Oscillation Criteria of second Order Non-Linear Differential Equations

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Abstract—In this paper we are concerned with the oscillation criteria of second order non-linear homogeneous differential equation. Example have been given to illustrate the results.

Keywords-component; Oscillatory, Second order differential equations, Non-Linear.

1. Introduction

The purpose of this paper is to establish a new oscillation criteria for the second order non-linear differential equation with variable coefficients of the form

$$x'' + f(x(t))(x'(t))^2 + g(x(t)) = 0, for\ t \geq t_0 \quad (1)$$

where $t_0 \geq 0$ is a fixed real number and $f(x)$ and $g(x)$ are continuously differentiable functions on the interval $[t_0, \infty)$.

The most studied equations are those equivalent to second order differential equations of the form

$$x'' + h(x) = 0, \quad (2)$$

where $h(x)>0$ is a continuously differentiable functions on the interval $[t_0, \infty)$ .Oscillation criteria for the second order nonlinear differential equations have been extensively investigated by authors(for example see[2], [3],[4],[5], [6],[8], [9] and the authors there in). Where the study is done by reducing the problem to the estimate of suitable first integral.

Definition1: A solution $x(t)$ of the differential equation (2) is said to be "nontrivial " if $x(t) \neq 0$ for at least one $t \in [t_0, \infty)$

Definition2: A nontrivial solution $x(t)$ of the differential equation (2) is said to be oscillatory if it has arbitrarily large zeros on $[t_0, \infty)$, otherwise it said to be " non oscillatory ".

Definition3: We say that the differential equation (1) oscillatory if an equivalent differential equation (2) is oscillatory.

2. Main Results

In [7] the author considered a class of systems equivalent to the second order non-linear differential equation (1). The standard equivalent system

$$x'(t) = y(t)$$
$$y'(t) = -g(x(t)) - f(x(t))(y(t))^2, \quad (3)$$

while he worked on a wider class of systems of the form

$$x'(t) = y(t)\alpha(t)$$
$$y'(t) = -\beta(x(t)) - \xi(x(t))(y(t))^2. \quad (4)$$

If $\alpha(t)>0$ then (4) is equivalent to a differential equation of the type (1). This allows to choose a modified system in order to be able to cope with different problems related to (2).Taking

$\xi(x(t)) \equiv 0, \alpha(x(t)) = e^{-F(x)}$ and $\beta(x) = g(x)e^{-F(x)}$,where

$$F(x) = \int_0^{x(t)} f(s))\, ds \; .$$

One obtains

$$x'(t) = e^{-F(x)}y(t)$$
$$y'(t) = -g(x(t))e^{F(x(t))}. \quad (5)$$

System (5) cab be transformed into

$$x'(t) = y(t)$$
$$y'(t) = -h(x(t)), \quad (6)$$

where

$h(x(t)) = g(x(t))e^{F(x(t))}$, which is equivalent to (2) where sufficient conditions for solutions of differential equation (1) to oscillate are given.

Remark: Assume that $f(x(t))$ and $g(x(t)) \in C^1(I, R)$ $where\ I = [t_0, \infty)\ with(\ possible\ I = R)$

Let us set

$$F(x(t)) = \int_0^{x(t)} f(v)dv, \qquad \phi(x(t)) = \int e^{F(s)} ds.$$

Since $\phi'(x(t)) > 0$, for all $t \in I.\ then\ \phi(x(t))$ is invertible on I, we define the transformation $u=\phi(x(t))$, acting on I. Accordion to Lemma1 in [7] any solution x(t) of (3) is a solution is a solution of (6).

Theorem1: Let $h(x)$ be continuous and continuously differentiable on $(-\infty,0) \cup (0, \infty)$ with $uh(u) \geq 0$ and let

$$\int_{x(t)}^{\infty} \frac{du}{h(u)} < \infty; \int_{-x(t)}^{-\infty} \frac{du}{h(u)} < \infty \ for\ every\ x(t) > 0 \quad , (7)$$

then any solution of the differential equation (2) is either oscillatory or tends monotonically to zero as $t \to \infty$.

Proof: Suppose that $x(t)$ is non-oscillatory solution of (2), and assume $x(t)>0$ for some $[t_0, \infty)$.

From (2) we get

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$$tx'' = -th(x) \;\; or \;\; ((tx'')/(h(x))) = -t.$$

Put

$$u = \left(\frac{t}{h(x)}\right); \;\; dv = x''$$

$$du = ((1/h) - tx'((h')/(h^2)))dt.$$

Then

$$\frac{tx'}{h(x)} - \frac{t_0 x'(t_0)}{h(x(t_0))} = \int_{t_0}^{t} \frac{x'(s)}{h(x(s))} ds$$

$$-\int_{t_0}^{t} \frac{s(x'(s)^2)h'(x(s))}{(h(x(s)))^2} ds - \int_{t_0}^{t} s ds \quad ,$$

then

$$\frac{tx'}{h(x)} - \frac{t_0 x'(t_0)}{h(x(t_0))} = \int_{t_0}^{t} \frac{x'(s)}{h(x(s))} ds$$

$$-\int_{t_0}^{t} \frac{s(x'(s)^2)h'(x(s))}{(h(x(s)))^2} ds - \frac{1}{2}(t^2 - t_0^2) + \frac{t_0 x'(t_0)}{h(x(t_0))}$$

Since $h'(x(t))>0$ then

$$\frac{tx'}{h(x)} \leq \int_{x(t_0)}^{x(t)} \frac{du}{h(u)} - \frac{1}{2}(t^2 - t_0^2) + \frac{t_0 x'(t_0)}{h(x(t_0))}$$

By hypothesis (7)we have

$$\frac{tx'}{h(x)} \to -\infty \; as \; t \to \infty$$

this means we obtain for some constant $k>0$

$$\frac{tx'}{h(x)} \leq -k \Rightarrow \frac{x'(t)}{h(x(t))} \leq -\frac{k}{t}.$$

Integrating from t_0 to t for $t_0>0$ we get

$$\int_{x(t_0)}^{x(t)} \frac{du}{h(u)} \leq ln\left(\frac{t_0}{t}\right)^k. \tag{8}$$

The right hand side is negative, since $x(t_0)>0$, $x(t)$ is positive. From (8) we conclude

$$\lim_{t\to\infty} x(t) = 0.$$

Thus $x(t)$ is oscillatory or tends monotonically to zero as $t \to \infty$

Theorem2: If In addition to hypotheses (7) we assume that for some $x(t)>0$

$$\lim_{\epsilon\to 0} \int_{\epsilon}^{x(t)} \frac{du}{h(u)} \leq \infty; \; \lim_{\epsilon\to 0^+} \int_{\epsilon}^{-x(t)} \frac{du}{h(u)} \leq \infty, \tag{9}$$

Then every solution $x(t)$ of the differential equation(2) is oscillatory.

Proof: As in theorem 1**,** we want to show that $x(t)$ doesn't tend monotonically to zero as $t \to \infty$.

Assume $x(t)>0$ for $a>0$ on $[t_0, \infty)$.

Since from (8) we have

$$\int_{x(a)}^{x(t)} \frac{du}{h(u)} \leq \ln\left(\frac{a}{t}\right)^k \;\; for \; t \geq a \geq t_0,$$

then there exists a positive real number m such that

$$\int_{x(a)}^{x(t)} \frac{du}{h(u)} > m > 0 \, . \quad ,$$

This means $\int_{x(a)}^{x(t)} \frac{du}{h(u)}$ is bounded below by a finite positive number, then by hypothesis (9), $x(t)$ doesn't tend monotonically to zero as $t \to \infty$.Then $x(t)$ is oscillatory.

Theorem3: Assume that $h(x)$ satisfies

$$i - \int_{1}^{\infty} \frac{du}{h(u)} < \infty \; ;$$

$$ii - \int_{-1}^{-\infty} \frac{du}{h(u)} < \infty \, ,$$

then every solution $x(t)$ of (2) is oscillatory.

Proof: Let $x(t)$ be non-oscillatory solution of (2), which without loss of generality, may be assumed to be positive for large t.

Define

$$w(t) = \frac{tx'(t)}{h(x)},$$

then

$$w'(t) = \frac{1}{h(x)}[x'(t) + tx''(t)] - \frac{th'(t)(x'(t))^2}{(h(x))^2}$$

or

$$= \frac{x'(t)}{h(x)} + t + \frac{h'(x)}{t}(w(t))^2 = 0. \tag{10}$$

Integrating (10) from α to t we get

$$w(t) = w(\alpha) + \int_{x(\alpha)}^{x(t)} \frac{du}{h(u)} + \int_{\alpha}^{t} s ds - \int_{\alpha}^{t} \frac{(w(s))^2 h'(x(s))}{s} ds$$

$$= w(\alpha) + \int_{x(\alpha)}^{x(t)} \frac{du}{h(u)} + \frac{1}{2}(t^2 - \alpha^2) + \int_{\alpha}^{t} \frac{(w(s))^2 h'(x(s))}{s} ds. \tag{11}$$

Since $\int_{x(\alpha)}^{x(t)} \frac{du}{h(u)} < \infty$ and $\frac{1}{2}(t^2 - \alpha^2) \to \infty$ from (11) we get $w(t)<0$ from which we get $x'(t)<0$ for large t which is a contradiction *(by lamma1II.I.8,[1])* where $x(t)>0$ *and then* $x'(t)>0$ *for large t.*

This completes the proof of the theorem.

EXAMPLES

Consider the second order nonlinear order differential

$$x'' + \frac{6x}{1+x^2}(x')^2 + (1+x^2)^{-1} = 0, \tag{12}$$

for this differential equation we have $f(x) = \frac{6x}{1+x^2}$ and $g(x) = (1+x^2)^{-1}$. Then the equivalent second order differential equation to (12) is

$$x'' + (1+x^2)^2 = 0, \tag{13}$$

where $h(x) = (1+x^2)^2$. To show the applicability of Theorem 1, the hypothesis is satisfied as follows

$$\int_{x(t)}^{\infty}\frac{du}{h(u)}=\lim_{b\to\infty}\int_{x(t)}^{b}\frac{2vdv}{(1+v^2)^2}=-\lim_{b\to\infty}[(1+v^2)^{-1}]_{x(t)}^{b}$$

$$=\frac{1}{1+(x(t))^2}<\infty.$$

Therefore the Theorem implies that the differential equation is oscillatory.

To show the applicability of Theorem 2 it is clear that the hypothesis is satisfied hence

$$\lim_{\epsilon\to 0^-}\int_{\epsilon}^{x(t)}\frac{du}{h(u)}=\lim_{\epsilon\to 0^-}\int_{\epsilon}^{x(t)}\frac{2vdv}{(1+v^2)^2}=-\lim_{\epsilon\to 0^-}[(1+v^2)^{-1}]_{\epsilon}^{x(t)}$$

$$-\lim_{\epsilon\to 0^-}[(1+(x(t))^2)^{-1}-(1+\epsilon^2)^{-1}]=\frac{(x(t))^2}{1+(x(t))^2}<\infty.$$

and

$$\lim_{\epsilon\to 0^+}\int_{\epsilon}^{x(t)}\frac{du}{h(u)}=\lim_{\epsilon\to 0^+}\int_{\epsilon}^{x(t)}\frac{2vdv}{(1+v^2)^2}=-\lim_{\epsilon\to 0^+}[(1+v^2)^{-1}]_{\epsilon}^{x(t)}$$

$$-\lim_{\epsilon\to 0^+}[(1+(x(t))^2)^{-1}-(1+\epsilon^2)^{-1}]=\frac{(x(t))^2}{1+(x(t))^2}<\infty.$$

Hence Theorem 2 is applicable.

To show the applicability of Theorem 3 the hypothesis is satisfied as follows

$$\text{i}-\int_{1}^{\infty}\frac{du}{h(u)}=\lim_{b\to\infty}\int_{1}^{b}\frac{2vdv}{(1+v^2)^2}=-\lim_{b\to\infty}[(1+v^2)^{-1}]_{1}^{b}$$

$$=ln\left(\frac{1}{2}\right)<\infty.$$

And

$$\text{ii}-\int_{-1}^{-\infty}\frac{du}{h(u)}=\lim_{b\to-\infty}\int_{-1}^{b}\frac{2vdv}{(1+v^2)^2}=-\lim_{b\to\infty}[(1+v^2)^{-1}]_{-1}^{b}$$

$$=ln\left(\frac{1}{2}\right)<\infty.$$

Hence Theorem 3 is applicable.

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