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The Optimal Hedging Ratio for Contingent Claims Based on Different Risk Aversions

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Abstract

Based on utility theory, this paper firstly combined different utility functions with risk aversion coefficient and constructed different kinds of optimizing problems for hedgers to hedge for stochastic-payment-typed contingent claim, and then, by the aid of dynamic programming theory, effective multi-stage hedging strategy is proposed for different risk-averse hedgers. Lastly, the research results that the optimal hedging ratios for three kinds of utility functions are equivalent and do not relate to the risk aversion coefficient.

Keywords

Risk Aversion, Contingent Claim, Hedging, The Optimal Hedging Ratio

1. Introduction

As an agreement to execute at the terminal moment, contingent claim is actually a measurable and non-negative random variable. And up to now, more and more enterprises introduce contingent claim to bestir their employee. For example, share option is a kind of contingent claim for a listed company to promote employee's operation effectiveness. For non-listed companies, they usually consent that their chief executives may be entitled to buy shares of the company with the present net asset value per-share, thus, if the net asset value per-share arise in future, the chief executives will be rewarded in perquisites.

As for hedging for contingent claims, in a complete market, one may completely hedge a contingent claim by self-financing method; however, in an incomplete market, as to a contingent claim, one can not completely hedge the potential risks. So, how to minimize the potential risks is what investors must think about, and many hedging methods were put forward; among them, Bouleau and Lamberton [1] firstly put forward mean-variance hedging method, which is to minimize the terminal expected value difference between the contingent claim and hedging positions, *i.e.*, to minimize, after this, many scholars have researched the hedging problem under mean-variance settings. Ales Cerny [2] and Gugushvili S. [3] studied the mean-variance hedging problem by using the dynamic programming theory and acquired the recursive-formatted optimal hedging strategies under discrete time model; Darrell Duffie *et al.* [4] and Jean Paul Laurent *et al.* [5] researched the mean-variance hedging problem under diffusion model and acquired the explicit solution of the optimal hedging strategies. Yang and Xiao [6] studied the risk-minimizing hedging problem by using of the Galtchouk-Kunita-Watanabe decomposing method and testified the existence and the Uniqueness of the optimal hedging strategies. Zhang W. *et al.* [7] and Ni J. *et al.* [8] studied the hedging problem about contingent claim produced during merchandise transaction.

As far as I know that most of those presented documents about hedging did not take the risk preference of investors into account, however, different investors have different risk preference and different ability of foreseeing financial risk, thus, for different investors, their hedging strategies must be not compatible with each other. In this paper, first off, we combine different utility functions with risk aversion coefficient and construct different kinds of optimizing problems for hedgers to hedge for stochastic-payment-typed contingent claim, and then, by the aid of dynamic programming theory, effective multi-stage hedging strategy is proposed for different risk-averse hedgers

2. Models

2.1. Utility Function

First, confirm that you have the correct template for your paper size. This template has been tailored for output on the custom paper size (21 cm * 28.5 cm). Risk aversion or risk preference is scale to measure investors' attitude to financial risk, Arrow (1971) [9] pointed out in his reference about financial risk that there were absolute risk aversion coefficient and relative risk aversion coefficient. Assuming that WN denotes the value of the terminal wealth while U(WN) denotes a utility function which is continuous and is two-order differentiable, we call $r_A(W_N) = -\frac{U''(W_N)}{U'(W_N)}$ be an absolute risk-hedging coefficient, and $r_R(W_N) = -\frac{W_N U''(W_N)}{U'(W_N)}$ be a relative risk-hedging coefficient. If $r_A(W_N) > 0$ and $r_R(W_N) > 0$, investors are of risk aversion, and if $r_A(W_N) < 0$ and $r_R(W_N) < 0$, investors are of risk preference, or else, if $r_A(W_N) = 0$ and $r_R(W_N) = 0$, investors are risk neutral. Under uncertain economic environment, anyone will try her best to maximize the expected utility of her terminal wealth to avoid financial risk [10] [11].

Here, we consider three common utility functions which relate to financial risk:

1) Quadratic utility function : $U(W_N) = W_N - \frac{\lambda}{2} W_N^2$, where λ is called as a

risk hedging coefficient. For a hedger with this quadratic utility function, she is of risk aversion and her risk aversion enhances while her wealth increases. The expected value of utility of wealth can be denotes as $E\left[U\left(W_N\right)\right]=E\left[W_N\right]-\frac{1}{2}\lambda Var\left[W_N\right]$.

2) Power utility function: $U\left(W_N\right)=\frac{W_N^{(1-\lambda)}}{1-\lambda}$, $0<\lambda<1$, which is also called as relative risk-hedging utility function. When W_N is a random variable, the expected utility value can be denoted as $E\left[U\left(W_N\right)\right]=\frac{1}{1-\lambda}E\left(W_N^{1-\lambda}\right)$.

3) negative exponential utility function: $U\left(W_N\right)=-\exp\left(-\lambda W_N\right)$, $\lambda>0$, which is also called as absolute risk-hedging utility function and the expected utility value can be denoted as $E\left[U\left(W_N\right)\right]=E\left[-\exp\left(-\lambda W_N\right)\right]$.

2.2. The Hedging Model

Assuming one people, such as an executive chief of a corporation, acquired a copy of contingent claim that will be executed at the terminal moment T according to his or her work rate, which is called as a kind of Stochastic-Payment-Typed Contingent Claim, in order to acquire the maximal profit, the executive chief, as a hedger, shall use the underlying asset related to the contingent claim to hedge the potential risk at discrete moments $t\in\{0,\Delta t,2\Delta t,\cdots,N\Delta t\}$ with time interval $\Delta t=T/N$ during period $[0,T]$.

Let St be the price of contingent claim at $t=n\Delta t$, and F_t be the price of the future contract underling on the claim, for a hedger, she use the future contract to hedge the risk of the contingent claim and ϑ_t denotes the position she holds at moment $t=n\Delta t$, while $W_t=S_t-\vartheta_t F_t$ denotes the value of hedging asset portfolio at moment $t=n\Delta t$, which will evolve as expressions (1).

$$W_t=\begin{cases} W_0 & t=1 \\ RW_{t-1}-\left(F_t-F_{t-1}\right)\vartheta_{t-1} & 2\le t\le N-1 \\ RW_{N-1}-\left(F_N-F_{N-1}\right)\vartheta_{N-1}+S_N & t=N \end{cases} \tag{1}$$

where, $R=\exp\left(r\Delta t\right)$ and r denotes the risk-free rate and.

Under the constraint of self-financing as (1), we can construct the hedging model just as following expressions (2).

$$\max_{\{\vartheta_t\}_{t=1}^{N-1}} E_1\left[U\left(W_N\right)\right] \tag{2}$$

where $E_1\left[U\left(W_N\right)\right]$ is an expected utility function, denoting the expected utility of the terminal wealth, and $E_t\left[\cdot\right]=E\left[\cdot\mid F_t\right],t=1,\cdots,N$ denote the conditional expectation, and similarly, $Var_t\left[\cdot\right]=Var\left[\cdot\mid F_t\right],t=1,\cdots,N$ denote the conditional variance.

3. The Solutions

3.1. The Basic Theory

According to the Bellman principal [12], the expressions (2) can be rewritten as (3) as following, and the optimal hedging ratios at each moment $t = n\Delta t$ may be acquired by using the backward recursion method.

$$\begin{aligned}
&\max_{(\vartheta_1,\cdots,\vartheta_{N-1})} E_1\left[U\left(W_N\right)\right] \\
&= \max_{(\vartheta_1,\cdots,\vartheta_{N-2})} E_1\left\{\max_{\vartheta_{N-1}} E_{N-1}\left[U\left(W_N\right)\right]\right\} \\
&= \cdots \\
&= \max_{\vartheta_1} E_1\left\{\max_{\vartheta_2} \cdots E\left\{\max_{\vartheta_{N-2}} E\left\{\max_{\vartheta_{N-1}} E\left[U\left(W_N\right)\right]\right\}\right\}\cdots\right\}
\end{aligned} \tag{3}$$

3.2. The Optimal Hedging Ratio

Proposition 1: In non-arbitrage market, the optimal hedging ratio of hedging problem $\max_{\{\vartheta_n\}_{n=1}^{N-1}} E_1\left[U\left(W_N\right)\right] = E_1\left(W_N\right) - \frac{\lambda}{2} Var_1\left(W_N\right)$ may be expressed as

$$\vartheta_n^* = \frac{1}{R^{N-n-1}}, n = 1,\cdots,N-1 \tag{4}$$

Proof: according to the dynamic programming principal, we can acquire the expressions (4) as following steps.

Step 1, At the moment $t = (N-1)\Delta t$, according to (1), there is

$$W_N = RW_{N-1} - \left(F_{N-1} - F_{N-1}\right)\vartheta_{N-1} + S_N \tag{5}$$

If substitute (5) into $E_{N-1}\left[U\left(W_N\right)\right] = E_{N-1}\left[W_N\right] - \frac{1}{2}\lambda Var_{N-1}\left[W_N\right]$, we can get the following optimizing problem (6):

$$\begin{aligned}
&\max_{\vartheta_{N-1}} E_{N-1}\left[U\left(W_N\right)\right] \\
&= \max_{\vartheta_{N-1}} E_{N-1}\left[U\left(RW_{N-1} - \left(F_N - F_{N-1}\right)\vartheta_{N-1} + S_N\right)\right] \\
&= \max_{\vartheta_{N-1}}\left\{E_{N-1}\left[RW_{N-1} - \left(F_N - F_{N-1}\right)\vartheta_{N-1} + S_N\right]\right. \\
&\quad \left. - \frac{1}{2} Var_{N-1}\left[RW_{N-1} - \left(F_N - F_{N-1}\right)\vartheta_{N-1} + S_N\right]\right\}
\end{aligned} \tag{6}$$

Only need to make derivative calculation on ϑ_{N-1} in (6), and let the differential coefficient equal to zero, we can acquire the optimal hedging ratio ϑ_{N-1}^* at the moment $t = (N-1)\Delta t$ as following expressions (7).

$$\vartheta_{N-1}^* = \frac{F_{N-1} - E_{N-1}\left(F_N\right) + \lambda Cov_{N-1}\left(F_N, S_N\right)}{\lambda Var_{N-1}\left(F_N\right)} \tag{7}$$

When in non-arbitrage market, there $E_{N-1}\left(F_N\right) = F_{N-1}$, and because the basic difference between the spot(contingent claim) and the future equals to zero, *i.e.*, $F_N = S_N$, the (7) may be expressed as (8).

$$\vartheta_{N-1}^* = \frac{0 + \lambda Cov_{N-1}\left(F_N, F_N\right)}{\lambda Var_{N-1}\left(F_N\right)} = \frac{1}{R^0} \tag{8}$$

At the moment $t = (N-2)\Delta t$, if substitute the optimal hedging ratio just as expressions (8) into $W_N = RW_{N-1} - (F_{N-1} - F_{N-2})\vartheta_{N-1} + S_N$, and according to (1), we can get

$$\begin{aligned} W_N &= RW_{N-1} - (F_N - F_{N-1})\vartheta^*_{N-1} + S_N \\ &= R^2 W_{N-2} - R(F_{N-1} - F_{N-2})\vartheta_{N-2} - (F_N - F_{N-1}) + S_N \end{aligned} \tag{9}$$

Similarly, if put (9) into $E_{N-2}\left[U(W_N)\right] = E_{N-2}[W_N] - \frac{1}{2}\lambda Var_{N-2}[W_N]$, we can get the optimizing problem (10).

$$\begin{aligned} &\max_{\vartheta_{N-2}} E_{N-2}\left[U(W_N)\right] \\ &= \max_{\vartheta_{N-2}} E_{N-2}\left[U\left(R^2 W_{N-2} - R(F_{N-1} - F_{N-2})\vartheta_{N-2} - (F_N - F_{N-1}) + S_N\right)\right] \\ &= \max_{\vartheta_{N-2}} \Big\{ E_{N-2}\left[R^2 W_{N-2} - R(F_{N-1} - F_{N-2})\vartheta_{N-2} - (F_N - F_{N-1}) + S_N\right] \\ &\quad -\frac{1}{2} Var_{N-2}\left[R^2 W_{N-2} - R(F_{N-1} - F_{N-2})\vartheta_{N-2} - (F_N - F_{N-1}) + S_N\right]\Big\} \end{aligned} \tag{10}$$

Only need to make derivative calculation on ϑ_{N-2} in (10), and let the differential coefficient equal to zero, we can acquire the optimal hedging ratio ϑ^*_{N-2} at the moment $t = (N-2)\Delta t$ as following expressions (11).

$$\begin{aligned} \vartheta^*_{N-2} &= \frac{F_{N-2} - E_{N-2}(F_{N-1}) - \lambda Cov_{N-2}(F_{N-1}, F_N - F_{N-1}) + \lambda Cov_{N-1}(F_{N-1}, S_N)}{\lambda Var_{N-1}(F_N)} \\ &= \frac{1}{R} \end{aligned} \tag{11}$$

Step 2, assume at any moment $t = (n+1)\Delta t$, the optimal hedging ratio can be expressed as $\vartheta^*_{n+1} = \frac{1}{R^{N-n-2}}$.

Step 3, we only need to prove that there is $\vartheta^*_n = \frac{1}{R^{N-n-1}}$ at moment $t = n\Delta t$.

In fact, according to (1), we can get the recursion expressions as following (12).

$$W_N = R^{N-n} W_n - \sum_{i=1}^{N-n} R^{N-n-i}(F_{n+i} - F_{n+i-1})\vartheta_{n+i-1} + S_N \tag{12}$$

Now, only need to put all $\vartheta^*_i = \frac{1}{R^{N-i-1}}(i = N-1, \cdots, n+1)$ into (12), there is the optimizing problem (13), and in non-arbitrage market, we can solve (13) and acquire the optimal hedging ratio at moment $t = n\Delta t$ as in (14).

$$\begin{aligned} &\max_{\vartheta_n} E_n\left[U(W_N)\right] = E_n[W_N] - \frac{1}{2}\lambda Var_n[W_N] \\ &= E_n\left[R^{N-n} W_n - R^{N-n-1}(F_{n+1} - F_n)\vartheta_n - \sum_{i=2}^{N-n}(F_{n+i} - F_{n+i-1}) + S_N\right] \\ &\quad -\frac{1}{2}\lambda Var_n\left[R^{N-n} W_n - R^{N-n-1}(F_{n+1} - F_n)\vartheta_n - \sum_{i=2}^{N-n}(F_{n+i} - F_{n+i-1}) + S_N\right] \end{aligned} \tag{13}$$

$$
\begin{aligned}
\vartheta_n^* &= \frac{F_n - E_n\left(F_{n+1}\right) - \lambda Cov_n\left[F_{n+1}, \sum_{i=2}^{N-n}\left(F_{n+i} - F_{n+i-1}\right)\right] + \lambda Cov_n\left[F_{n+1}, F_N\right]}{\lambda R^{N-n-1} Var_n\left(F_{n+1}\right)} \\
&= \frac{0 - \lambda Cov_n\left[F_{n+1}, \sum_{i=2}^{N-n}\left(F_{n+i} - F_{n+i-1}\right) - F_N\right]}{\lambda R^{N-n-1} Var_n\left(F_{n+1}\right)} \\
&= \frac{0 - \lambda Cov_n\left[F_{n+1}, \sum_{i=2}^{N-n}\left(F_{n+i} - F_{n+i-1}\right) - F_N\right]}{\lambda R^{N-n-1} Var_n\left(F_{n+1}\right)} \\
&= \frac{0 - \lambda Cov_n\left[F_{n+1}, -F_{n+1}\right]}{\lambda R^{N-n-1} Var_n\left(F_{n+1}\right)} \\
&= \frac{1}{R^{N-n-1}}
\end{aligned} \tag{14}
$$

Proposition 2: In non-arbitrage market, the optimal hedging ratio for $\max_{\{\vartheta_n\}_{n=1}^{N-1}} E_1\left[U\left(W_N\right)\right] = \frac{1}{1-\lambda} E_1\left(W_N^{1-\lambda}\right)$ can be expressed as in (15).

$$
\vartheta_n^* = \frac{1}{R^{N-n-1}}, n = 1, \cdots, N-1 \tag{15}
$$

Proof, Firstly, at moment $t = (N-1)\Delta t$, denoting $\lambda' = 1 - \lambda$, then, there is

$$
\max_{\vartheta_{N-1}} E_{N-1}\left[U\left(W_N\right)\right] = \frac{1}{\lambda'} E_{N-1}\left(W_N^{\lambda'}\right), \quad 0 < \lambda' < 1 \tag{16}
$$

If substituting $W_N = RW_{N-1} - \left(F_N - F_{N-1}\right)\vartheta_{N-1} + S_N$ into (16), and make derivative calculation on ϑ_{N-1} in (16), and let the differential coefficient equal to zero, there is

$$
\begin{aligned}
& F_{N-1} E_{N-1}\left[\left(RW_{N-1} - \left(F_N - F_{N-1}\right)\vartheta_{N-1} + S_N\right)^{\gamma'-1}\right] \\
& = E_{N-1}\left[F_N\left(RW_{N-1} - \left(F_N - F_{N-1}\right)\vartheta_{N-1} + S_N\right)^{\gamma'-1}\right]
\end{aligned} \tag{17}
$$

Assuming the market is complete, *i.e.*, $E_{N-1}\left(F_N\right) = F_{N-1}$ and the basic difference between the spot(contingent claim) and the future equals to zero, we can acquire the optimal hedging ratio ϑ_{N-1}^* at the moment $t = (N-1)\Delta t$ as following expressions (18).

$$
\vartheta_{N-1}^* = 1 = \frac{1}{R^0} \tag{18}
$$

Then, assume at any moment $t = (n+1)\Delta t$, the optimal hedging ratio can be expressed as $\vartheta_{n+1}^* = \frac{1}{R^{N-n-2}}$. now, only need to proof $\vartheta_n^* = \frac{1}{R^{N-n-1}}$.

In fact, according to (1), we can get the recursion expressions as following (19).

$$
W_N = R^{N-n} W_n - \sum_{i=1}^{N-n} R^{N-n-i}\left(F_{n+i} - F_{n+i-1}\right)\vartheta_{n+i-1} + S_N \tag{19}
$$

Only substitute all $\vartheta_i^* = \frac{1}{R^{N-i-1}} (i = N-1, \cdots, n+1)$ into (19), there is

$$W_N = R^{N-n} W_n - R^{N-n-1} (F_{n+1} - F_n) \vartheta_n - \sum_{i=2}^{N-n} (F_{n+i} - F_{n+i-1}) + S_N \quad (20)$$

Just as acquiring ϑ_{N-1}^*, we can solve problem (21),

$$\begin{aligned} &\max_{\vartheta_n} E_n \left[U(W_N) \right] = \frac{1}{\lambda'} E_n \left[W_N^{\lambda'} \right] \\ &= \frac{1}{\lambda'} E_n \left[\left(R^{N-n} W_n - R^{N-n-1} (F_{n+1} - F_n) \vartheta_n - \sum_{i=2}^{N-n} (F_{n+i} - F_{n+i-1}) + S_N \right)^{\lambda'} \right] \end{aligned} \quad (21)$$

There is $\vartheta_n^* = \frac{1}{R^{N-n-1}}$.

Proposition 3: In non-arbitrage market, the optimal hedging ratio for $E\left[U(W_N)\right] = E\left[-\exp(-\lambda W_N)\right]$ can be expressed as in (22).

$$\vartheta_n^* = \frac{1}{R^{N-n-1}}, n = 1, \cdots, N-1 \quad (22)$$

The prove is similar to proposition 1.

4. Conclusion

In this paper, based on utility theory, we researched the hedging problem for stochastic-payment-typed contingent claim. Firstly, we combined different utility functions with risk aversion coefficient and constructed different kinds of optimizing problems for hedgers to hedge for stochastic-payment-typed contingent claim, and then, by the aid of dynamic programming theory, effective multi-stage hedging strategy is proposed for different risk-averse hedgers. Lastly, the research results that the optimal hedging ratios for different kinds of utility functions are equivalent and do not relate to the risk aversion coefficient.

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Conflicts of Interest

The author declares no conflicts of interest regarding the publication of this paper.

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