A View on Stochastic Finite Element and Geostatistics for Resource Parameters Estimation

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ABSTRACT

The resource parameter estimation using stochastic finite element, geostatistics etc. is a key point on uncertainty, risk analysis, optimization [1-5] etc. In this view, the paper presents some consideration on: 1) Stochastic finite element estimation. The concept of random element is simplified as a stochastic finite element (SFE) taking into account a parallelepiped element with eight nodes in which are given the probability density functions (pdf) on its point supports. In this context it is shown: a—the stochastic finite element is a linear interpolator, related to the distributions given at each nodes; b—the distribution pdf in whatever point \( x \in V \); c—the estimation of the mean value of \( Z(x) \); 2) Volume integrals calculus; 3) SFE in geostatistics approaches; 4) SFE in PDE solution. Finally, some conclusions are presented underlying the importance of SFE applications

Keywords: Parameter Estimation; SFE; Geostatistics; Kriging; Risk Analysis; Optimisation

1. Introduction

Many physical phenomena and processes are mathematically modeled by partial differential equations (PDE). The data required by PDE’s models as resource and material parameters are in practice subject to uncertainty due to different errors or modeling assumptions, the lack of knowledge and information. In this view the parameters are (not deterministic) stochastic ones [6].

The considerable attention that stochastic finite element (SFE) received over the last decade [7-9] is mainly attributed to the spectacular growth of computing power, rendering possible the efficient treatment of large scale problems in dynamics of processes etc.

Fundamental issue in SFE is the parameter estimation and reserves. The most outstanding method for the approximate solution of a SPDE is the MONTE CARLO method [10]. On the other hand, the geostatistics is a useful discipline to make the inference about the spatial risk phenomenon (processes) [11].

2. A View on the Random Element

Let’s be defined a fixed probability space \((\Omega, A, P)\) [7], where \(\Omega\) is a nonempty set of “outcomes” or elementary events”, \(A\) is a \(\sigma\) algebra of subsets of \(\Omega\) (the “random events”) and \(P\) is a probability measure on the measurable space \((\Omega, A)\). If \((\chi, S_\chi)\) is another measurable space, then a random element \(X\) in \(\chi\) is a measurable mapping from \((\Omega, A, P)\) into \((\chi, S_\chi)\) i.e. it holds \(X: \Omega \rightarrow \chi\) with:

\[
X^{-1}(B) := \{X \in B\} := \{\omega \in \Omega: \chi(\omega) \in B\}, \quad \forall B \in S_\chi
\]

with each random element \(X: \Omega \rightarrow \chi\), \(P_\chi\) is a probability measure of \((\chi, S_\chi)\) connected with the distribution of random elements. It is defined by:

\[
P_\chi(B) := P\{X \in B\} = P\{\omega \in \Omega: \chi(\omega) \in B\}, \quad \forall B \in S_\chi
\]

A random element \(X\) with values in \(X\) is called a simple random element if the range is a finite nonempty set in \(X\), where exists a partition \([4,12]\) of the probability space \(\Omega = \bigcup_{k=1}^{N} \Omega_k\) with measurable sets

\[
\Omega_k \in A \quad k = 1, 2, \ldots, N \quad \left( N \in \mathbb{N} \right)
\]

such like: \(X(\omega) = x_k\) for \(\omega \in \Omega_k\).

The corresponding probabilities are:

\[
P(\Omega_k) = p_k, \quad p_k \geq 0, \quad k = 1, 2, \ldots, N
\]

\[
\sum_{k=1}^{N} p_k = 1
\]

The distribution of a simple random element is a discrete
probability measure on \((X, S)\) that might be written as:
\[
P_x = \sum_{k=1}^{N} p_k \delta_{x_k},
\]
where: \(\delta_{x_k}\) the Dirac measure
\[
\delta_{x_k}(B) = \begin{cases} 
1 & \text{if } x_k \in B \\
0 & \text{otherwise}
\end{cases}
\]


Even though this is a general concept [7,14] we will present some considerations in the viewpoint its applications in the parameter estimation of different phenomena and processes.

Let’s consider a zone \(V \subset R^3\) and a random function \(Z(x), x \in V\). The zone \(V\) is sorted out into blocks \(v_i\) by a parallelepiped grid:
\[
V = \bigcup v_i
\]
where: \(v_i\) is a parallelepiped element with eight nodes.

At each node, the random function \(Z(x)\) is known, in other words it is given the probability density function (pdf) on its point support (Figure 1). It is required:

The distribution pdf in whatever point \(x \in V\).

The estimation of the mean value
\[
z_x = \frac{1}{V} \int Z(x) dx \text{ over the domain } V
\]

We define a stochastic element as a block, with the random function \(Z(x), x \in v_i\).

Let us consider a reference element \(w_i\) in the co-ordinate system \(s_1, s_2, s_3\). If we choose an incomplete base [15]:
\[
P(s) = \{1, s_1, s_2, s_3, s_1 s_2, s_1 s_3, s_2 s_3, s_1 s_2 s_3\}
\]
Then the function \(Z(x)\) could be presented as a linear combination:
\[
Z(x) = Z(s_1 s_2 s_3) = (P(s))^{-1} \cdot \{N(s)\} \cdot \{Z^s\} \tag{4}
\]
where:
\([P(s)]^{-1}\) is the matrix, whose elements are the polynomials base values at the nodes
\([Z^s]\) — is the vector of the distributions of the nodes;
\([N(s)]\) — is the vector of the shape functions;
\(N_i, i = 1, 2, \ldots, 8\)
\[
N(s) = \{N_i(s), N_2(s), \ldots, N_8(s)\}
\]
\(N_i(s) = \{1 + s_{s_i}^1\} \{1 + s_{s_i}^2\}\)

In the formula (5), "the exponent \(i^r\)" is not a variable. It indicates only the sign within the parentheses.

3.1. The Mean Value

To calculate the mean value \(z_{x_i} = \frac{1}{V} \int Z(x) dx\), we consider the deterministic transformation:
\[
X_i(s) = \{N_i(s)\} \{X^s\} \quad i = 1, 2, \ldots, 8
\]
Therefore [13]
\[
Z_{x_i} = \frac{1}{V} \int \left[ Z(x_i (S_1, S_2, S_3)) \cdot x_i (S_1, S_2, S_3) \right] \det J dS_1 dS_2 dS_3
\]
\[
= \frac{8}{i = 1} H_i Z_i(x)
\]

The coefficients \(a_{ij}, b_{ij}, c_{ij}, d_{ij}\), \(i, j = 1, 2, 3\) are depend only on the node coordinates. Knowing the above coefficient we can calculate [13] the weight coefficient \(H_i, i = 1, \ldots, 8\) as for example for \(H_2\):
\[
H_2 = 8/3c_{21}a_{32}c_{13} + 8/9 d_{23} a_{32} a_{31} + 8/27 c_{12} c_{23} d_{13}
+ 8/27 d_{23} c_{12} b_{13} + 8/9 a_{12} b_{31} c_{13} + 8/9 b_{12} b_{31} a_{13}
+ 8/27 a_{23} d_{12} b_{31} + 8/27 a_{12} d_{12} c_{31} + 8/9 c_{12} a_{23} d_{31}
+ 8/9 d_{12} a_{23} a_{33} + 8/9 d_{12} b_{23} c_{33} + 8/9 b_{12} b_{23} a_{33}
+ 8/27 c_{12} d_{12} d_{31} + 8/27 d_{12} b_{23} b_{31} + 8/9 a_{12} c_{23} c_{31}
+ 8/9 b_{12} c_{23} a_{31} + 8/27 a_{12} d_{12} c_{23} + 8/27 d_{12} d_{12} a_{31}
+ 8/9 a_{12} a_{12} a_{33} + 8/9 c_{13} a_{22} c_{33} + 8/9 a_{13} b_{13} b_{31}
+ 8/9 c_{13} b_{22} a_{31} + 8/9 d_{13} a_{22} c_{31} + 8/9 d_{13} d_{22} a_{31} + 8/9 c_{21} a_{12} c_{33}
+ 8/9 d_{21} d_{12} a_{31} + 8/9 a_{21} c_{22} c_{33} + 8/9 b_{21} b_{22} a_{33}
+ 8/27 c_{21} c_{12} d_{31} + 8/27 a_{12} a_{22} d_{31} + 8/27 a_{12} d_{12} d_{31}
+ 8/27 b_{23} a_{12} b_{23} + 8/9 c_{12} a_{23} c_{23} + 8/9 d_{12} a_{23} a_{23}
+ 8/9 a_{12} b_{23} c_{23} + 8/27 b_{12} b_{23} a_{23} + 8/9 c_{12} b_{23} b_{23}
+ 8/27 d_{12} a_{23} b_{12} + 8/27 a_{23} d_{12} d_{23} + 8/27 b_{12} b_{23} b_{13}
\]

Figure 1. Parallelepiped element.
It is known that:
\[ \sum_{i=4}^{8} H_i = 1 \tag{8} \]

Thus, the coefficients \( H_i \) are the distribution weights. In other words they make the weighted average of the given distributions at the nodes.

Thus, the mentioned stochastic finite element estimator is a linear interpolator, regarding to the distributions given at its nodes \([13]\).

Taking into account that averaging process is one of the most frequently employed concept in computational techniques at finite element and geostatistics, below are presented two integral estimation procedures, which are key points on the estimation of the stiffness matrices in SFE and kriging, cokriging, covariance matrices in geostatistics \([11,16,17]\).

4. Volume Integrals within Polyedras \([18]\)

Let’s take a function \( u(x_1, x_2, x_3) \) in a coordinate system \( x_1, x_2, x_3 \). The integral of volume \( V \) will be estimated:
\[ \int_V u(x_1, x_2, x_3) \, dv \tag{9} \]

We will construct a vector \( \hat{\phi} \) \((x_1, x_2, x_3)\) that will satisfy:
\[ u = \operatorname{div} \hat{\phi} \tag{10} \]
where:
\[ \varphi = \varphi_{i} \hat{i} + \varphi_{j} \hat{j} + \varphi_{k} \hat{k} \tag{11} \]
\( i_1, i_2, i_3 \) is the system of the unit vector along the coordinate directions.

Let’s suppose that the boundary surface \( S \) of the volume \( V \) is composed of \( k \) plane polygonal faces \( S_i \) \((i = 1, 2, \ldots, k)\). Applying the divergence theorem we find:
\[ \int_V u(x_1, x_2, x_3) \, dv = \sum_{j=1}^{k} \int_{S_j} u(x_1, x_2, x_3) \, dx \, dS_j \tag{12} \]
where: the projected area \( dS_j \) is perpendicular to \( \hat{i} \) and lies in the \((x_2, x_3)\) plane.

The equation of the plane face \( dS_j \) can be expressed as:
\[ x_1 = x_1^{(j)}(x_2, x_3) = a_1^{(j)} + a_2^{(j)} x_2 + a_3^{(j)} x_3 \]
so the right-hand side of Equation (12) can be simplified to be:
\[ \int_F u(x_1, x_2, x_3) \, dV = \sum_{j=1}^{k} \int_{S_j} \phi \, u(x_1, x_2, x_3) \, dS_j \tag{13} \]
where: the surface \( S_j \) is a polygon in the \((x_2, x_3)\), in which the function \( \phi \) is to be integrated for \( F = 1, 2, \ldots, k \).

In this way, the computation of the volume integral is a procedure to integrate an arbitrary function within a polygon. Further repeating the above mentioned procedure we could find:
\[ \int_{\Omega} V(x_1, x_2) \, d\Omega = \sum_{j=1}^{k} \int_{S_j} \Psi(x_1, x_2) n_j \, dS = \sum_{j=1}^{k} \int_{S_j} \Psi(x_1, x_2) n_j \, dS \tag{14} \]

where, the perimeter \( T \) is a collection of the straight lines
\[ T_j, \ j = 1, \ldots, k, \]
while,
\[ x_i = x_i^{(j)}(x_2) \tag{15} \]
\[ n_j \, dT = dx_2 \tag{15_2} \]

Let the \( x_2 \) coordinates for \( a \) side \( x_2^{(a)} \) and \( x_2^{(e)} \). So:
\[ \int_{T_j} \Psi_1(x_1, x_2) n_j \, dT = \int_{x_2^{(a)}}^{x_2^{(e)}} \Psi_1(x_1, x_2) n_j \, dx_2 = \int_{x_2^{(a)}}^{x_2^{(e)}} y_j(x_2) \, dx_2 \tag{16} \]

Finally the above integral could be estimated by the Gaussian scheme quadrature. It is to be noted that volume integral is a deterministic procedure, but if the \( \omega = v \) and \( X(\omega) = u \), then it could be estimated as a stochastic finite element using Monte-Carlo method.

Parallelly if \( u(x), x \in R^n \) is a random function (RF) then the integral \( 1/|v| \int_{\Omega} u(x) \, dv \) could be treated in the geo-statistical view as a mean value.

5. Geostatistical Approach

5.1. Variograms

Geostatistics are based on the theory of the regionalized variables \([2]\) with assumption that data are observations of stochastic variables. The central tool of geostatistics is the variogram or semivariance function which is a structure describing the spatial dependence of the spatial variable \([11]\).

The following formula is the most frequently used for the variogram (semivariance) calculations:
\[ \gamma(h) = \frac{1}{2N} \sum_{i=1}^{N} \left[ Z(x_i) - Z(x_i + h) \right]^2 \tag{17} \]
where:
\( x_i \) is a data location, \( h \) is a log vector, \( z(x_i) \) is the data value at location \( x_i \), \( N \) is the number of data pairs spaced a distance and direction \( h \) units apart.

Semivariance calculations can also be performed with
data from RS images for example as a cross variogram. It is
defined as half of the average product of the log distance
to the two variables Z and Y.

\[ \gamma_{xy}(h) = \frac{1}{2N} \sum_{i=1}^{N} \left[ \left( Z(x_i) - Z(x_i + h) \right) \left( Y(x_i) - Y(x_i + h) \right) \right] \]

(17)

where:
- \( Z(x_i) \) and \( Y(x_i) \) are the data value in point \( x_i \) for two
  variables (profiles);
- \( N \) is the number of data separated by length of the
  vector \( h \);
- A variogram usually is characterized by three parameters
  \([2]\):
  - Sill—the plateau that the semivariogram reaches;
  - Range—the distance at which two data points are
    uncorrelated;
  - Nugget—the vertical discontinuity at the origin.

Usually the application of the semivariograms requires
that the data accomplish the intrinsic hypothesis for a
regionalized variable. In other words a random function
\( Z(x) \) is said to be intrinsic when:
the mathematical expectation exists and does not depend
on the support \( x \)

\[ E\{Z(x)\} = m \forall (x) \quad \text{(18)} \]

for all vectors \( h \) the increment \( Z(x+h) - Z(x) \) has a
finite variance which does not depend on \( x \)

\[ \text{Var}[Z(x+h) - Z(x)] = E\left[ (Z(x+h) - Z(x))^2 \right] \quad \forall (x) \quad \text{(19)} \]

where:
- \( Z(x) \) is a random function \( i.e. \) locally at a point \( x_i \), \( Z(x_i) \)
is a random variable and \( Z(x_i) \) and \( Z(x_i + h) \) are generally
independent but are related by a correlation expressing
the spatial structure of the initial regionalized variable
\( Z(x) \). Experimental variograms are approximated by dif-
f erent models like: spherical, exponential, Gaussian, circular,
tetraspherical, pentaspherical, Hole effect, \( K—Bessel \) etc. \([2,16,18]\).

5.2. Kriging in SFE View \([13]\]

Let be \( Z(x) \) the random function and the estimation of the
mean value:

\[ Z_v = \frac{1}{v} \int_{V} Z(x) \, dx \quad \text{(20)} \]

over a given domain \( V \) is required knowing a support of discrete
values \( Z_{\alpha} \), \( \alpha = 1, \ldots, n \).

According to the Kriging approach \([2]\) the linear esti-
mator \( Z_k^i \) of the \( n \) data values is considered:

\[ Z_k^i = \sum_{\alpha=1}^{n} \lambda_\alpha Z_\alpha \quad Z_\alpha = \frac{1}{v_\alpha} \int_{V_\alpha} Z(x) \, dx \quad \text{(21)} \]

The \( n \) weights \( \lambda_\alpha \) are calculated under the classic
hypothesis of the moments:

\[ E\{Z(x)\} = m \]

\[ E\{Z(x+h)Z(x)\} - m^2 = C(h) \quad \text{or} \]

\[ E\{Z(x+h) - Z(x)\}^2 = 2\gamma(h) \quad \text{(22)} \]

We must be assure that the estimator is unbiased as
well as the variance is minimal. Let us suppose that one
(or both) of two hypotheses are not accomplished and
both the expectation of \( Z(x) \) and the covariance depend
on \( x \):

\[ E\{Z(x)\} = m(x) \]

\[ C(x,h) = E\{Z(x+h)Z(x)\} - m(x+h)m(x) \quad \text{(23)} \]

Before taking into consideration this hypothesis, it
should be underlined, whatever the moment functions are
going to be, they should always lead to a positive vari-
ance. Also, we will show the calculation of Kriging solution
using SFE but without considering its existence and
uniqueness (It is not the aim of this paper). To ensure
that estimator is unbiased we impose the condition:

\[ \sum_{\alpha=1}^{n} \lambda_\alpha m_\alpha - m_v = 0 \quad \text{(24)} \]

With

\[ m_v = E\{Zv(x)\} = E\left[ \frac{1}{v} \int_{V} Z(x) \, dx \right] \]

\[ m_\alpha = E\{Z(v_\alpha)\} = E\left[ \frac{1}{v_\alpha} \int_{V_\alpha} Z(x) \, dx \right] \quad \alpha = 1, \ldots, n \quad \text{(25)} \]

The estimation variance is:

\[ E\left[ Z_v - Z_v^* \right]^2 = E\left[ Z_v^2 \right] - 2E\{Z,v\} + E\{v\} \quad \text{(26)} \]

Taking into account the expression of \( E\{Z_v^2\} \) we have:

\[ E\{Z_v^2\} = E\left[ \frac{1}{v^2} \int_{V} \int_{V} Z(x)Z(y) \, dy \, dx \right] \]

\[ = \sum_{i=1}^{n} \sum_{j=1}^{n} c_{ij} E\{Z_i(x_i)Z_j(y_j)\} \quad \text{(27)} \]

Also

\[ \sum_{i=1}^{n} \sum_{j=1}^{n} c_{ij} E\{Z_i(x_i)Z_j(y_j)\} \]

\[ = \sum_{i=1}^{n} \sum_{j=1}^{n} \left[ C(v_i,v_j) + m_{\alpha_i}m_{\alpha_j} \right] \quad \text{(28)} \]

where:
m_\alpha is the expectation of \( Z(x_i) \) at the node \( i \),
\[ \bar{C}(v, v) \] is the covariation depending not only by the distance \( h \), but also on \( x \).

Carrying out other means and substituting to the estimated variance we obtain:
\[ E \left[ Z - Z_\alpha \right]^2 = \bar{C}(v, v) - 2 \sum_{\alpha=1}^{n} \lambda_\alpha \bar{C}(v, v_\alpha) + \sum_{\alpha=1}^{n} \lambda_\alpha \sum_{\beta=1}^{n} \lambda_\beta \bar{C}(v_\beta, v_\alpha) \]

(29)

Now the problem is to find the weights \( \lambda_\alpha \), \( \alpha = 1, \ldots, k \) which minimize the estimation under non-bias conditions:
\[ \sum_{\alpha=1}^{n} \left( \lambda_\alpha - \frac{1}{n} \right) m_\alpha = 0 \]

(30)

For this reason, we use the Lagrange multiplier’s method, according to which we need to take the derivatives of:
\[ F = \bar{C}(v, v) - 2 \sum_{\alpha=1}^{n} \lambda_\alpha \bar{C}(v, v_\alpha) + \sum_{\alpha=1}^{n} \lambda_\alpha \sum_{\beta=1}^{n} \lambda_\beta \bar{C}(v_\beta, v_\alpha) + 2 \mu \sum_{\alpha=1}^{n} \left( \lambda_\alpha - \frac{1}{n} \right) m_\alpha \]

This procedure provides the Kriging system of \( n + 1 \) linear equation equations in \( \lambda_\alpha, \mu \):
\[ \sum_{\alpha=1}^{n} \lambda_\alpha \bar{C}(v_\beta, v_\alpha) - \mu m_\alpha = \bar{C}(v, v_\alpha) \]
\[ \sum_{\alpha=1}^{n} \lambda_\alpha m_\alpha = e, \quad e = \sum_{\alpha=1}^{n} \frac{1}{n} m_\alpha \]

which can be expressed in matrix form:
\[ [K] \{ \lambda_\alpha \} = \{ M \} \]
\[ [K] = \begin{bmatrix}
    c(v_1v_1) & c(v_1v_2) & \cdots & c(v_1v_n) \\
    c(v_2v_1) & c(v_2v_2) & \cdots & c(v_2v_n) \\
    \vdots & \vdots & \ddots & \vdots \\
    c(v_nv_1) & c(v_nv_2) & \cdots & c(v_nv_n)
\end{bmatrix} \]
\[ \{ \lambda_\alpha \} = \begin{bmatrix}
    \lambda_1 \\
    \lambda_2 \\
    \vdots \\
    \lambda_n
\end{bmatrix}, \quad \{ M \} = \begin{bmatrix}
    \bar{C}(v, v_1) \\
    \bar{C}(v, v_2) \\
    \vdots \\
    \bar{C}(v, v_n)
\end{bmatrix} \]

(32)

(33)

Let us suppose that solution of system (33) exists and it is unique. In this situation, it is quite clear that system (33) is general, in the sense of so-called Kriging system.

**Example 1**

In **Figure 2** it is shown a structure with 3 blocs: \( v_1 = 1 \times 1, v_2 = 1 \times 1, v_2 = 2 \times 2 \) in a contaminated (radioactive, oil, gas etc.) zone.

The equation of the variogram is \( \gamma(x) = 4h \) and the means of the parameter measured in the blocs \( v_1, v_2 \) are respectively:
\[ E(Z_1(x)) = 0.590 \quad E(Z_2(x)) = 0.409. \]

Let’s estimate the parameter \( Z(x) \) in the block \( v_3 \) resolving the Kriging system using finite element.

According to Kriging approach we have:
\[ Z = \lambda_1 Z_1 + \lambda_2 Z_2 \]

where \( \lambda_1, \lambda_2 \) parameters of the Kriging system:
\[ \begin{bmatrix}
    \gamma(v_1, v_1) & \gamma(v_1, v_2) & \cdots & \gamma(v_1, v_n) \\
    \gamma(v_2, v_1) & \gamma(v_2, v_2) & \cdots & \gamma(v_2, v_n) \\
    \vdots & \vdots & \ddots & \vdots \\
    \gamma(v_n, v_1) & \gamma(v_n, v_2) & \cdots & \gamma(v_n, v_n)
\end{bmatrix} \begin{bmatrix}
    \lambda_1 \\
    \lambda_2 \\
    \vdots \\
    \lambda_n
\end{bmatrix} = \begin{bmatrix}
    \gamma(x_1, x_1) \\
    \gamma(x_2, x_2) \\
    \gamma(x_n, x_n)
\end{bmatrix} \]

The solution is \( \lambda_1 = 0.5906, \lambda_2 = 0.409, \lambda_3 \approx 0. \)

Therefore,
\[ Z = \lambda_1 Z_1 + \lambda_2 Z_2 = 0.5906 \times 5 + 0.409 \times 7 \approx 5.81. \]

**Example 2**

In the **Figure 3**, it is presented a profile in a waste zone in which a parameter has been measured using a constant step \( h \).

The respective variogram shown in **Figure 4** has been approximated by a spheric model:

![Figure 2. Contaminated zone with three blocks.](image1)

![Figure 3. Profile of measured parameter.](image2)

![Figure 4. Variogram of diameters depending on distance between sample plots.](image3)
with \( c = 1 \) and \( a = 4h \) (the range).

The parabolic form of the variogram around the origin shows it is homogeneous [2].

6. SFE in Partial Differential Equations

The parameters of partial differential equations in many cases are not deterministic but stochastic ones. In this view let’s have a look on a PDE. First our starting point is the second order elliptic boundary value problem:

\[
y_{\omega}(h) = \begin{cases} \frac{3h}{2a} - \frac{1}{2a^2} & h \leq a \\ 0 & h \geq a \end{cases}
\]

\( \omega \) is the index variable fields, \( \omega \) is the output variables measure.

The diffusion coefficient \( T \) or the source term \( p \) respectively.

In many applications, only limited information about the random diffusion convection parameters, resulting from their synthetic and real distributions.

As a simple example let’s take a glance at the stochastic finite element on diffusion-convection equation [5, 6, 12, 19]:

\[
\begin{align*}
\frac{\partial}{\partial t} \left( k_x \frac{\partial \varphi}{\partial x} + k_y \frac{\partial \varphi}{\partial y} \right) + V_x \frac{\partial \varphi}{\partial x} + V_y \frac{\partial \varphi}{\partial y} + Q &= C \frac{\partial \varphi}{\partial \tau} \\
\end{align*}
\]

using the Crack-Nickolson algorithm with \( 0 < \theta \leq 1 \):

\[
\varphi^{n+1}_{i,j} = a_t \varphi^n_{i,j} + a_{11} \varphi^n_{i+1,j} + a_{12} \varphi^n_{i+1,j+1} + a_{21} \varphi^n_{i+1,j+1} + a_{22} \varphi^n_{i+1,j+1} + b Q^n_{i,j}
\]

where:

\( C \) —the solute concentration, \( x, y \)—spatial co-ordinate, \( t \)—time coordinate, \( V \) —the flow velocity vector with its components \( V_x, V_y \), \( D \)—the diffusion coefficient, \( a_t, i = 1, 5 \) and \( b \) are the coefficients depending on the mentioned coefficients, \( \Delta x, \Delta y \) spatial steps, \( \Delta \tau \) time step. Below we are presenting a river plane zone contaminated by a point pollutant source Figure 5, placed in the left side of the node 13.

In this scheme, it was operated with mean values of the random diffusion convection parameters, resulting from their synthetic and real distributions.

The components \( V_x \) and \( V_y \) has been measured in an interval of time. The component of \( V_x \) is positive over the line 13 - 18 and negative under this one.

To illustrate the idea, it is shown below a partial solution of the contaminant concentration in the step \( st = 5 \) for a simple non stationary flow problem (Dirichle—Newman conditions). Using \( q = 1, K_x = 1, K_y = 1, V_x = 1, \text{ and } V_y = 1 \).

![Figure 5. A contaminated river zone.](image-url)
As it was expected the solution is symmetric.

There are simple resemblances between different concepts and operators in geostatistics and SFE as for example: blocs, interpolation operator, minimization of the variance (energy).

7. Conclusion

SFE and Geostatistic applications are of the great importance in environmental resources, nuclear and renewable energy, ecology, forestry, geology, climate, water and air pollution, mapping as well as on their uncertainty, risk analysis and optimization [1,5,14,15,20,21].

REFERENCES


