Open Journal of Discrete Mathematics, 2012, 2, 145-148
http://dx.doi.org/10.4236/ojdm.2012.24029 Published Online October 2012 (http://www.SciRP.org/journal/ojdm)

Mean Cordial Labeling of Graphs

Raja Ponraj[1], Muthirulan Sivakumar[2], Murugesan Sundaram[1*]
[1]Department of Mathematics, Sri Paramakalyani College, Alwarkurchi, India
[2]Department of Mathematics, Unnamalai Institute of Technology, Kovilpatti, India
Email: ponrajmaths@indiatimes.com, sivamaths.vani_r@yahoo.com, sundaram_spkc@rediffmail.com

Received August 12, 2012; revised September 4, 2012; accepted September 17, 2012

ABSTRACT

Let f be a map from $V(G)$ to $\{0,1,2\}$. For each edge uv assign the label $\left\lceil \frac{f(u)+f(v)}{2} \right\rceil$. f is called a mean cordial labeling if $|v_f(i)-v_f(j)| \le 1$ and $|e_f(i)-e_f(j)| \le 1$, $i, j \in \{0,1,2\}$, where $v_f(x)$ and $e_f(x)$ denote the number of vertices and edges respectively labelled with x ($x = 0,1,2$). A graph with a mean cordial labeling is called a mean cordial graph. We investigate mean cordial labeling behavior of Paths, Cycles, Stars, Complete graphs, Combs and some more standard graphs.

Keywords: Path; Star; Complete Graph; Comb

1. Introduction

The graphs considered here are finite, undirected and simple. The vertex set and edge set of a graph G are denoted by $V(G)$ and $E(G)$ respectively. The cardinality of $V(G)$ and $E(G)$ are respectively called order and size of G Labelled graphs are used in radar, circuit design, communication network, astronomy, cryptography etc. [1]. The concept of cordial labeling was introduced by Cahit in the year 1987 in [2]. Let f: $V(G)$ to $\{0,1\}$ be a function. For each edge uv assign the label $|f(u)-f(v)|$. f is called a cordial labeling if $|v_f(i)-v_f(j)| \le 1$ and $|e_f(i)-e_f(j)| \le 1$, $i, j \in \{0,1\}$ where $v_f(x)$ and $e_f(x)$ denote the number of vertices and edges respectively labelled with x (x = 0, 1). A graph with admits a cordial labeling is called a cordial graph. Product cordial labeling was introduced by M. Sundaram, R. Ponraj and Somasundaram [3]. Here we introduce a new notion called mean cordial labeling. We investigate the mean cordial labeling behavior of some standard graphs. The symbol $\lceil x \rceil$ stands for smallest integer greater than or equal to x. Terms not defined here are used in the sense of Harary [4].

2. Mean Cordial Labeling

Definition 2.1. Let f be a function from $V(G)$ to $\{0,1,2\}$.

For each edge uv *of* G, assign the label $\left\lceil \frac{f(u)+f(v)}{2} \right\rceil$. f is called a mean cordial labeling of G *if* $|v_f(i)-v_f(j)| \le 1$ and $|e_f(i)-e_f(j)| \le 1$, $i, j \in \{0,1,2\}$ where $v_f(x)$ and $e_f(x)$ denote the number of vertices and edges labelled with x ($x = 0,1,2$) respectively. A graph with a mean cordial labeling is called mean cordial graph.

Remarks 2.2: If we restrict the range set of f to $\{0,1\}$,the definition 2.1 coincides with that of product cordial labeling.

Remarks 2.3: If we try to extend the range set of f to $\{0,1,\cdots,k\}(k>2)$, the definition 2.1 shall not workout since $v_f(0)$ becomes very small.

Theorem 2.4: Every graph is a sub graph of a connected mean cordial graph.

Proof: Let G be a given (p,q) graph. Take three copies of K_p. Let G_1, G_2, G_3 respectively denote the first, second and third copies of K_p. Let $u \in V(G_1), v \in V(G_2), w \in V(G_3)$. Let G^* be the graph with

$V(G^*) = V(G_1) \cup V(G_2) \cup V(G_3)$

and $E(G^*) = E(G_1) \cup E(G_2) \cup E(G_3) \cup \{uv, vw\}$.

Clearly G^* is a super graph of G. Assign the label 0 to all the vertices of G_1,1 to all the vertices of G_2, 2 to all the vertices of G_3. Then $v_f(0) = v_f(1) = v_f(2) = p$ and $e_f(0) = \binom{p}{2}$, $e_f(1) = \binom{p}{2}+1$, $e_f(2) = \binom{p}{2}+1$.

Therefore this labeling is a mean cordial labeling of G^*.

Theorem 2.5: Any Path P_n is a mean cordial.

[*]Associate professor (retired).

Proof: Let P_n be the Path $u_1u_2\cdots u_n$.

Case (1): $n \equiv 0 \pmod 3$

Let $n = 3t$. Define $f(u_i) = 2, 1 \le i \le t$, $f(u_{t+i}) = 1, 1 \le i \le t$, $f(u_{2t+i}) = 0, 1 \le i \le t$. Then $v_f(0) = v_f(1) = v_f(2) = t$ and $e_f(0) = t-1, e_f(1) = e_f(2) = t$. Hence *f* is a mean cordial labeling.

Case (2): $n \equiv 1 \pmod 3$

Let $n = 3t+1$. Assign labels to the vertices $u_i (1 \le i \le n-1)$ as in case (1). Then assign the label 0 to the vertex u_n. Here. $v_f(0) = v_f(1) = v_f(2) = t$ and $e_f(0) = t$, $e_f(1) = e_f(2) = t$. Hence *f* is a mean cordial labeling.

Case (3): $n \equiv 2 \pmod 3$

Let $n = 3t+2$. Assign labels to the vertices $u_i (1 \le i \le n-1)$ as in case (2). Then assign the label 1 to the vertex u_n. Here $v_f(0) = v_f(1) = t+1, v_f(2) = t$, $e_f(1) = t+1$. and $e_f(0) = e_f(2) = t$. Hence *f* is a mean cordial labeling.

Illustration 2.6: Mean cordial labeling of P_6 is shown in **Figure 1**.

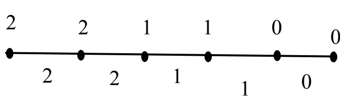

Figure 1. Mean cordial labeling of *P*$_6$.

Theorem 2.7: The Star $K_{1,n}$ is a mean cordial iff $n \le 2$.

Proof: Let $V(K_{1,n}) = \{u, u_i : 1 \le i \le n\}$ and $E(K_{1,n}) = \{uu_i : 1 \le i \le n\}$.

For $n \le 2$, the result follows from Theorem 2.5. Assume $n > 2$. If possible let there be a mean cordial labeling *f*.

Case (1): $f(u) = 0$.

Then $f(u) + f(v) \le 2$ for all edge *uv*. This forces $e_f(2) = 0$. a contradiction.

Case (2): $f(u) = 2$.

In this case $e_f(0) = 0$, again a contradiction.

Case (3): $f(u) = 1$.

Here also $e_f(0) = 0$, a contradiction.

Hence $K_{1,n}$ is not a mean cordial for all $n > 2$.

Theorem 2.8: The cycle C_n is mean cordial iff $n \equiv 1, 2 \pmod 3$.

Proof: Let C_n be the cycle $u_1u_2\cdots u_nu_1$

Case (1): $n \equiv 0 \pmod 3$

Let $n = 3t$. Then $v_f(0) = v_f(1) = v_f(2) = t$. In this case $e_f(0) \le t-1$, a contradiction

Case (2): $n \equiv 1 \pmod 3$

Let $n = 3t+1$. Define $f(u_i) = 0, 1 \le i \le t+1$ $f(u_{t+1+i}) = 1, 1 \le i \le t$, $f(u_{2t+1+i}) = 2, 1 \le i \le t$

Then $v_f(0) = t+1$, $v_f(1) = v_f(2) = t$ and $e_f(1) = t+1$, $e_f(0) = e_f(2) = t$. Hence *f* is a mean cordial labeling.

Case (3): $n \equiv 2 \pmod 3$

Let $n = 3t+2$. Define $f(u_i) = 0, 1 \le i \le t+1$, $f(u_{t+1+i}) = 1, 1 \le i \le t$, $f(u_{2t+1+i}) = 2, 1 \le i \le t+1$

Then $v_f(1) = t$, $v_f(0) = v_f(2) = t+1$ and $e_f(1) = t+1$, $e_f(0) = e_f(2) = t$. Hence *f* is a mean cordial labeling.

Theorem 2.9: The Complete graph K_n is mean cordial iff $n \le 2$.

Proof: Clearly K_1 and K_2 are mean cordial by Theorem 2.5. Assume $n > 2$. If possible let there be a mean cordial labeling *f*.

Case (1): $n \equiv 0 \pmod 3$

Let $n = 3t, t \ge 1$. Then $v_f(0) = v_f(1) = v_f(2) = t$

$e_f(0) = \binom{t}{2}$, $e_f(1) = \binom{t}{2} + t^2 + t^2$, and

$e_f(2) = \binom{t}{2} + t^2$. Then $e_f(1) - e_f(0) = 2t^2 > 1$, a contradiction.

Case (2): $n \equiv 1 \pmod 3$

Let $n = 3t+1$

Subcase (i): $v_f(0) = t+1$ $v_f(1) = v_f(2) = t$

$e_f(0) = \binom{t+1}{2}$, $e_f(1) = \binom{t}{2} + t(t+1) + t(t+1)$ and

$e_f(2) = \binom{t}{2} + t^2$, $e_f(1) - e_f(2) = t^2 + 2t > 1$, a contradiction.

Subcase (ii): $v_f(1) = t+1$ $v_f(0) = v_f(2) = t$

$e_f(0) = \binom{t}{2}$, $e_f(1) = \binom{t+1}{2} + t(t+1) + t^2$ and

$e_f(2) = \binom{t}{2} + t(t+1)$. $e_f(2) - e_f(0) = t^2 + t > 1$, a contradiction.

Subcase (iii): $v_f(2) = t+1$ $v_f(0) = v_f(1) = t$

$e_f(0) = \binom{t}{2}$, $e_f(1) = \binom{t}{2} + t(t+1) + t^2$ and

$e_f(2) = \binom{t+1}{2} + t(t+1)$. Then

$e_f(1) - e_f(0) = 2t^2 + t > 1$, a contradiction.

Case (3): $n \equiv 2 \pmod 3$

Let $n = 3t+2$

Subcase (i): $v_f(0) = t$ $v_f(1) = v_f(2) = t+1$

$e_f(0) = \binom{t}{2}$, $e_f(1) = \binom{t+1}{2} + t(t+1) + t(t+1)$ and

$e_f(2) = \binom{t+1}{2} + (t+1)^2$. Then

$e_f(1) - e_f(0) = 2t^2 + 3t > 1$, a contradiction

Subcase (ii): $v_f(1) = t \quad v_f(0) = v_f(2) = t+1$

$e_f(0) = \binom{t+1}{2}$, $e_f(1) = \binom{t}{2} + t(t+1) + (t+1)^2$ and

$e_f(2) = \binom{t+1}{2} + t(t+1)$. Then

$e_f(2) - e_f(0) = t^2 + t > 1$, a contradiction.

Subcase (iii): $v_f(2) = t \quad v_f(0) = v_f(1) = t+1$

$e_f(0) = \binom{t+1}{2} \quad e_f(1) = \binom{t+1}{2} + (t+1)^2 + t(t+1)$ and

$e_f(2) = \binom{t}{2} + t(t+1)$

Then $e_f(1) - e_f(0) = (t+1)^2 + t^2 + t > 1$, a contradiction.

Theorem 2.10: The Wheel W_n is not a mean cordial graph for all $n \geq 3$.

Proof: Let $W_n = C_n + K_1$ where C_n is the cycle $u_1u_2 \cdots u_nu_1$ and $V\{K_1\} = u$. If possible let there be a mean cordial labeling *f*.

Case (1): $n \equiv 0 \pmod 3$

Let $n = 3t$ Then the size of the wheel is $6t$.

Subcase (i): $f(u) = 0$,

Here $e_f(0) \leq t + t - 1 \leq 2t - 1$, a contradiction.

Subcase (ii): $f(u) = 1$ or 2.

In this case $e_f(0) \leq t$, again a contradiction.

Case (2): $n \equiv 1 \pmod 3$

Let $n = 3t+1$. Then $e_f(0) \leq 2t-1$ or $e_f(0) \leq t$ according as $f(u) = 0$, or $f(u) \neq 0$, this is a contradiction.

Case (3): $n \equiv 2 \pmod 3$

Similarly to case (1), we get a contradiction.

Theorem 2.11: $S(K_{1,n})$ is mean cordial, where *S*(*G*) denotes subdivision of *G*.

Proof:

Let $V(S(K_{1,n})) = \{u, u_i, v_i : 1 \leq i \leq n\}$ and

$E(S(K_{1,n})) = \{uu_i, u_iv_i : 1 \leq i \leq n\}$

Case (i): $n \equiv 0 \pmod 3$

Let $n = 3t$.

Define $f(u) = 0$, $f(u_i) = 0, 1 \leq i \leq t$,

$f(u_{t+i}) = 1, 1 \leq i \leq 2t$, $f(v_i) = 0, 1 \leq i \leq t$,

$f(v_{t+i}) = 2, 1 \leq i \leq 2t$, Then $v_f(0) = 2t+1$,

$v_f(1) = v_f(2) = 2t$ and $e_f(0) = e_f(1) = 2t$, $e_f(2) = 2t$.

Hence *f* is a mean cordial labeling.

Case (2): $n \equiv 1 \pmod 3$

Let $n = 3t+1$ Assign labels to the vertices u, u_i and $v_i \, (1 \leq i \leq n-1)$ as in case (1). Then assign the label 1 and 2 to the vertices u_n and v_n respectively. Here $v_f(0) = v_f(1) = v_f(2) = 2t+1$ and $e_f(0) = 2t$, $e_f(1) = e_f(2) = 2t+1$, Hence *f* is a mean cordial labeling.

Case (3): $n \equiv 2 \pmod 3$

Let $n = 3t+2$. Assign labels to the vertices u, u_i and v_i $(1 \leq i \leq n-1)$ as in case (2). Then assign the label 0 and 2 to the vertices u_n, v_n respectively. Here $v_f(0) = v_f(2) = 2t+2$, $v_f(1) = 2t+1$ and $e_f(1) = 2t+2$, $e_f(0) = e_f(2) = 2t+1$, Hence f is a mean cordial labeling.

Illustration 2.12: Mean cordial labeling of $S(K_{1,6})$ is shown in **Figure 2**.

Theorem 2.13: The comb $P_n \Theta K_1$ is mean cordial where $G_1 \Theta G_2$ denotes the corona of G_1 and G_2.

Proof: Let P_n be the Path $u_1u_2 \cdots u_n$. Let

$V(P_n \Theta K_1) = V(P_n) \cup \{v_i : 1 \leq i \leq n\}$,

$E(P_n \Theta K_1) = E(P_n) \cup \{u_iv_i : 1 \leq i \leq n\}$

Case (1): $n \equiv 0 \pmod 3$

Let $n = 3t$.

$f(u_i) = 2, 1 \leq i \leq t$, $f(u_{t+i}) = 1, 1 \leq i \leq t$

$f(u_{2t+i}) = 0, 1 \leq i \leq t$, $f(v_i) = 2, 1 \leq i \leq t$

$f(v_{t+i}) = 1, 1 \leq i \leq t$, $f(v_{2t+i}) = 0, 1 \leq i \leq t$

Then $v_f(0) = v_f(1) = v_f(2) = 2t$ and

$e_f(1) = e_f(2) = 2t, e_f(0) = 2t-1$,

Hence *f* is a mean cordial labeling.

Case (2): $n \equiv 1 \pmod 3$

Let $n = 3t+1$. Assign labels to the vertices u_i and v_i $(1 \leq i \leq n-1)$ as in case (1). Then assign the label 0 and 1 to the vertices u_n and v_n. Here

$v_f(0) = v_f(1) = 2t+1, v_f(2) = 2t$, and

$e_f(0) = e_f(2) = 2t$, and $e_f(1) = 2t+1$

Hence *f* is a mean cordial labeling.

Case (3): $n \equiv 2 \pmod 3$

Let $n = 3t+2$. Assign labels to the vertices u_i & v_i $(1 \leq i \leq n-2)$ as in case (1). Then assign the label 0, 2 and 0, 1 to the vertices u_{n-1}, u_n and v_{n-1}, v_n respectively. Here $v_f(0) = 2t+2$, $v_f(1) = v_f(2) = 2t+1$ and $e_f(0) = e_f(1) = e_f(2) = 2t+1$. Hence f is a mean cordial labeling.

Theorem 2.14: $P_n \Theta 2K_1$ is mean cordial.

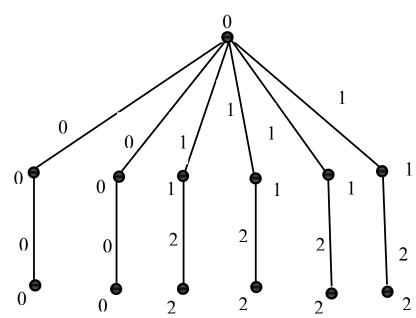

Figure 2. Mean cordial labeling of $S(K_{1,6})$.

Proof: Let P_n be the Path $u_1u_2\cdots u_n$. Let v_i and w_i be the pendant vertices which are adjacent to $u_i, 1\le i\le n$.

Case (1): n is even.

Define $f(u_i)=0,\ 1\le i\le \frac{n}{2}$

$f\left(u_{\frac{n}{2}+i}\right)=1,\ 1\le i\le \frac{n}{2}$

$f(v_i)=0,\ 1\le i\le \frac{n}{2}$

$f(w_i)=1,\ 1\le i\le \frac{n}{2}$

$f\left(v_{\frac{n}{2}+i}\right)=2,\ 1\le i\le \frac{n}{2}$

$f\left(w_{\frac{n}{2}+i}\right)=2,\ 1\le i\le \frac{n}{2}$

Then $v_f(0)=v_f(1)=v_f(2)=n$ and $e_f(0)=n-1, e_f(1)=e_f(2)=n.$

Hence f is a mean cordial labeling.

Case (2): n is odd.

Define $f(u_i)=0,\ 1\le i\le \frac{n-1}{2}$

$f\left(u_{\frac{n-1}{2}+i}\right)=1,\ 1\le i\le \frac{n+1}{2}$

$f(v_i)=0,\ 1\le i\le \frac{n-3}{2}$

$f(w_i)=1,\ 1\le i\le \frac{n-3}{2}$

$f\left(v_{\frac{n-1}{2}}\right)=f\left(w_{\frac{n-1}{2}}\right)=0$

$f\left(v_{\frac{n+1}{2}}\right)=1$, $f\left(w_{\frac{n+1}{2}}\right)=2$

$f\left(v_{\frac{n+1}{2}+i}\right)=f\left(w_{\frac{n+1}{2}+i}\right)=2,\ 1\le i\le \frac{n-1}{2}$

Then $v_f(0)=v_f(1)=v_f(2)=n$ and $e_f(0)=n-1, e_f(1)=e_f(2)=n.$

Hence f is a mean cordial labeling.

Theorem 2.15: The $K_{2,n}$ is a mean cordial iff $n\le 2$.

Proof:

Let $V(K_{2,n})=\{u,v,u_i:1\le i\le n\}$ and $E(K_{2,n})=\{uu_i,vu_i:1\le i\le n\}$. $K_{2,1}$ and $K_{2,2}$ are mean cordial by Theorem 2.5 and 2.8 respectively. Assume $n>2$. Suppose f is a mean cordial labeling of $K_{2,n}$. Clearly either $f(u)=0$ or $f(v)=0$. Without loss generality we can assume $f(u)=0$ so that $f(v)\ne 0$.

Case (1): $n\equiv 0 \pmod 3$

Let $n=3t$. Then $e_f(0)=t$ or $t-1$, a contradiction since the size of $K_{2,n}$ is $6t$.

Case (2): $n\equiv 1 \pmod 3$

Let $n = 3t + 1$. Here $ce_f(0)=t$, again a contradiction.

Case (3): $n\equiv 2 \pmod 3$

Let $n = 3t + 2$. Here $e_f(0)=t$ or $t+1$, again a contradiction to the size of $K_{2,n}$.

3. Conclusion

In this paper we introduced the concept of mean cordial labeling and studied the mean cordial labeling behavior of few standard graph. The authors are of the opinion that the study of mean cordial labeling behavior of graph obtained from standard graphs using the graph operation shall be quite interesting and also will lead to newer results.

4. Acknowledgements

The authors are thankful to the referee for their valuable comments and suggestions.

REFERENCES

[1] J. A. Gallian, "A Dynamic Survey of Graph Labeling," *Electronic Journal of Combinatorics*, Vol. 18, 2011, pp. 1-219.

[2] I. Cahit, "Cordial Graphs: A Weaker Version of Graceful and Harmonious Graphs," *Ars Combinatoria*, Vol. 23, No. 3, 1987, pp. 201-207.

[3] M. Sundaram, R. Ponraj and S. Somosundram, "Product Cordial Labeling of Graph," *Bulletin of Pure and Applied Sciences*, Vol. 23, No. 1, 2004, pp. 155-162.

[4] F. Harary, "Graph Theory," Addision Wisely, New Delhi, 1969.

