

Stability Analysis of Periodic Solutions of Some Duffing's Equations

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How to cite this paper: Eze, E.O., Obasi, U.E. and Agwu, E.U. (2019) Stability Analysis of Periodic Solutions of Some Duffing's Equations. *Open Journal of Applied Sciences*, 9, 198-214.

<https://doi.org/10.4236/ojapps.2019.94017>

Received: October 10, 2018

Accepted: April 20, 2019

Published: April 23, 2019

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Abstract

In this paper, some stability results were reviewed. A suitable and complete Lyapunov function for the hard spring model was constructed using the Cartwright method. This approach was compared with the existing results which confirmed a superior global stability result. Our contribution relies on its application to high damping door constructions. (2010 Mathematics Subject Classification: 34B15, 34C15, 34C25, 34K13.)

Keywords

Stability Analysis, Duffing's Equations, Periodic Solutions, MATCAD, Cartwright Method

1. Introduction

In real life, most problems that occur are non-linear in nature and may not have analytic solutions except by approximations or simulations and so trying to find an explicit solution may in general be complicated and sometimes impossible. Duffing's equation is a second order non-linear differential equation used to model such problems of non-linear nature [1]. In particular, it is used to model damped and driven oscillators, for example, modelling of the brain [2], in prediction of earthquake occurrences [3], signal processing [4] and crash analysis [5]. In [6], differential equation which describes a non-linear oscillation was first introduced by Duffing's with cubic stiffness constant. The general form of Duffing's equation is:

$$\ddot{x} + c\dot{x} + g(t, x) = p(t) \quad (1.1)$$

where $p(t)$ is continuous and 2π -periodic in $t \in \mathbb{R}$.

The study of periodic solution of Duffing's equation is like that of the study of classical Hamiltonian equation of motion which is characterized by multiplicity

of periodic solution. Various techniques for investigating the stability of periodic solutions of Duffing's equation have been reported, (see [7] [8] [9] [10] [11]). Our approach to study stability of periodic solutions is by the use of the Lyapunov direct method. Many researchers have obtained useful and valid results using the Lyapunov second method (direct method) for stability analysis and construction of appropriate Lyapunov functions for other Duffing equation but the use and construction of Lyapunov function using Cartwright method are rare in literature, for instance [12] [13] [14] [15] [16]. The application of this method is in constructing a scalar function and its derivatives with peculiar characterizations. When these characterizations are satisfied, the stability behaviour of the system is solved. The difficulties in the construction of suitable Lyapunov functions in nonlinear systems have attracted the attention of many researchers and have been summarized in [17] [18] [19] that discussed the general approach to the stability study of periodic solution which is related to the classical Lyapunov theorem based on linear approximations. This reduces the stability study of periodic solutions to the stability of the system linearized at the periodic motion. Since linearized systems contain periodic coefficients, the theory of parametric resonance can be applied. Such approach with the analysis of Floquet multiplier is used in [9] [20] [21]. The other traditional approach to the study of stability of periodic solutions is the approximate average and multiple scales method, which reduces original time dependent dynamical systems to autonomous system. In this case stability study is the analysis of fixed point, (see [22]).

Our construction of suitable Lyapunov function rests squarely on the approach of [23]. Others who used this approach are [16] [24]. Very few nonlinear systems can be solved explicitly and so we must rely on numerical schemes to approximate the solutions, (see [25] [26] [27] [28]). This paper is motivated by studying [11] [29] where stability analysis of the unforced and damped cubic-quintic Duffing oscillator of the form

$$\ddot{u} + \delta\dot{u} + \alpha u + \beta u^3 + \mu u^5 = 0 \quad (1.2)$$

was carried out. Using Equation (1.2) the existence of a three term valid solution was obtained using derivative expansion method. The derivative expansion method is one of the perturbative methods which require the existence of a small parameter and is therefore not valid in principle for the Duffing equation in which the nonlinearity is large. Its stability analysis of the equilibrium point was carried out using the eigenvalue approach. Secondly, our motivation was augmented by reading the works of [10] where new criteria for existence and asymptotic stability of periodic solutions of a Duffing equation of the form

$$\ddot{x} + c\dot{x} + g(t, x) = 0 \quad (1.3)$$

were derived. This approach exposed by [10] is useful when the non-linearity does not admit the decomposition of $g(t, x) = g(x) + h(t)$. In line with the above review and ongoing research in this direction, the objective of this paper is to investigate the stability analysis of periodic solutions of the Duffing type

$$\ddot{x} + c\dot{x} + ax + bx^2 + 2x^3 = h(t) \quad (1.4)$$

with boundary conditions as:

$$x(0) = x(2\pi) \quad (1.5)$$

$$\dot{x}(0) = \dot{x}(2\pi)$$

where a, b, c are real constants and $h: [0, 2\pi] \rightarrow \mathbb{R}^n$ is continuous. In Equation (1.4), a is the stiffness constant, c is the coefficient of viscous damping and $bx^2 + 2x^3$ represents the nonlinearity in the restoring force acting like a hard spring. Equation (1.4) has received wide interest in neurology, modelling of mechanical systems such as shock absorbers in most vehicles. It can be used to model plant stems to better understand the effect of nonlinear stiffness or resonant behavior of plants. It is a model arising in many branches of Physics and Engineering such as oscillation of rigid pendulum using moderately large amplitude motion [29], Vibration of buckled beam [30] etc. This equation together with Van der Pol's has been reported in textbooks as examples of nonlinear oscillation. See [31] [32] [33].

2. Preliminaries

Definition 2.1. (Properties of Lyapunov Function)

The Lyapunov function has the following properties:

- Continuity: $V(t, X)$ is continuous and single valued under the condition $t \geq T$ and $|x_i| < H$ and $V(t, 0) \equiv 0$.
- $V(t, X)$ is positive definite; and
- $\dot{V} = \frac{\partial V}{\partial x_1} \dot{x}_1 + \frac{\partial V}{\partial x_2} \dot{x}_2 + \dots + \frac{\partial V}{\partial x_n} \dot{x}_n + \frac{\partial V}{\partial t}$, representing the total derivative with respect to t is negative definite.

Definition 2.2. (Complete Lyapunov function)

A Lyapunov function V defined as $V: I \times \mathbb{R}^n \rightarrow \mathbb{R}$ is said to be COMPLETE if for $X \in \mathbb{R}^n$,

- $V(t, X) \geq 0$;
- $V(t, X) = 0$ if and only if $X = 0$ and
- $\dot{V}(t, X) \leq -c|X|$ where c is any positive constant and $|X|$ given by

$$|X| = \left(\sum_{i=1}^n (x_i^2) \right)^{\frac{1}{2}} \rightarrow \infty$$

It is INCOMPLETE if 3) is not satisfied.

Theorem 2.3. Consider a system of differential equations

$$\dot{X} = f(x) \quad (1.6)$$

$f: D \rightarrow \mathbb{R}^2$. Let $X = 0$ be an equilibrium point of Equation (1.6).

Let $V: D \rightarrow \mathbb{R}$ be a continuously differentiable function; such that:

- $V(0) = 0$;
- $V(x) > 0$ in $D - \{0\}$;

$$3) \dot{V}(x) \leq 0 \text{ in } D - \{0\}.$$

Then $X = 0$ is stable.

Theorem 2.4. Let $V : D \rightarrow \mathbb{R}$ be a continuously differentiable function such that:

- 1) $V(0) = 0$;
- 2) $V(x) > 0$ in $D - \{0\}$;
- 3) $\dot{V}(x) < 0$ in $D - \{0\}$.

Then $X = 0$ is “asymptotically stable”.

Theorem 2.5. Let $V : D \rightarrow \mathbb{R}$ be a continuously differentiable function such that:

- 1) $V(0) = 0$;
- 2) $V(x) > 0$ in $D - \{0\}$;
- 3) $V(x)$ is “radially bounded”;
- 4) $\dot{V}(x) < 0$ in $D - \{0\}$.

Then $X = 0$ is “globally asymptotically stable”.

Theorem 2.6. Suppose all conditions for asymptotic stability are satisfied. In addition to it, suppose there exists constants K_1, K_2, K_3, P

- 1) $K_1 \|X\|^P \leq V(X) \leq K_2 \|X\|^P$;
- 2) $\dot{V}(X) \leq -K_3 \|X\|^P$.

Then the origin $X = 0$ is “exponentially stable”.

Moreover, if this conditions hold globally, then the origin $X = 0$ is globally exponentially stable.

3. Results

3.1. Construction of Lyapunov Function for the Duffing Equation Using Cartwright Method

We adapt the method of construction of Lyapunov function used in [23] and extend it to the second order non-linear differential equation of the Duffing type of the form (3.1). In the sequel, [34] asserted that Lyapunov functions are vital in determining stability, instability, boundedness and periodicity of ordinary differential. Considering the Duffing equation of the form

$$\ddot{x} + c\dot{x} + ax + bx^2 + 2x^3 = p(t) \quad (1.7)$$

The equivalent systems of Equation (1.7) is

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= -cx_2 - ax_1 - h(x_1) \end{aligned} \quad (1.8)$$

where $h(x_1) = bx_1^2 + 2x_1^3$ and $p(t) = 0$.

Writing the equivalent systems of Equation (1.7) in compact form, we have

$$\dot{X} = AX \quad (1.9)$$

where $A = \begin{bmatrix} 0 & 1 \\ -a - h(x) & -c \end{bmatrix}$, $X = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$.

The method discussed here is based on the fact that the matrix A has all its ei-

genvalues as negative real parts. Then from the general theory which corresponds to any positive quadratic form $U(x)$, there exists another positive definite quadratic form $V(x)$ such that

$$\dot{V} = -U \tag{1.10}$$

Choosing the most general quadratic form of order two and picking the coefficient in the quadratic form to satisfy Equation (1.10) along the solution paths of Equation (1.10) we assume V to be defined by

$$2V = Ax^2 + By^2 + 2Kxy \tag{1.11}$$

$$\begin{aligned} \dot{V} &= Ax\dot{x} + By\dot{y} + K(\dot{x}y + y\dot{x}) \\ &= Axy + B(-cy - ax - h(x)) + Ky^2 + Kx(-cy - ax - h(x)) \\ &= Axy - Bcy^2 - Byax - Byh(x) + Ky^2 - Kcxy - Kax^2 - Kxh(x) \end{aligned}$$

Simplifying the coefficients we have

$$\dot{V} = (A - Ba - Kc)xy + (K - Bc)y^2 - (Ka)x^2 - (Kx + By)h(x) \tag{1.12}$$

To make \dot{V} negative definite, we equate the coefficient of mixed variable to zero and the coefficients of x^2 and y^2 to any positive constant (say δ) as outlined in **Table 1**

$$A - Ba - Kc = 0 \tag{1}$$

$$K - Bc = \delta \tag{2}$$

$$-Ka = \delta \tag{3}$$

$$-(Kx + By) = 0 \tag{4}$$

From Equation (3) above

$$-Ka = \delta$$

$$K = -\frac{\delta}{a} \tag{1.13}$$

Then substituting the value of K into Equation (2) we obtain

$$-\frac{\delta}{a} - Bc = \delta$$

$$-Bc = \delta + \frac{\delta}{a}$$

$$B = -\frac{\delta(a+1)}{ca} \tag{1.14}$$

Substituting for K and B in (1) we have

$$A = Ba + Kc = -\frac{\delta(a+1)}{c} - \frac{\delta c}{a} \tag{1.15}$$

which by further simplification gives that

$$A = -\frac{\delta}{ca} [(a+1)a + c^2] \tag{1.16}$$

Substituting for the values of the constant A, B, K in Equation (1.11) gives

$$2V = -\frac{\delta}{ca} [(a+1)a + c^2] x^2 - \frac{\delta}{ca} [a+1] y^2 - 2\frac{\delta}{a} xy$$

Table 1. A table showing terms and coefficients of Equation (1.12).

Terms	Coefficient
xy	$A - Ba - Kc$
y^2	$K - Bc$
x^2	$-Ka$
$h(x)$	$-(Kx + By)$

$$V = -\frac{\delta}{2ca} \left[((a+1)a + c^2)x^2 + (a+1)y^2 + 2cxy \right] \quad (1.17)$$

By choosing $\frac{\delta}{ca} = -1$

$$V(x) = \frac{1}{2} \left[((a+1)a + c^2)x^2 + (a+1)y^2 + 2cxy \right] > 0 \quad (1.18)$$

Using Equation (1.19) and the fact that $\dot{V} < 0$, the equilibrium point is asymptotically stable.

3.2. Stability Analysis of Periodic Solution of Duffing Equation Using the Eigenvalue Approach

Considering the Duffing equation of the form (1.7), the first equivalent systems of (1.7) is given by

$$\begin{aligned} \dot{x} &= y \\ \dot{y} &= -cy - ax - bx^2 - 2x^3 + h(t) \end{aligned} \quad (1.19)$$

For the unforced case, Equation (1.19) is reduced to

$$\begin{aligned} \dot{x} &= y \\ \dot{y} &= -cy - ax - bx^2 - 2x^3 \end{aligned} \quad (1.20)$$

At fixed points, $\dot{x} = y = 0$.

So that $y = 0$ and $\dot{y} = -ax - bx^2 - 2x^3 = x(-ax - bx - 2x^2) = 0$.

Giving us $x = 0, x_1 = \frac{-b + \sqrt{b^2 - 8a}}{4}$ and $x_2 = \frac{-b - \sqrt{b^2 - 8a}}{4}$ which correspond to $(0, 0)$, $(x_1, 0)$ and $(x_2, 0)$ at fixed point. Analysis of the stability of the fixed points can be done by linearizing Equation (1.20) which gives

$$\begin{aligned} \ddot{x} &= \dot{y} \\ \dot{y} &= -cy - (a + 2bx + 6x^2)\dot{x} \end{aligned} \quad (1.21)$$

The matrix equation for (1.21) can be written as

$$\begin{bmatrix} \ddot{x} \\ \dot{y} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -(a + 2bx + 6x^2) & -c \end{bmatrix} \begin{bmatrix} \dot{x} \\ \dot{y} \end{bmatrix} \quad (1.22)$$

Examining the stability at the point $(0, 0)$ gives

$$\begin{bmatrix} 0-\lambda & 1 \\ -(a+2bx+6x^2)-\lambda & -c-\lambda \end{bmatrix}$$

$$\lambda c + \lambda^2 + a + 2bx + 6x^2 + \lambda = 0$$

$$\lambda^2 + \lambda(1+c) + g = 0 \tag{1.23}$$

$$\lambda = \frac{-(1+c) \pm \sqrt{(1+c)^2 - 4g}}{2} \tag{1.24}$$

where $\lambda_1 = \frac{1}{2}(- (1+c) + \sqrt{(1+c)^2 - 4g})$ and $\lambda_2 = \frac{1}{2}(- (1+c) - \sqrt{(1+c)^2 - 4g})$ are the roots of Equation (1.23). λ_1 and λ_2 can be written as

$$\lambda_1 = \frac{1}{2}(-\delta + \sqrt{\delta^2 - 4g}) \text{ and } \lambda_2 = \frac{1}{2}(-\delta - \sqrt{\delta^2 - 4g}) \text{ where } \delta = 1+c \text{ and}$$

$$g = a + 2bx + 6x^2$$

The coefficient of β shows that the Duffing equation is highly damped representing a hard spring. This hard spring is represented in Equation (1.21) where the coefficient of x^2 is 6.

At the origin, $\lambda_{\pm}^{(0,0)} = \frac{1}{2}(-\delta \pm \sqrt{\delta^2 - 4g})$.

With $\delta = 0$, $\lambda_{1,2} = \frac{1}{2}(\pm\sqrt{-4g})$.

For this, we consider the following cases:

- 1) When $g = 0$, $\lambda_{1,2} = 0$ and this implies that b, x, a are all zero;
- 2) When $g > 0$, $\lambda_{1,2} = \pm i\sqrt{g}$ which corresponds to critical points that are centres for which stability is ensured;
- 3) When $g < 0$, $\lambda_{1,2} = \pm\sqrt{g}$ which corresponds to saddles giving rise to instability [11].

With $\delta > 0$, $\lambda_{1,2} = \frac{1}{2}(-\delta \pm \sqrt{\delta^2 - 4g})$.

For this, we consider the following cases:

- 1) When $g = 0$, $\lambda_{1,2} = 0, -\delta$;
- 2) When $g > 0$, $\lambda_{1,2} = \frac{1}{2}(-\delta \pm \sqrt{\delta^2 - 4g})$.

For the discriminate, we have the following cases:

When $\delta^2 < 4g$, $\lambda_{1,2} = \frac{1}{2}(-\delta \pm i\sqrt{\delta^2 - 4g})$ shows that the roots are imaginary which to lead to spiral and asymptotic stability.

When $\delta^2 > 4g$, $\lambda_{1,2} = \frac{1}{2}(-\delta \pm \sqrt{\delta^2 - 4g})$ shows that the roots are real which leads to saddles and instability.

When $\delta^2 = 4g$, $\lambda_{1,2} = \pm\sqrt{g}$ shows that the roots are real corresponding to saddles and instability.

When $g < 0$, $\lambda_{1,2} = \frac{1}{2}(\delta \pm \sqrt{\delta^2 + 4g})$.

For the discriminate, we consider the following cases:

1) When $\delta^2 + 4g < 0$, $\lambda_{1,2} = \frac{1}{2}(\delta \pm i\sqrt{\delta^2 + 4g})$ which corresponds to spirals and asymptotic stability.

2) When $\delta^2 + 4g > 0$, $\lambda_{1,2} = \frac{1}{2}(\delta \pm i\sqrt{\delta^2 + 4g})$ which leads to instability.

3) When $\delta^2 + 4g = 0$, $\lambda_{1,2} = \pm(i\sqrt{g})$ which leads to centres and instability.

Interestingly, for special case when $c = 0$ with no forcing term equation (1.20) becomes

$$\dot{x} = y \quad (1.25)$$

$$\dot{y} = -ax - bx^2 - 2x^3 \quad (1.26)$$

The above can be integrated by quadrature, differentiating (1.25) and plugging in (1.26) gives

$$\dot{x} = \dot{y} = -ax - bx^2 - 2x^3 \quad (1.27)$$

Multiplying both sides by \dot{x} gives

$$\dot{x}\dot{x} - ax\dot{x} - b\dot{x}x^2 - 2\dot{x}x^3 = 0 \quad (1.28)$$

Equation (1.28) can be written as

$$\frac{d}{dt} \left[\frac{1}{2}\dot{x}^2 - \frac{1}{2}ax^2 - \frac{1}{3}bx^3 - \frac{1}{2}x^4 \right] = 0 \quad (1.29)$$

So we have a variant of motion

$$h = \frac{1}{2}\dot{x}^2 - \frac{1}{2}ax^2 - \frac{1}{3}bx^3 - \frac{1}{2}x^4 \quad (1.30)$$

solving for \dot{x}^2 gives $\dot{x}^2 = \left(\frac{dx}{dt}\right)^2 = 2h + ax^2 + \frac{2}{3}bx^3 + x^4$

$$\frac{dx}{dt} = \sqrt{2h + ax^2 + \frac{2}{3}bx^3 + x^4}$$

$$t = \int dt = \int \frac{dt}{\sqrt{2h + ax^2 + \frac{2}{3}bx^3 + x^4}} \quad ([35], \text{ p. } 29)$$

Note that the invariant of motion h satisfies $\dot{x} = \frac{\partial h}{\partial \dot{x}} = \frac{\partial h}{\partial y}$ where

$$\frac{\partial h}{\partial x} = ax + bx^2 + 2x^3 = \dot{y} \quad (1.31)$$

So the equation of Duffing oscillator are given by the Hamiltonian system

$$\dot{x} = \frac{\partial h}{\partial y} \quad \text{and} \quad \dot{y} = -\frac{\partial h}{\partial x} \quad ([35], \text{ p. } 31)$$

We consider the stability analysis of Duffing's equation for different values of a, b, c , as illustrated in **Table 2**.

Table 2. The stability analysis and numerical solutions of duffing’s equation at different values a, b, c .

$a = 0.1$	$x_{12} = \frac{-0.2 \pm \sqrt{-0.76}}{4}$	-0.05	Spiral	Asymptotically stable
$b = 0.2$		-0.05	Spiral	Asymptotically stable
$c = 0.1$	$x_{21} = \frac{-0.2 \pm \sqrt{-0.76}}{4}$	0.168	Saddle	Unstable
		-1.07	Spiral	Stable
$a = 0.6$	$x_{12} = \frac{-0.2 \pm \sqrt{-4.71}}{4}$	-0.0075	Spiral	Asymptotically stable
$b = 0.3$		-0.0075	Spiral	Asymptotically stable
$c = 0.5$	$x_{21} = \frac{-0.2 \pm \sqrt{4.71}}{4}$	0.468	Saddle	Unstable
		-0.618	Spiral	Stable
$a = 0.01$	$x_{12} = \frac{-0.2 \pm \sqrt{-0.0796}}{4}$	-0.02	Spiral	Asymptotically stable
$b = 0.02$		-0.02	Spiral	Asymptotically stable
$c = 0.03$	$x_{21} = \frac{-0.2 \pm \sqrt{0.0796}}{4}$	0.066	Saddle	Unstable
		-0.076	Spiral	Stable
$a = 0.2$	$x_{12} = \frac{-0.2 \pm \sqrt{-1.56}}{4}$	-0.2	Spiral	Asymptotically stable
$b = 0.2$		-0.2	Spiral	Asymptotically stable
$c = 0.03$	$x_{21} = \frac{-0.2 \pm \sqrt{1.56}}{4}$	0.262	Saddle	Unstable
		-0.362	Spiral	Stable

Field work 2017.

3.3. Numerical Solution of Duffing’s Equation Using Mathcad

$$a := 0.1 \quad b := 0.2 \quad c := 0.1$$

$$t_0 := 0 \quad t_1 := 150 \text{ Solution interval endpoints;}$$

$$ic := \begin{bmatrix} 0 \\ 1 \end{bmatrix} \text{ Initial condition vector;}$$

$$N := 1500 \text{ Number of solution values on } [t_0, t_1].$$

$$D(t, X) := \begin{bmatrix} X_1 \\ -a \cdot X_0 - b \cdot (X_0)^2 - 2 \cdot (X_0)^3 - c \cdot X_1 \end{bmatrix} \text{ Derivative function.}$$

$$S := rkfixed(ic, t_0, t_1, N, D).$$

$$T := S^{(0)} \text{ Independent variable values.}$$

$$X_1 := S^{(1)} \text{ Solution function values.}$$

$$X_2 := S^{(2)} \text{ Derivative function values.}$$

3.4. Stability Analysis of Duffing Equation under Oyesanya and Nwamba (2013)

The unforced and damped cubic-quintic Duffing oscillator is given by

$$\ddot{u} + \delta \dot{u} + \alpha u + \beta u^3 + \mu u^5 = 0 \tag{1.32}$$

From Equation (1.32) we obtain the autonomous dynamical system

$$\left. \begin{aligned} y &= \dot{u} \\ \dot{y} &= -\delta \dot{u} - \alpha u - \beta u^3 - \mu u^5 \end{aligned} \right\} \tag{1.33}$$

From the condition $\dot{u} = 0$ we obtain $y = 0$ which gives us

$$\alpha u + \beta u^3 + \mu u^5 = 0 \tag{1.34}$$

Equation (1.34) can be written as

$$u[\alpha + \beta u^2 + \mu u^4] = 0 \quad (1.35)$$

We obtain the roots of (1.35) as

$$u_0 = 0, u_{1,2,3,4} = \pm \sqrt{\frac{1}{2\mu}[-\beta \pm \sqrt{\beta^2 - 4\mu\alpha}]}$$

Then the equation satisfied by the eigenvalues of our systems stability matrix is

$$\lambda^2 + \delta\lambda = -\alpha - 3\beta u^2 - 5\mu u^4 \quad (1.36)$$

where \underline{u} denotes $u_1 u_2 u_3$ and u_4 or the u -coordinate of an equilibrium point. Whether our eigenvalues will be complex, real or imaginary will be determined by the values of δ and $g = 5\mu u^4 + 3\beta u^2 + \alpha$ with $= 0$, $\lambda_{1,2} = \frac{1}{2}[\pm\sqrt{-(4g)}]$.

For this we consider the following cases:

1) $g = 0$: for this case $\lambda_{1,2} = 0$.

This contradicts our assumption that $\det J \neq 0$ and it also implies that α, β, μ are all zero.

2) $g > 0$: for this case $\lambda_{1,2} = \pm i\sqrt{g}$.

This corresponds to critical points that are centers for which stability is ensured.

3) $g < 0$: for this case $\lambda_{1,2} = \pm\sqrt{g}$ which corresponds to saddles giving rise to instability in (1.35) with $\delta > 0$,

$$\lambda_{1,2} = \frac{1}{2}[-\delta \pm \sqrt{\delta^2 - 4g}]$$

Then considering the case below we have:

1) $g = 0$: giving the values $\lambda_{1,2} = 0, -\delta$ which goes contrary to our assumption that $\det j \neq 0$. It also implies that α, β, μ are all zero.

2) $g > 0$: for this case

$$\lambda_{1,2} = \frac{1}{2}[-\delta \pm \sqrt{\delta^2 - 4g}]$$

Considering the discriminant for this case, we have three cases.

a) $\delta^2 < 4g$, $\lambda_{1,2} = \frac{1}{2}[-\delta \pm \sqrt{\delta^2 - 4g}]$ which corresponds to spirals and asymptotic stability.

b) The case $\delta^2 > 4g$ gives the values of the form $\lambda_{1,2} = \frac{1}{2}[-\delta \pm \sqrt{p}]$, $p > 0$ which corresponds to nodes resulting in asymptotic stability if $\delta > \sqrt{p}$ and to saddles and consequently instability if $\delta < \sqrt{p}$.

c) $\delta^2 = 4g$ gives $\lambda_{1,2} = \pm\sqrt{g}$. For this case we have saddles and hence instability.

3) $g < 0$ leads us to have $\lambda_{1,2} = \frac{1}{2}[\delta \pm \sqrt{\delta^2 - 4g}]$.

Considering the discriminate we consider the three cases as follows:

a) $\delta^2 + 4g < 0$: this case gives $\lambda_{1,2} = \frac{1}{2}[\delta \pm i\sqrt{\delta^2 - 4g}]$ and we have spirals and consequently asymptotic stability.

b) $\delta^2 + 4g > 0$ gives $\lambda_{1,2} = \frac{1}{2}[\delta \pm i\sqrt{Q}]$, $Q > 0$ and this leads us to the existence of node and instability if $\delta > \sqrt{Q}$ or saddles and instability if $\delta < \sqrt{Q}$.

c) $\delta^2 + 4g = 0$, $\lambda_{1,2} = \pm[i\sqrt{g}]$ which yield centers and stability.

We now consider the stability analysis of the dynamics for a few choices of α, β, μ and δ by using employing Equations (1.35) and (1.36). These are illustrated in **Table 3**.

3.5. Stability Analysis of Torres (2004)

We consider the Duffing's equation of the form

$$\ddot{x} + c\dot{x} + g(t, x) = 0 \tag{1.37}$$

with $c > 0$ and $g : [0, 2\pi] \times R \rightarrow R$ is a L^1 -caratheodory function such that the partial derivative g_x exists and it is:

L^1 -caratheodory. For a given $1 \leq p \leq +\infty$, let us define the set

$$\Omega_{p,c} = \left\{ a \in L^p(0, 2\pi) : a > 0, \|a\|_p < \left(1 + \frac{c^2}{4}\right)k(2p^*) \right\}$$

with p^*, k defined. Let φ be a 2π periodic solution of Equation (1.37), we

Table 3. Stability analysis of the dynamics of the oscillator for few choices of parameter α, β and μ .

$\alpha = \beta = \mu = 1$	$U_{1,2} = \pm\sqrt{\frac{-1+i\sqrt{3}}{2}}$	-1	Spirals	Asymptotically stable
		-1	Spirals	Unstable
	$U_{3,4} = \pm\sqrt{\frac{-1-i\sqrt{3}}{2}}$	1	Spirals	Unstable
		-1		
$\alpha = 1, \beta = \mu = -1$	$U_{1,2} = \pm\sqrt{\frac{-1-i\sqrt{5}}{2}}$	-1	Saddles	Unstable
		-1	Saddles	Unstable
	$U_{3,4} = \pm\sqrt{\frac{-1+i\sqrt{5}}{2}}$	1	Saddles	Unstable
		-1	Saddles	Unstable
$\alpha = 1, \beta = -1, \mu = 0.01$	$U_{1,2} = \pm\sqrt{\frac{-1+\sqrt{0.96}}{0.02}}$	-1	Spirals	Asymptotically stable
		-1	Spirals	Unstable
	$U_{3,4} = \pm\sqrt{\frac{1-\sqrt{0.96}}{0.02}}$	1	Saddle	Unstable
		-1		Unstable
$\alpha = -1, \beta = \mu = 1$	$U_{1,2} = \pm\sqrt{\frac{-1+\sqrt{5}}{2}}$	-1	Spirals	Asymptotically stable
		-1	Spirals	Unstable
	$U_{3,4} = \pm\sqrt{\frac{-1-\sqrt{5}}{2}}$	1	Spirals	Asymptotically stable
		-1	Spirals	Unstable
$\alpha = -1, \beta = 1, \mu = -0.02$	$U_{1,2} = \pm\sqrt{\frac{1-\sqrt{0.92}}{0.04}}$	-1	Spirals	Asymptotically stable
		-1	Saddles	Unstable
		1	Saddles	Unstable
		-1	Saddles	Unstable

assume that $a \in \Omega_{p,c}$ exists such that

$$g_x(t, \varphi(x)) < a(t) \text{ for a.e } t \in [0, 2\pi] \quad (1.38)$$

For such an a , the operator L as defined in maximum principle is inversely positive. By defining the operator.

$$F : C[0, 2\pi] \rightarrow L^1(0, 2\pi), \quad H : L^1(0, 2\pi) \rightarrow W$$

$$G_x = a(x) \rightarrow g(x) \quad H = L^{-1}F.$$

The solution φ can be seen as a fixed point of the operator H . Next, we recall that an isolated 2π -periodic solution has a well-defined index $\gamma(\varphi)$ and $|\gamma(\varphi) \leq 1|$.

4. Discussion

Figure 1 is the MATCAD showing the behavior of the Duffing's equation when $a = 0.1, b = 0.2$ and $c = 0.1$ is periodic. We observe asymptotically stable at both saddle and spiral point *i.e.* $\left(\frac{-0.2 \pm \sqrt{-0.76}}{4}, 0\right)$ which are three equilibrium points observed. This situation is seen in Duffing's equation.

Figure 2 shows the dynamics of Duffing's equation when $a = 0.6, b = 0.3, c = 0.5$. We observed asymptotically stable at spiral point *i.e.* $\left(\frac{-0.3 \pm \sqrt{-4.71}}{4}, 0\right)$. This shows that the phase is revolving round it. At (0.468) the point was saddle showing that it is unstable.

Figure 3: the MATCAD shows the dynamics of the Duffing's equation when $a = 0.01, b = 0.02, c = 0.03$. Three equilibrium points was observed. Saddles were obtained at $(-0.02, 0.066)$ showing that the phases are toward the origin.

Figure 4: the MATCAD were obtained for the values of $a = -0.2, b = 0.2, c = 0.03$. In this case it is important to note that the phase line tend to converge toward the equilibrium points.

Figure 5: The trajectory profile of Duffing's equation were obtained for the values $a = -0.2, b = 0.2, c = 0.03$. In this case, there is an increase in oscillation which is as a result of decrease in the damping coefficient. That is decrease in damping leads to increase in oscillation.

Figure 6: The phase portrait of Duffing's equation was obtained. We observed that the phase line was depicting a center of non-stable node.

Figure 7: The trajectory profile of Duffing's equation were obtained for the values $a = 0.6, b = 0.3, c = 0.5$. In this case, there is a decrease in the maximum displacement from the origin which leads to decrease in oscillation. The decrease in oscillation is as a result of increase in the damping coefficient. That is increase in damping leads to decrease in oscillation.

Figure 8: In this case, the phase portrait was depicting the center as an unstable node.

5. Conclusion

The study of the stability analysis of periodic solution using Lyapunov direct

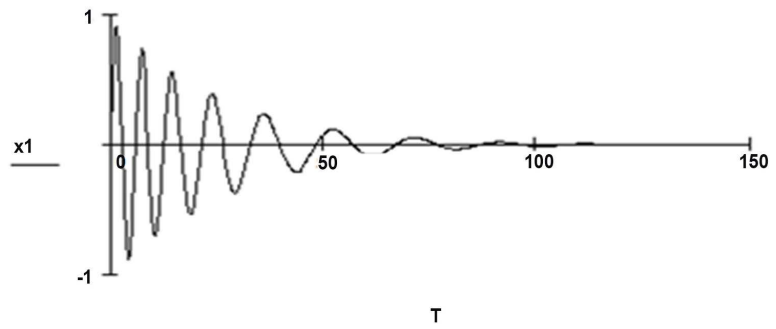


Figure 1. Trajectory profile of Duffing equation for values $a = 0.1, b = 0.2, c = 0.1$.

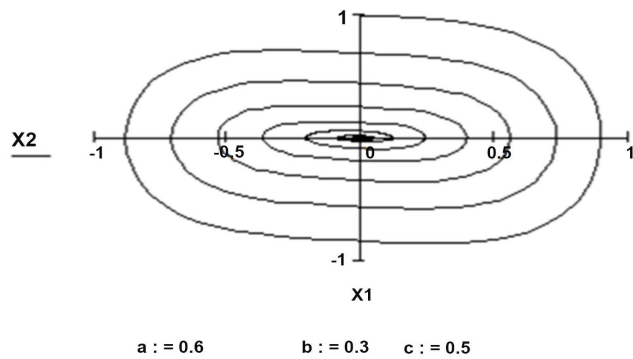


Figure 2. Phase portrait of Duffing's equation showing asymptotic stability of solution as a spiral sink.

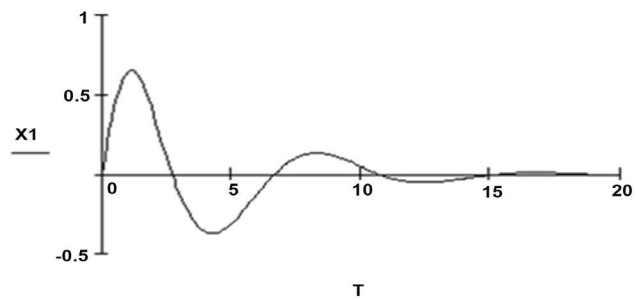


Figure 3. Trajectory profile of Duffing equation for values $a = 0.6, b = 0.3, c = 0.5$.

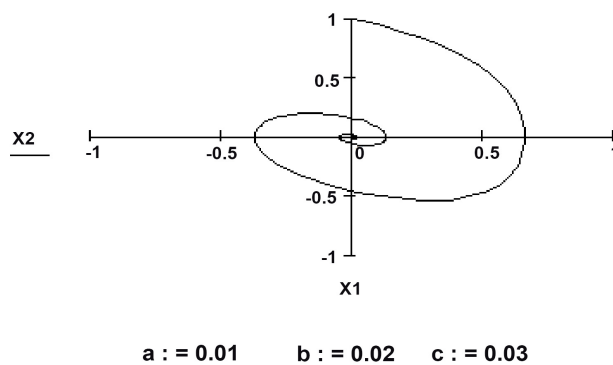


Figure 4. Phase portrait of Duffing's equation depicting asymptotic stability of solution as a spiral sink.

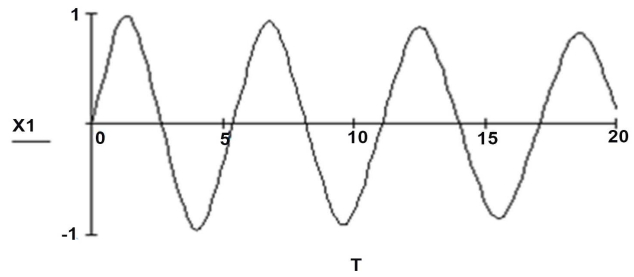


Figure 5. Oscillatory profile of Duffing equation for parameters $a = -0.2, b = 0.2, c = 0.03$.

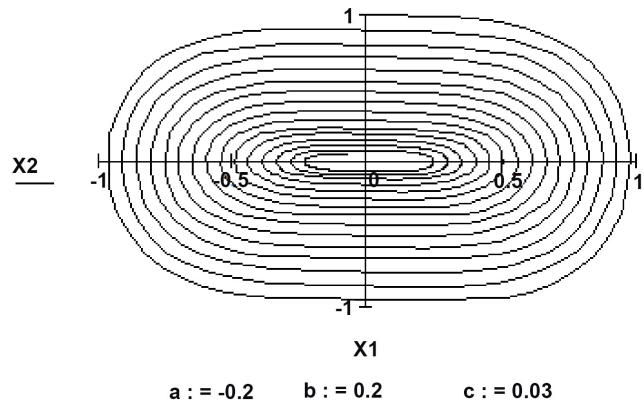


Figure 6. Phase portrait of Duffing's equation depicting a centre of a non-stable node.

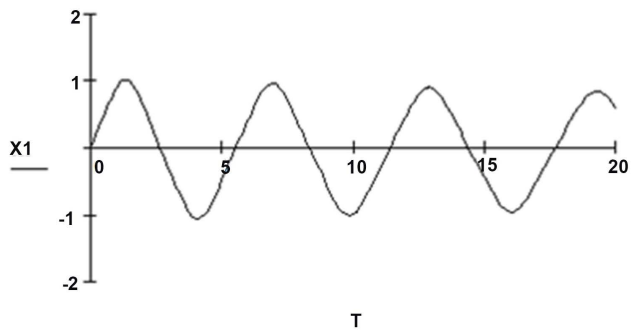


Figure 7. Oscillatory profile of Duffing equation for values $a = 0.6, b = 0.3, c = 0.5$.

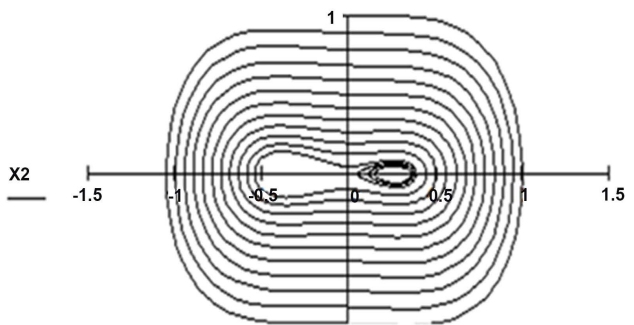


Figure 8. Phase portrait depicting the centre as an unstable node (parameters).

method and Matcad has yielded successful results. We discovered that global stability of periodic solution was achieved through the construction of a suitable and complete Lyapunov function for the hard spring system using Cartwright method. From the above tables discussed and outlined, we could see that the stability behaviours of the solutions are similar despite the different methods. Our technique showed superiority above others because stability is a property of the equilibrium point and of the system.

Conflicts of Interest

The authors declare no conflicts of interest regarding the publication of this paper.

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