Boundaries of Smooth Strictly Convex Sets in the Euclidean Plane $\mathbb{R}^2$

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Abstract
We give a characterization of the boundaries of smooth strictly convex sets in the Euclidean plane $\mathbb{R}^2$ based on the existence and uniqueness of inscribed triangles.

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Strict Convexity, Smoothness, Supporting Lines, Inscribed Triangles

1. Introduction
The reader unfamiliar with the theory of convex sets is referred to the books [1] [2] [3] [4] [5]. Let $M$ be a set in the $n$-dimensional Euclidean space $\mathbb{R}^n$. In the following we shall denote by $\text{int} M$, $cl M$, $\partial M$, $\text{conv} M$ the interior, the closure, the boundary and respectively the convex hull of the set $M$. With $d(x,y)$ we denote the Euclidean distance of the points $x$ and $y$ and with $L(x,y)$ the line determined by the points $x$ and $y$. The diameter $\text{diam} M$ of a set $M$ is $\text{diam} M = \sup \{d(x,y) : x, y \in M\}$. For a point $p \in \mathbb{R}^2$ and a real number $r$ we shall denote with $C(p,r)$ and $D(p,r)$ the circle and respectively the disc with center $p$ and radius $r$. The distance $d(p,M)$ between a point $p$ and a set $M$ in $\mathbb{R}^2$ is $d(p,M) = \inf \{d(p,x) : x \in M\}$. With $]x,y[\}$ we denote the open line segment with endpoints $x$ and $y$, that is $]x,y[\} = \text{conv} \{x,y\} \setminus \{x,y\}$. For 3 nonlinear points $x,y$ and $z$ in $\mathbb{R}^2$ we denote with $\max \angle(x,y,z)$ the maximum angle of the triangle $\Delta(x,y,z)$. A convex curve is a connected subset of the boundary of a convex set.

2. Preliminaries
In the chapter 8 of the book [4] of F.A. Valentine the author says the following: “It is interesting to see what kind of strong conclusions can be obtained from
weak suppositions about any triplet of points of a plane set $S$. In [6] Menger gives such a characterization of the boundary of a convex plane set $S$ based on intersection properties of $S$ with the seven convex sets in which the space $\mathbb{R}^2$ is subdivided by the lines $L(x_1, x_2), L(x_2, x_3)$ and $L(x_3, x_1)$ determined by an arbitrary triplet of noncollinear points $\{x_1, x_2, x_3\}$ from $S$. In [7] K. Juul proved the following:

**Theorem 1.** A plane set $S$ fulfills

1) $\forall x, y, z \in S : S \cap \text{int} \{\text{conv} \{x, y, z\}\} = \emptyset$, if and only if $S$ is either a subset of the boundary of a convex set, or an $X$-set, that is a set $\{x_1, x_2, x_3, x_4, x_5\}$ with $\{x_1, x_2, x_3\} = \{x_5\}$.

A survey of different characterizations of convex sets is given in the paper [8]. The results of K. Menger and that of K. Juul give characterizations of the boundaries of convex sets.

In the years 1978 [9] and 1979 [10] we have proved the following two theorems giving a characterization of the boundaries of smooth strictly convex sets:

**Theorem 2.** A plane compact set $S$ is the boundary of a smooth strictly convex set if and only if the following two conditions hold:

1) $\forall x, y, z \in S : S \cap \text{int} \{\text{conv} \{x, y, z\}\} = \emptyset$.

2) For every triangle $\Delta(p_1, p_2, p_3)$ in $\mathbb{R}^2$ there is one and only one triangle $\Delta(p_1', p_2', p_3')$ homothetic to the triangle $\Delta(p_1, p_2, p_3)$ inscribed in the set $S$, i.e. such that $p_1', p_2', p_3' \in S$.

**Theorem 3.** A plane compact set $S$ is the boundary of a smooth strictly convex set if and only if the following two conditions hold:

1) For every $p \in S$ and every $\epsilon > 0$ there is a positive number $\delta(p, \epsilon)$ such that for every triplet of nonlinear points $r, s, t$ in $S \cap \text{int} \{D(p, \delta(p, \epsilon))\}$ we have $\max \angle(r, s, t) > \pi - \epsilon$.

2) For every triangle $\Delta(p_1, p_2, p_3)$ in $\mathbb{R}^2$ there is one and only one triangle $\Delta(p_1', p_2', p_3')$, homothetic to the triangle $\Delta(p_1, p_2, p_3)$ inscribed in the set $S$, i.e. such that $p_1', p_2', p_3' \in S$.

3. Main Results

The main result of this paper is Theorem 4 giving another characterization of the boundaries of smooth strictly convex sets in the Euclidean plane $\mathbb{R}^2$ which uses also condition (2) of the Theorem 2 and Theorem 3.

**Theorem 4.** A compact set $S$ in the Euclidean plane $\mathbb{R}^2$ is the boundary of a smooth strictly convex set if and only if there are verified the following three conditions:

1) For every triangle $\Delta(p_1, p_2, p_3)$ in $\mathbb{R}^2$ there is one and only one triangle $\Delta(p_1', p_2', p_3')$ homothetic to the triangle $\Delta(p_1, p_2, p_3)$ inscribed in the set $S$, i.e. such that $p_1', p_2', p_3' \in S$.

2) For any two distinct points $p \in S$ and $q \in S$ there are at least two points $t_1$ and $t_2$ such that $t_1 \in S \cap H_1$ and $t_2 \in S \cap H_2$, where $H_1$ and $H_2$ are the two open halfplanes generated in $\mathbb{R}^2$ by the line $L(p, q)$.

3) The set $S$ does not contain three collinear points.
For the proof of Theorem 4 we need the following theorem from the paper [11] and three lemmas:

**Theorem 5.** Let \( \Delta(a,b,c) \) be a triangle in the Euclidean plane \( \mathbb{R}^2 \). Suppose that \( S \) is a strictly convex closed arc of class \( C^1 \). Then there exists a single triangle \( \Delta(a,b,c) \) homothetic to the triangle \( \Delta(a,b,c) \) inscribed in the set \( S \), in the sense that \( a_i, b_i, c_i \in S \).

**Lemma 1.** The convex hull \( \text{conv} S \) of a compact set \( S \) in the Euclidean plane \( \mathbb{R}^2 \) verifying the condition (2) from Theorem 4 is a strictly convex set.

**Proof.** Let us suppose the contrary. Then there are two distinct points \( p, q \in \partial \{ \text{conv} S \} \) such that the line segment \( \text{conv} \{ p, q \} \subset \text{conv} S \). The convex hull of a compact set is also a compact set (see [5] Theorem 2.2.6). The line \( L(p,q) \) is thereby a supporting line for the compact set \( \text{conv} S \). Denote with \( H_1 \) and \( H_2 \) the two open halfplanes generated by the line \( L(p,q) \) such that \( \text{conv} S \subset \text{cl} \{ H_1 \} \) and \( H_2 \cap \text{conv} S = \emptyset \). By Carathodory’s Theorem (see [5] or [12] Theorem 2.2.4) the point \( p \in \text{conv} S \) can be expressed as a convex combination of 3 or fewer points of \( S \).

If the point \( p \) can be expressed only as a convex combination of three (and not of fewer) points \( x_1, x_2, x_3 \) of \( S \) then it follows that we must have \( p \in \text{int} \{ \text{conv} \{ x_1, x_2, x_3 \} \} \subset \text{int} \{ \text{conv} S \} \) in contradiction to the fact that \( p \in \partial \{ \text{conv} S \} \).

If the point \( p \) can be expressed only as a convex combination of 2 (and not of fewer) points of \( S \), there are \( x_1 \in S \) and \( x_2 \in S \) such that \( p \in \text{conv} \{ x_1, x_2 \} \subset \text{conv} S \subset \text{cl} \{ H_1 \} \). Then the points \( x_1 \) and \( x_2 \) must be on the supporting line \( L(p,q) \). As \( H_2 \cap \text{conv} S = \emptyset \) this is in contradiction with property (2) of the set \( S \).

Thereby we must have \( p \in S \). By an analog reasoning for the point \( q \) we can conclude that we have also: \( q \in S \). Thus we have proved the existence of at least 2 different points of \( S \) on the supporting line \( L(p,q) \) of \( \text{conv} S \) in contradiction to the property (2) of the set \( S \).

**Lemma 2.** The boundary \( \partial \{ \text{conv} S \} \) of the convex hull of a compact set \( S \) in the Euclidean plane \( \mathbb{R}^2 \) verifying the condition (2) from Theorem 4 is a subset of the set \( S \), i.e. \( \partial \{ \text{conv} S \} \subset S \).

**Proof.** Let \( p \in \partial \{ \text{conv} S \} \) be an arbitrary point from the boundary of the convex hull of the compact set \( S \). Each boundary point of the compact convex set \( \text{conv} S \) in \( \mathbb{R}^2 \) is situated on at least one supporting line of the set \( \text{conv} S \) (see for instance [3] pp. 6). We distinguish now the following two cases:

1) There is only one supporting line \( L_1 \) of the set \( \text{conv} S \) going through the point \( p \), i.e. the boundary \( \partial \{ \text{conv} S \} \) is smooth in the point \( p \). By Lemma 1 it follows that the convex hull \( \text{conv} S \) is a strictly convex set and thereby we have \( \text{conv} S \cap L_1 = p \).

Let us now suppose the point \( p \notin S \). From \( \text{conv} S \cap L_1 = p \) and \( p \notin S \) follows then \( S \cap L_1 = \emptyset \). Denote with \( H_o \) the open halfplane generated by the line \( L_1 \), which contains the set \( S \). As \( S \) is a compact set we have then \( r = \min \{ d(x,L_1) : x \in S \} > 0 \). Consider then in the open halfplane \( H_o \) a line
$L'$ parallel to the line $L$ at distance $r$ to the line $L$. Denote with $H'$ the open halfplane generated by the line $L'$ and such that $H' \subset H$. It is evident that $p \notin cl \{H'\}$. From the definition of the constant, $r$ follows $S \subset cl \{H'\}$ and $\partial \{ \text{conv}S \} \subset cl \{H'\}$ in contradiction to $p \in \partial \{ \text{conv}S \}$. Thereby our supposition $p \notin S$ is false, i.e. we must have $p \in S$.

2) There are two supporting lines $L_1$ and $L_2$ of the set $\text{conv}S$ going through the point $p$. Denote then with $L'_1$ and $L'_2$ the two halflines with endpoint $p$ of the line $L_1$ and respectively $L_2$ such that $\text{conv}S \subset \text{conv} \{L'_1 \cup L'_2\}$.

Let us suppose that $p \notin S$. From the compactness of $S$ follows then the existence of a real number $r > 0$ such that for the disc $D(p,r)$ with the center $p$ and the radius $r$ we have: $D(p,r) \cap S = \emptyset$. Consider then the points $q_1 = C(p,r) \cap L_1$ and $q_2 = C(p,r) \cap L_2$, where $C(p,r)$ is the circle with center $p$ and radius $r$. Let $H_1$ be the open halfplane generated by the line $L(q_1,q_2)$, which contains the point $p$ and $H_2$ the other open halfplane generated by the line $L(q_1,q_2)$. We have then evidently $S \cap cl H_1 = \emptyset$ and thereby $S \subset H_2$. From the inclusion $S \subset H_2$ it follows also that $\text{conv}S \subset H_2$. As $\partial \{ \text{conv}S \} \subset S$ we have also: $\partial \{ \text{conv}S \} \subset H_2$ in contradiction to our supposition $p \notin \partial \{ \text{conv}S \}$. Therefore the point $p$ must belong to the set $S$.

So we have proved in both cases (1) and (2) that $p \in \partial \{ \text{conv}S \}$ implies $p \in S$, i.e. $\partial \{ \text{conv}S \} \subset S$.

A characterization of compact sets $S$ in the Euclidean plane $R^2$ for which we have $S = \partial \{ \text{conv}S \}$ is given in the following:

**Lemma 3.** A compact set $S$ in the Euclidean plane $R^2$ has a strictly convex hull and coincides with the boundary of its convex hull $\partial \{ \text{conv}S \}$ if and only if there are verified the conditions (2) and (3).

**Proof.** Let $S$ be a compact set in the Euclidean plane $R^2$, which has a strictly convex hull $\text{conv}S$ and such that $S = \partial \{ \text{conv}S \}$. Consider then two arbitrary points $p_1$ and $p_2$ of the set $S$ and the two open halfplanes generated by the line $L(p_1,p_2)$ in $R^2$. Because $S$ has a strictly convex hull it is then evident that we have verified condition (2) and (3).

To prove the only if part of the lemma let us consider a compact set $S$ in the Euclidean plane $R^2$, which verifies conditions (2) and (3). By Lemma 1 the convex hull $\text{conv}S$ of $S$ is a strictly convex set. By Lemma 2 we have then for the set $S$ the inclusion $\partial \{ \text{conv}S \} \subset S$. Let us now suppose that we would have $S \subset \partial \{ \text{conv}S \}$, i.e. there is a point $p \in S$ such that $p \notin \partial \{ \text{conv}S \}$. Then the point $p$ must be an interior point of the convex hull $\text{conv}S$. Let $L$ be an arbitrary line such that $p \in L$. Then it is obvious that the line $L$ intersects $\partial \{ \text{conv}S \}$ in two different points $t_1$ and $t_2$ such that $p \in \text{conv} \{t_1,t_2\}$. From $\partial \{ \text{conv}S \} \subset S$ it follows that $t_1 \in S$ and $t_2 \in S$ in contradiction to the condition (3) of the set $S$. So we conclude that $S \subset \partial \{ \text{conv}S \}$. This inclusion together with the inclusion $\partial \{ \text{conv}S \} \subset S$ gives then $S \subset \partial \{ \text{conv}S \}$.

A similar result as that of Lemma 3 without the compactness of the set $S$ but with the additional assumption of the connectedness of the set $S$ was obtained.
Theorem 6. A connected set $S$ in $\mathbb{R}^2$ is a convex curve if and only if it verifies condition (1) from Theorem 1.

Proof of Theorem 4.

For the proof of the if-part of the theorem let $S$ be the boundary of a compact smooth strictly convex set in the Euclidean plane $\mathbb{R}^2$. It is then easy to verify conditions (2) and (3) for the set $S$. Condition (1) follows immediately from Theorem 5.

For the proof of the “only if”—part of the theorem let $S$ be a compact set in the Euclidean plane $\mathbb{R}^2$, which verifies conditions (1), (2) and (3). By Lemma 3 it follows that the convex hull $\text{conv}S$ of the set $S$ is strictly convex and that $S = \partial \{\text{conv}S\}$.

It remains to prove that $\text{conv}S$ is also a smooth set. Let us assume the contrary: there is a point $a_i \in \partial \{\text{conv}S\}$, which is not a smooth point of the boundary of $S$, i.e. there exist two supporting lines $L_1$ and $L_2$ for the set $\text{conv}S$ at the point $a_1$. For $i \in \{1, 2\}$ denote with $H_i$ the closed half-plane generated by the supporting line $L_i$, which contains the set $S$. Denote with $C$ the convex cone $C = H_1 \cap H_2$. We have then evidently the inclusions: $S \subset C$ and $\text{conv}S \subset C$. As $\text{conv}S$ is a strictly convex set we also have the inclusion $S \setminus a_1 \subset \text{int} C$. For $i \in \{1, 2\}$ denote with $L_i'$ the closed halfline of the line $L_i$ with origin $a_1$ such that $L_i' \subset \partial C$. Consider then the isosceles triangle $\Delta(a_1, a_2, a_3)$ with $d(a_1, a_2) = d(a_1, a_3)$ and such that angle $\angle a_2 a_1 a_3$ has the same angle bisector as the boundary angle of the cone $C$ formed by the halflines $L_1'$ and $L_2'$ with the vertex $a_1$ and such that the angle $\angle a_2 a_1 a_3$ is greater than the boundary angle of the cone $C$. By condition (1) there exists then three points $a_i' \in S, i = 1, 2, 3$ such that triangle $\Delta(a_1', a_2', a_3')$ is homothetic to the triangle $\Delta(a_1, a_2, a_3)$. Because the angle $\angle a_2 a_1 a_3$ is greater than the boundary angle of the cone $C$ the point $a_i'$ cannot coincide with the point $a_i$. From this fact and the inclusion $S \setminus a_1 \subset \text{int} C$ we can conclude that we have: $a_i' \in \text{int} C$ for $i = 1, 2, 3$. From the homothety of the triangles $\Delta(a_1', a_2', a_3')$ and $\Delta(a_1, a_2, a_3)$ it follows then that $a_i' \in \text{int} \{\text{conv} \{a_1, a_2', a_3'\}\} \subset \text{int} \{\text{conv}S\}$ in contradiction to $a_i' \in S = \partial \{\text{conv}S\}$.

So we have proved that the convex hull $\text{conv}S$ is a smooth strictly convex set.

4. Conclusions

As we have seen condition (1) is used and is essential in the proofs of the Theorem 2, Theorem 3 and Theorem 4. We emit now the following:

Conjecture: A compact set $S$ in the Euclidean plane $\mathbb{R}^2$ is the boundary of a smooth strictly convex set if and only if there is verified the condition:

For every triangle $\Delta(p_1, p_2, p_3)$ in $\mathbb{R}^2$ there is one and only one triangle $\Delta(p_1', p_2', p_3')$ homothetic to the triangle $\Delta(p_1, p_2, p_3)$ and inscribed in the set $S$, i.e. such that $p_i', p_i' \in S$.

P. Mani-Levitska cites in his survey [8] the papers [7] and [9] and says referring to these, that he has not encountered extensions of these results to higher
dimensions. We also don’t know generalizations of our results to higher dimensions.

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