An Algorithm for Traffic Equilibrium Flow with Capacity Constraints of Arcs

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Abstract

In the traffic equilibrium problem, we introduce capacity constraints of arcs, extend Beckmann's formula to include these constraints, and give an algorithm for traffic equilibrium flows with capacity constraints on arcs. Using an example, we illustrate the application of the algorithm and show that Beckmann's formula is a sufficient condition only, not a necessary condition, for traffic equilibrium with capacity constraints of arcs.

Keywords

The Traffic Equilibrium Problem with Capacity Constraints of Arcs, Equilibrium Flow, Algorithm, Capacity of Arc, Saturated Path

1. Introduction

In [1], Wardrop introduced the traffic equilibrium problem and proposed a scalar equilibrium principle. In [2], Beckmann et al. gave a mathematical programming problem that was equivalent to Wardrop’s traffic equilibrium problem. Using Beckmann’s work, it is possible to find the traffic equilibrium flow if the cost function is a scalar. In [3], Chen and Yen generalized Wardrop’s equilibrium principle to a (weak) vector equilibrium principle. In [4] [5], we extended the vector equilibrium principle to capacity constraints along arcs and derived existence and stability results for (weak) vector equilibrium flows. In this paper, we introduce the traffic equilibrium problem with capacity constraints of arcs (TEPCCA), extend Beckmann’s transformation to cover capacity constraints along arcs, and give an algorithm for traffic equilibrium flows with capacity constraints of arcs for scalar cost functions. As an example, we illustrate the algorithm and show that Beckmann’s transformation is a sufficient condition only, not a necessary condition, for traffic equilibria with capacity constraints of arcs. For other results with respect to traffic equilibrium with capacity constraints of arcs, we refer to [6], and for other results with respect to algorithms of equilibrium flows, we refer to [7]-[9] and the references therein.

For a traffic network, let \( V \) denote the set of nodes, \( E \) the set of directed arcs, and \( W \) the set of origin-destination O-D pairs. For each \( \omega \in W \), let \( P_\omega \) denote the set of available paths joining O-D pair \( \omega \) and denote by \( K = \bigcup_{\omega \in W} P_\omega, m = |K| \). Let \( D = (d_\omega)_{\omega \in W} \) denote the demand vector, with \( d_\omega > 0 \) denoting the traffic demand on O-D pair \( \omega \). For each \( \omega \in W \) and \( k \in P_\omega \), let \( f_k \geq 0 \) denote the traffic flow on path \( k \). The flow vector \( f \) is said to be a feasible path flow (flow). Clearly, for each \( a \in E \), the amount of flow on arc \( a \) needs to satisfy the capacity constraint: \( c_a \geq f_a \geq 0 \). For each \( \omega \in W \), the flow \( f \) needs to satisfy the demand constraint: \( d_\omega \) is the traffic demand on O-D pair \( \omega \). A traffic network is usually denoted by \( \{V, E, W, D, C\} \).

2. Preliminaries

For the following definitions, see [4] [5].

**Definition 2.1.** Assume that a flow \( f \in A \).
1) for each \( a \in E \), if \( f_a = c_a \), then \( a \) is said to be a saturated arc of flow \( f \), otherwise a nonsaturated arc of flow \( f \).
2) for each \( k \in \bigcup_{\omega \in W} P_\omega \), if there exists a saturated arc \( a \) of flow \( f \) such that \( a \) belongs to path \( k \), then \( k \) is said to be a saturated path of flow \( f \), otherwise a nonsaturated path of flow \( f \).

**Definition 2.2.** (Equilibrium principle with capacity constraints of arcs). A flow \( f \in A \) is said to be in equilibrium if:
\[
(\forall \omega \in W, k \in P_\omega) f_k = d_\omega, \quad \forall \omega \in W, k \in P_\omega, \quad f_k \geq 0, \quad \forall \omega \in W, k \in P_\omega.
\]
In this paper, we assume that for each \( \omega \in W \), the demand \( d_\omega \) is fixed and \( A \neq \emptyset \). It is easy to verify that \( A \) is convex and compact. For each \( \omega \in W \), the flow on arc \( a \) needs to satisfy the capacity constraint: \( c_a \geq f_a \). For each \( a \in E \), the flow on arc \( a \) needs to satisfy the demand constraint: \( d_\omega \) is the traffic demand on O-D pair \( \omega \). A flow \( f \) satisfying the demand and capacity constraints is called a feasible path flow (a feasible flow for short). Let \( \Gamma = \{N, A, t\} \).

3. A Generalization of Beckmann’s Formula

For the TEPCCA \( \Gamma = \{N, A, t\} \), construct the following mathematical programming problem \( Q \):

\[
\text{Min } z(f) = \sum_{a \in E} \int_{0}^{t_a} t_a(x) \, dx
\]

\[
\sum_{k \in P_\omega} f_k = d_\omega, \quad \forall \omega \in W, k \in P_\omega
\]

\[
f_a = \sum_{k \in P_\omega} f_k c_a \leq c_a, \quad \forall a \in E, \omega \in W, k \in P_\omega
\]

\[
f_k \geq 0, \quad \forall \omega \in W, k \in P_\omega.
\]

The above formula is a generalization of Beckmann’s formula. The next theorem shows that each solution of the generalization of Beckmann’s formula is an equilibrium flow for \( \Gamma \).

**Theorem 3.1.** Consider the TEPCCA. Assume that for each \( a \in E \), \( t_a(f) \) is continuous on \( R^n \), then the flow \( f \in A \) is in equilibrium if \( f \) solves the mathematical programming problem \( Q \).

**Proof.** Set \( h_\omega = \sum f_k - d_\omega \) and \( g_a = c_a - f_a \). The Kuhn-Tucker conditions for the problem \( Q \) are:
\[
\frac{\partial z}{\partial f_k} - \sum_{a} \rho_a \frac{\partial h_a}{\partial f_k} - \sum_{a} \lambda_a \frac{\partial g_{a}}{\partial f_k} - \beta_k = 0, \quad \forall \omega \in W, k \in P_o
\]

where \( \rho_a, \lambda_a \) and \( \beta_k \) are Lagrange multipliers. Since for each \( a \in E \), \( t_a(f) \) is continuous on \( R^n \), we have

\[
\frac{\partial z}{\partial f_k} = \frac{\partial}{\partial f_k} \left( \sum_{a} t_a(x) \right) = \sum_{a} \frac{\partial}{\partial f_k} t_a(x) = \sum_{a} \frac{\partial h_a}{\partial f_k} = \sum_{a} \frac{\partial g_{a}}{\partial f_k} = \beta_k - \sum_{a} \rho_a \frac{\partial h_a}{\partial f_k} = \rho_a.
\]

When path \( k \) is a nonsaturated path of flow \( f \), for each \( a \in k \), we have \( c_a - f_a > 0 \). Note that \( \lambda_a (c_a - f_a) = 0 \), we have \( \lambda_a = 0 \). Thus,

\[
\sum_{a} \frac{\partial g_{a}}{\partial f_k} = \sum_{a} \frac{\partial h_a}{\partial f_k} = \sum_{a} \lambda_a = \begin{cases} 0 & \text{if path } k \text{ is a nonsaturated path of flow } f \\ \le 0 & \text{otherwise.} \end{cases}
\]

Hence, when \( k \) is a nonsaturated path, we have \( f_k (t_k - \rho_a) = 0 \), i.e.,

- if \( f_k > 0 \), \( t_k = \rho_a \) \( \forall \omega \in W, k \in P_o \)
- if \( f_k = 0 \), \( t_k \ge \rho_a \) \( \forall \omega \in W, k \in P_o \)

and when \( k \) is a saturated path, we have \( f_k (t_k - \rho_a + \sum_{a} \lambda_a) = 0 \), i.e.,

- if \( f_k > 0 \), \( t_k (\ge 0) = \rho_a - \sum_{a} \lambda_a \le \rho_o \) \( \forall \omega \in W, k \in P_o \)
- if \( f_k = 0 \), \( t_k \ge 0 \) \( \forall \omega \in W, k \in P_o \)

In other words, if paths \( k \) is a nonsaturated path, then \( t_k \ge \rho_o \), and if paths \( k \) such that \( f_k > 0 \), then \( t_k \le \rho_o \). Thus, for \( \forall \omega \in W, \forall k, j \in P_o, t_k(f) - t_j(f) > 0 \) and \( j \) is a nonsaturated path, then \( f_k = 0 \), otherwise \( f_k > 0 \), which implies that \( t_k (f) \le \rho_o \le t_j (f) \), a contradiction. By Definition 2.2, the proof is finished.

From the generalization of Beckmann’s formula, it is easy to construct an algorithm to calculate the equilibrium flow for the TEPCCA \( \Gamma = \{N, A, t\} \).

4. An Algorithm for the Traffic Equilibrium Flow with Capacity Constraints of Arcs

For the TEPCCA \( \Gamma = \{N, A, t\} \), because there are usually many paths in \( K = \bigcup_{a \in P_o} \omega \), implying that there are many variable in the generalization of Beckmann’s formula, it is often difficult to compute its solution. Note that there are many paths for which the flow is zero in an equilibrium flow. If we delete these from \( K \), it does not cause any change in the equilibrium flow. For this season, we construct the following algorithm to compute the equilibrium flow with capacity constraints of arcs. Assume that for each \( a \in E \), \( t_a(f) \) is continuous on \( R^n \).

Step 1. Find a feasible flow \( f^0 \in A \) and denote by \( H^0 = \{l \in K : f_i^0 > 0\} \). Let \( i = 0 \).

Step 2. Solve the restricted problem \( \overline{Q} \):

\[
\begin{align*}
\text{Min} & \quad z(f) = \sum_{a} \int_{0}^{f_i} t_a(x) \, dx \\
\text{s.t.} & \quad f_a = \sum_{l \in a} t_l, \quad \forall \omega \in W, k \in H^i \\
& \quad f_i \ge 0, \quad \forall k \in H^i.
\end{align*}
\]

We obtain solution \( f^i \in A \). For each O-D pair \( \omega \in W \), denote by \( t^i_\omega = \max_{l \in H^i} \{t_l(f^i) : f_i > 0\} \), where \( t_l(f^i) \) denotes the cost of path \( l \) when flow is \( f^i \) on the network \( N \).

Step 3. After deleting all saturated arcs of the flow \( f^i \) in the network \( N \), we compute its shortest path for
each O-D pair. For each O-D pair \( \omega \in W \), let \( S^{(1)}_{\omega} = \{ l \in P_{\omega} : l \) is a shortest path for \( \omega \) and \( t_{\omega}(f_{\omega}) < t_{\omega}^{(1)} \} \).

Step 4. If \( S^{(1)}_{\omega} = \bigcup_{\omega \in W} S^{(1)}_{\omega} = \emptyset \), go to Step 5; otherwise let \( H^{(1)} = H^{(1)} \cup S^{(1)}_{\omega}, i = i + 1 \) and go to Step 2.

Step 5. The equilibrium flow is \( f^{(1)} \) for the TEPPCA and stop.

The following example shows the calculation process of the algorithm.

**Example 4.1.** Consider the TEPPCA (see Figure 1), where

\[
V = \{1, 2, 3, 4, 5, 6\}, \quad E = \{e_1, e_2, e_3, e_4, e_5, e_6, e_7, e_8, e_9\}, \quad C = (6, 7, 10, 8, 7, 5, 9, 7, 11, 7),
\]

\[
W = \{\omega_1, \omega_2\} = \{(1, 4), (3, 6)\}, \quad D = (d_{\omega_1}, d_{\omega_2}) = (9, 8),
\]

and

\[
t_{\omega_1}(f_{\omega_1}) = 4f_{\omega_1}^2 + 17, t_{\omega_2}(f_{\omega_2}) = 3f_{\omega_2}^2 + 18, t_{\omega_1}(f_{\omega_1}) = 30f_{\omega_1}^2 + 120, t_{\omega_2}(f_{\omega_2}) = 2f_{\omega_2}^2 + 84.
\]

For O-D pair \( \omega_1 = (1, 4) \): \( P_{\omega_1} \) contains paths \( l_1 = (e_1) \), \( l_2 = (e_1e_2) \), \( l_3 = (e_1e_3) \), and \( l_4 = (e_1e_4e_5) \), and for O-D pair \( \omega_2 = (3, 6) \): \( P_{\omega_2} \) contains paths \( l_1 = (e_9) \), \( l_2 = (e_9e_10) \), \( l_3 = (e_9e_11) \), and \( l_4 = (e_9e_11e_1) \).

Let \( f = (f_1, f_2, f_3, f_4, f_5, f_6, f_7, f_8)^T \in R^8 \), where \( f_j \) denotes the flow on path \( l_j, (j = 1, 2, 3, 4, 5, 6, 7, 8) \). Thus, we have

\[
f_{\omega_1} = f_{1} + f_{4}, f_{\omega_2} = f_{5} + f_{8}, f_{\omega_1} = f_{1} + f_{4}, f_{\omega_2} = f_{5} + f_{8},
\]

Next, we compute the equilibrium flow with capacity constraints of arcs.

1) It is easy to verify that

\[
f^0 = (f_1, f_2, f_3, f_4, f_5, f_6, f_7, f_8)^T = (9, 0, 0, 8, 0, 0, 0)^T \in A.
\]

Let \( i = 0 \).

2) Solve the restricted problem \( Q^0 \):

\[
\text{Min} z(f) = \sum_{\omega \in W} \int_0^{t_{\omega}} t_{\omega}(x)dx = 10f_1^2 + 120f_1 + \frac{1}{3}f_2^3 + 18f_5
\]

\[
\text{s.t.} \begin{cases}
    f_1 = 9 \\
    f_5 = 8
\end{cases}
\]

We obtain solution \( f^1 = f^0 \in A \). For O-D pair \( \omega_1 = (1, 4) \), \( t_{\omega_1}^1 = 2550 \), and for O-D pair \( \omega_2 = (3, 6) \), \( t_{\omega_2}^1 = 82 \).

![Figure 1. A traffic network.](image-url)
3) There is no saturated arc of flow $f^1$ in the network $\mathbb{N}$. For O-D pair $\omega_1 = (1,4)$, it is easy to verify that the shortest path is $l_4$, whereas for O-D pair $\omega_2 = (3,6)$, the shortest path is $l_6$. Note that $t_{l_6}(f^1) = 50 < r_{\omega_1}^2 = 2550, t_{l_6}(f^1) = 46 < r_{\omega_2}^2 = 82$.

Thus $S_{\omega_1}^1 = \{l_4\}, S_{\omega_2}^1 = \{l_6\}$.

4) Since $S^1 = S_{\omega_1}^1 \cup S_{\omega_2}^1 = \{l_4, l_6\} \neq \emptyset$, let $H^1 = H^0 \cup S^1 = \{l_1, l_4, l_5, l_9\}$ and solve the restricted problem $Q^1$:

$$\min z(f) = \sum_{a \in A} \int_0^{t_a} (x) \, dx = 10 f_1^3 + 120 f_4 + \frac{4}{3} f_4^3 + 17 f_4 + (f_4 + f_k)^2 + 18(f_4 + f_k)$$

$$+ \frac{1}{3} f_4 + 15 f_4 + \frac{1}{3} f_4^3 + 18 f_4 + f_k^3 + 18 f_4 + \frac{2}{3} f_k^3 + 10 f_k$$

$$= 10 f_1^3 + 120 f_4 + \frac{5}{3} f_4^3 + 50 f_4 + (f_4 + f_k)^2 + \frac{1}{3} f_4^3 + 18 f_4 + \frac{5}{3} f_k^3 + 46 f_k$$

s.t.

$$\begin{align*}
  f_1 + f_4 &= 9 \\
  f_4 + f_k &= 8 \\
  f_4 + f_k &\leq 5 \\
  10 &\geq f_1 \geq 0, 6 \geq f_4 \geq 0, 9 \geq f_k \geq 0, 7 \geq f_k \geq 0
\end{align*}$$

We obtain solution $f^1 = (4,0,0,5,8,0,0,0)^T \in A$. For O-D pair $\omega_1 = (1,4)$, $r_{\omega_1} = 600$, and for O-D pair $\omega_2 = (3,6)$, $r_{\omega_2} = 82$.

5) After deleting saturated arc $e_k$ of flow $f^2$ in the network $\mathbb{N}$. For O-D pair $\omega_1 = (1,4)$, it is easy to verify that the shortest path is $l_4$, whereas for O-D pair $\omega_2 = (3,6)$, the shortest path is $l_5$. Note that $t_{l_5}(f^2) = 124 < r_{\omega_1}^2 = 600, t_{l_5}(f^2) = 82 = r_{\omega_2}^2 = 82$.

Thus $S_{\omega_1}^2 = \{l_4\}, S_{\omega_2}^2 = \emptyset$.

6) Since $S^2 = S_{\omega_1}^2 \cup S_{\omega_2}^2 = \{l_4\} \neq \emptyset$, let $H^2 = H^1 \cup S^2 = \{l_1, l_4, l_5, l_9\}$ and solve the restricted problem $Q^2$:

$$\min z(f) = \sum_{a \in A} \int_0^{t_a} (x) \, dx = 10 f_1^3 + 120 f_4 + \frac{4}{3} f_4^3 + 17 f_4 + (f_4 + f_k)^2 + 18(f_4 + f_k)$$

$$+ \frac{1}{3} (f_1 + f_4)^3 + 15 (f_2 + f_4) + \frac{2}{3} f_4^3 + 84 f_4 + \frac{1}{3} f_4^3 + 18 f_4 + f_k^3 + 18 f_4 + \frac{2}{3} f_k^3 + 10 f_k$$

$$= 10 f_1^3 + 120 f_4 + \frac{4}{3} f_4^3 + 50 f_4 + (f_4 + f_k)^2 + \frac{1}{3} (f_1 + f_4)^3$$

$$+ \frac{2}{3} f_4^3 + 99 f_4 + \frac{1}{3} f_4^3 + 18 f_4 + \frac{5}{3} f_k^3 + 46 f_k$$

s.t.

$$\begin{align*}
  f_1 + f_4 + f_k &= 9 \\
  f_4 + f_k &= 8 \\
  f_4 + f_k &\leq 5 \\
  10 &\geq f_1 \geq 0, 6 \geq f_4 \geq 0, 9 \geq f_k \geq 0, 7 \geq f_k \geq 0
\end{align*}$$

We obtain solution $f^2 = (1.44, 3.64, 0.392, 6.92, 0.0, 1.08)^T \in A$. For O-D pair $\omega_1 = (1,4)$, $r_{\omega_1} = 182.20$, and for O-D pair $\omega_2 = (3,6)$, $r_{\omega_2} = 65.89$.

7) After deleting saturated arc $e_k$ of flow $f^3$ in the network $\mathbb{N}$. For O-D pair $\omega_1 = (1,4)$, it is easy to verify that the shortest path is $l_5$, whereas for O-D pair $\omega_2 = (3,6)$, the shortest path is $l_5$. Note that $t_{l_5}(f^3) = 182.20 = r_{\omega_1}^3 = 182.20, t_{l_5}(f^3) = 65.89 = r_{\omega_2}^3 = 65.89$, thus $S_{\omega_1}^3 = \emptyset, S_{\omega_2}^3 = \emptyset$. 

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8) Because \( S^3 = S_{tol}^3 \cup S_{to2}^3 = \emptyset \), the equilibrium flow is \( f^3 = (1.44, 3.64, 0, 3.92, 6.92, 0, 0, 1.08)^T \), hence stop.

Note that

\[
\begin{align*}
\int_0^{f_{t_1}} x \, dx &= \int_0^{f_{t_1}} (4x^2 + 17) \, dx = \frac{4}{3} f_{t_1}^3 + 17 f_{t_1} = \frac{4}{3} (f_1 + f_4)^3 + 17 (f_5 + f_6), \\
\int_0^{f_{t_2}} x \, dx &= \int_0^{f_{t_2}} (3x^2 + 18) \, dx = f_{t_2}^3 + 18 f_{t_2} = (f_1 + f_4)^3 + 18 (f_5 + f_6), \\
\int_0^{f_{t_3}} x \, dx &= \int_0^{f_{t_3}} (30x^2 + 120) \, dx = 10 f_{t_3}^3 + 120 f_{t_3} = 10 (f_5 + f_6), \\
\int_0^{f_{t_4}} x \, dx &= \int_0^{f_{t_4}} (2x^2 + 84) \, dx = \frac{2}{3} f_{t_4}^3 + 84 f_{t_4} = \frac{2}{3} f_5^3 + 84 f_5, \\
\int_0^{f_{t_5}} x \, dx &= \int_0^{f_{t_5}} (x^3 + 112) \, dx = \frac{1}{3} f_{t_5}^3 + 112 f_{t_5} = \frac{1}{3} f_5^3 + 112 f_5, \\
\int_0^{f_{t_6}} x \, dx &= \int_0^{f_{t_6}} (2x + 18) \, dx = f_{t_6}^3 + 18 f_{t_6} = (f_1 + f_4)^3 + 18 (f_5 + f_6), \\
\int_0^{f_{t_7}} x \, dx &= \int_0^{f_{t_7}} (8x^2 + 62) \, dx = \frac{8}{3} f_{t_7}^3 + 62 f_{t_7} = \frac{8}{3} f_6^3 + 62 f_6, \\
\int_0^{f_{t_8}} x \, dx &= \int_0^{f_{t_8}} (6x^2 + 65) \, dx = 2 f_{t_8}^3 + 65 f_{t_8} = 2 f_5^3 + 65 f_5, \\
\int_0^{f_{t_9}} x \, dx &= \int_0^{f_{t_9}} (x^3 + 18) \, dx = \frac{1}{3} f_{t_9}^3 + 18 f_{t_9} = \frac{1}{3} f_5^3 + 18 f_5, \\
\int_0^{f_{t_{10}}} x \, dx &= \int_0^{f_{t_{10}}} (x^2 + 15) \, dx = \frac{1}{3} f_{t_{10}}^3 + 15 f_{t_{10}} = \frac{1}{3} (f_1 + f_4)^3 + 15 (f_5 + f_6), \\
\int_0^{f_{t_{11}}} x \, dx &= \int_0^{f_{t_{11}}} (2x^2 + 10) \, dx = \frac{2}{3} f_{t_{11}}^3 + 10 f_{t_{11}} = \frac{2}{3} (f_1 + f_4)^3 + 10 (f_5 + f_6).
\end{align*}
\]

Thus the generalization of Beckmann’s formula \( Q \) is:

\[
\text{Min} z(f) = \frac{4}{3} (f_1 + f_4)^3 + 17 (f_5 + f_6) + (f_1 + f_4)^3 + 18 (f_5 + f_6) + 10 (f_5 + f_6) + 120 f_1 +
\frac{2}{3} f_5^3 + 84 f_5 + \frac{1}{3} f_5^3 + 112 f_5 + (f_4 + f_6)^2 + 18 (f_4 + f_6) + \frac{8}{3} f_6^3 + 62 f_6

+ 2 f_5^3 + 65 f_5 + \frac{1}{3} f_5^3 + 18 f_5 + \frac{1}{3} (f_2 + f_4)^3 + 15 (f_2 + f_4) + \frac{2}{3} (f_6 + f_8)^3 + 10 (f_6 + f_8))
\]

\[
\begin{align*}
f_1 + f_2 + f_3 + f_4 &= 9 \\
f_5 + f_6 + f_7 + f_8 &= 8 \\
f_1 + f_4 &\leq 11 \\
f_5 + f_8 &\leq 6 \\
\text{s.t.} \\
f_1 + f_5 &\leq 7 \\
f_2 + f_6 &\leq 7 \\
f_3 + f_7 &\leq 5 \\
10 &\geq f_1 \geq 0.8, 8 \geq f_2 \geq 0.6, f_3 \geq 0.5, f_4 \geq 0, \\
9 &\geq f_5 \geq 0.7, f_6 \geq 0.7, f_7 \geq 0.5, f_8 \geq 0.
\end{align*}
\]

It is easy to verify that \( f = (1.44, 3.64, 0, 3.92, 6.92, 0, 0, 1.08)^T \) is the solution of the generalization of Beckmann’s formula \( Q \) (\( \text{Min} z(f) = 1327.31 \)) . Clearly, \( f \) is an equilibrium flow for the TEPCCA.

Note that \( g = (1.46, 3.74, 0, 3.80, 6.80, 0, 0, 1.20)^T \) is also an equilibrium flow for the TEPCCA, but it is not a solution of the generalization of Beckmann’s formula \( Q \), i.e., Theorem 3.1 is a sufficient condition only, not a necessary condition.
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