tau})_q}}{|R|_{(\underline{x},\underline{\tau})_q}}+\theta_1\right). \quad (3.34)$$

Using (1.8) in (3.34), we get,

$$\mathrm{MSE}\left(t_{\mathrm{MRVAN}(3.2)}\right)=\bar{Y}^2C_y^2\left(\left(\theta_2-\theta_1\right)\left(1-\rho^2_{y(\underline{x},\underline{\tau})}\right)+\theta_1\right). \quad (3.35)$$

Simplifying (3.35), we get,

$$\mathrm{MSE}\left(t_{\mathrm{MRVAN}(3.2)}\right)=\bar{Y}^2C_y^2\left(\theta_2\left(1-\rho^2_{y(\underline{x},\underline{\tau})}\right)+\theta_1\rho^2_{y(\underline{x},\underline{\tau})}\right). \quad (3.36)$$

3.4. Bias and Consistency of Mixture Ratio Estimators

These mixture ratio estimators using multiple auxiliary variables and attributes in two-phase sampling are biased. However, these biases are negligible for large samples that is $(n \geq 30)$. It's easily shown that the mixture ratio estimators are consistent estimators using multiple auxiliary variables since they are linear combinations of consistent estimators it follows that they are also consistent.

4. Simulation, Result and Discussion

We carried out data simulation experiments to compare the performance of mixture ratio estimators using multiple auxiliary variables and attributes in two-phase sampling with ratio estimator using one auxiliary variable and one auxiliary attribute or ratio estimator using multiple auxiliary variable or multiple auxiliary attributes in two-phase sampling estimators for finite population.

All the results were obtained after carrying out two hundred simulations and taking their average.

1) Study variable $N = 800,\ n = 96,\ n_2 = 34,\ \text{mean} = 50,\ \text{standard deviation} = 6.$

2) For ratio estimator the auxiliary variable is positively correlated with the study variable and the line passes through the origin.

$$N = 500,\ n = 96,\ n_2 = 34,\ \text{mean} = 29,\ \text{standard deviation} = 6.712.$$

$$N = 500,\ n = 96,\ n_2 = 34,\ \text{mean} = 38,\ \text{standard deviation} = 8.612.$$

Correlation coefficients $\rho_{yx_1} = 0.6820,\ \rho_{yx_2} = 0.5788.$

3) For ratio estimator the auxiliary attributes is positively correlated with the study variable and the line passes through the origin.

$$N = 500,\ n = 96,\ n_2 = 34,\ \text{mean} = 0.5612.$$

$$N = 500,\ n = 96,\ \text{mean} = 0.4826.$$

Correlation coefficients $\rho_{y\tau_2} = 0.7031$, $\rho_{y\tau_1} = 0.6614$.

In order to evaluate the efficiency gain we could achieve by using the proposed estimators, we have calculated the variance of mean per unit and the mean squared error of all estimators we have considered. We have then calculated percent relative efficiency of each estimator in relation to variance of mean per unit. We have then compared the percent relative efficiency of each estimator, the estimator with the highest percent relative efficiency is considered to be the more efficient than the other estimators. The percent relative efficiency is calculated using the following formulae.

$$\text{eff}\left(\hat{\bar{Y}}\right) = \frac{\text{Var}\left(\hat{\bar{y}}\right)}{\text{MSE}\left(\hat{\bar{Y}}\right)} * 100 \tag{4.0}$$

Table 1 shows percent relative efficiency of proposed estimator with respect to mean per unit estimator for single-phase sampling. It is very clear from **Table 1** that our proposed mixture ratio estimator using multiple auxiliary variables and multiple auxiliary attributes simultaneously is the most efficient compared to ratio estimator using one auxiliary variable and one auxiliary attribute or ratio estimator using multiple auxiliary variable or multiple auxiliary attributes in two-phase sampling.

Table 2 compares the efficiency of full information case and partial case to no information case and full to partial information case of proposed mixture ratio estimators. It is observed that the full information case and partial information case are more efficient than no information case because they have higher percent relative efficiency than no information case. In addition, the full information case is more efficient than the partial information case because it has a higher percent relative efficiency than partial information case.

5. Conclusion

The proposed mixture ratio estimator under full information case is recommended for estimating the finite population mean since it is the most efficient estimator compared to mean per unit, ratio estimator using one auxiliary variable, ratio estimator using one auxiliary attribute, ratio estimator using multiple auxiliary variable and ratio estimator using multiple auxiliary attributes in two-phase sampling. In case some auxiliary variables or attributes are unknown, we recommend mixture ratio estimator under partial information case since it is more

Table 1. Relative efficiency of existing and proposed estimator with respect to mean per unit estimator for two-phase sampling.

Estimator	Relative percent efficiency with respect to mean per unit in two-phase sampling
$\text{var}(\bar{y})$	100
t_{RV}	146
t_{RA}	185
t_{MRV}	212
t_{MRA}	233
$t_{\text{MRVAF(3.0)}}$ (proposed)	285

Table 2. Comparisons of full, partial and no information cases for proposed ratio-cum-product estimator using multiple auxiliary variables.

Population	Percent relative efficiency of full and partial to no information			Percent relative efficiency of full to partial in formation case	
Estimator	$t_{\text{MRVAN(3.2)}}$	$t_{\text{MRVAP(3.1)}}$	$t_{\text{MRVAF(3.0)}}$	$\text{t}_{\text{MRVAP(3.1)}}$	$t_{\text{MRVAF(3.0)}}$
Relative percent efficiency	100	139	172	100	127

efficient than the mixture ratio estimator under no information case and if all are unknown, we recommend the mixture ratio estimator under no information case to estimate finite population mean.

References

[1] Neyman, J. (1938) Contribution to the Theory of Sampling Human Populations. *Journal of the American Statistical Association*, **33**, 101-116. http://dx.doi.org/10.1080/01621459.1938.10503378

[2] Cochran, W.G. (1940) The Estimation of the Yields of the Cereal Experiments by Sampling for the Ratio of Grain to Total Produce. *Journal of Agricultural Science*, **30**, 262-275. http://dx.doi.org/10.1017/S0021859600048012

[3] Hansen, M.H. and Hurwitz, W.N. (1943) On the Theory of Sampling from Finite Populations. *Annals of Mathematical Statistics*, **14**, 333-362. http://dx.doi.org/10.1214/aoms/1177731356

[4] Watson, D.J. (1937) The Estimation of Leaf Areas. *Journal of Agricultural Science*, **27**, 474. http://dx.doi.org/10.1017/S002185960005173X

[5] Olikin, I. (1958) Multivariate Ratio Estimation for Finite Population. *Biometrika*, **45**, 154-165. http://dx.doi.org/10.1093/biomet/45.1-2.154

[6] Raj, D. (1965) On a Method of Using Multi-Auxiliary Information in Sample Surveys. *Journals of the American Statistical Association*, **60**, 154-165. http://dx.doi.org/10.1080/01621459.1965.10480789

[7] Singh, M.P. (1967) Ratio-Cum-Product Method of Estimation. *Metrika*, **12**, 34-42. http://dx.doi.org/10.1007/BF02613481

[8] Robson, D.S. (1952) Multiple Sampling of Attributes. *Journal of the American Statistical Association*, **47**, 203-215. http://dx.doi.org/10.1080/01621459.1952.10501164

[9] Ahmad, Z. (2008) Generalized Multivariate Ratio and Regression Estimators for Multi-Phase Sampling. Unpublished Ph.D. Thesis, National College of Business Administration and Economics, Lahore.

[10] Zahoor, A., Muhhamad, H. and Munir, A. (2009) Generalized Regression-Cum-Ratio Estimators for Two Phase Sampling Using Multiple Auxiliary Variables. *Pakistan Journal of Statistics*, **25**, 93-106.

[11] Jhajj, H.S., Sharma, M.K. and Grover, L.K. (2006) A Family of Estimator of Population Mean Using Information on Auxiliary Attributes. *Pakistan Journal of Statistics*, **22**, 43-50.

[12] Naik, V.D. and Gupta, P.C. (1996) A Note on Estimation of Mean with Known Population of Auxiliary Character. *Journal of the Indian Society of Agricultural Statistics*, **48**, 151-158.

[13] Bahl, S. and Tuteja, R.K. (1991) Ratio and Product Type Estimator. *Information and Optimization Sciences*, **12**, 159-163.

[14] Hanif, M., Haq, I.U. and Shahbaz, M.Q. (2009) On a New Family of Estimator Using Multiple Auxiliary Attributes. *World Applied Science Journal*, **11**, 1419-1422.

[15] Kung'u, J. and Odongo, L. (2014) Ratio-Cum-Product Estimator Using Multiple Auxiliary Attributes in Single Phase Sampling. *Open Journal of Statistics*, **4**, 239-245. http://dx.doi.org/10.4236/ojs.2014.44023

[16] Kung'u, J. and Odongo, L. (2014) Ratio-Cum-Product Estimator Using Multiple Auxiliary Attributes in Two-Phase Sampling. *Open Journal of Statistics*, **4**, 246-257. http://dx.doi.org/10.4236/ojs.2014.44024

[17] Moeen, M., Shahbaz, Q. and Hanif, M. (2012) Mixture Ratio and Regression Estimators Using Multi-Auxiliary Variable and Attributes in Single Phase Sampling. *World Applied Sciences Journal*, **18**, 1518-1526.

[18] Simiuddin, M. and Hanif, M. (2007) Estimation of Population Mean in Single and Two Phase Sampling with or without Additional Information. *Pakistan Journal of Statistics*, **23**, 99-118.

[19] Arora, S. and Lal, B. (1989) New Mathematical Statistics. Satya Prakashan, New Delhi.