Support-Limited Generalized Uncertainty Relations on Fractional Fourier Transform

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Abstract
This paper investigates the generalized uncertainty principles of fractional Fourier transform (FRFT) for concentrated data in limited supports. The continuous and discrete generalized uncertainty relations, whose bounds are related to FRFT parameters and signal lengths, were derived in theory. These uncertainty principles disclose that the data in FRFT domains may have much higher concentration than that in traditional time-frequency domains, which will enrich the ensemble of generalized uncertainty principles.

Keywords
Discrete Fractional Fourier Transform (DFRFT), Uncertainty Principle, Frequency-Limiting Operator

1. Introduction
In information processing, the uncertainty principle plays an important role in elementary fields, and data concentration is often considered carefully via the uncertainty principle [1]-[8]. In continuous signals, the supports are assumed to be infinite, based on which various uncertainty relations [1] [2] [9]-[21] [22] have been presented. However, in practice, both the supports of time and frequency are often limited for N-point discrete signals. In such case, the infinite support fails to hold true. In limited supports, some papers such as [23]-[26] have discussed the uncertainty principle in conventional time-frequency domains for continuous and discrete cases and some conclusions are achieved that can be taken as our special cases in the following sections. However, none of them has covered the FRFT in terms of Heisenberg uncertainty principles that have been widely used in various

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fields [4]-[6]. Therefore, there has a great need to discuss the uncertainty relations in FRFT domains. As the rotation of the traditional FT [27], FRFT [5] [6] [28]-[30] has some special properties with its transform parameter and sometimes yields the better results such as the detection of LFM signal [31]. Readers can see more details on FRFT in [6] and [32] and so on.

In this paper, we extend the Heisenberg uncertainty principle in FRFT domain for both discrete and continuous cases for the ε-concentrated signals or the signals with finite supports. It is shown that these bounds are connected with lengths of the supports and FRFT parameters. In a word, there have been no reported papers covering these results and conclusions, and most of them are new or novel.

2. Preliminaries

2.1. Definition of DFRFT

Here, we first briefly review the definition of FRFT. For given continuous signal
\( x(t) \in L^1(R) \cap L^2(R) \) and
\( \|x(t)\|_2 = 1 \), its FRFT [6] is defined as

\[
X_\alpha(u) = F_\alpha(x(t)) = \int_{-\infty}^{\infty} x(t) K_\alpha(u,t) dt
\]

\[
= \begin{cases} 
  x(t) & \alpha = 2n\pi \\
  x(-t) & \alpha = (2n \pm 1)\pi 
\end{cases}
\]

where \( n \in \mathbb{Z} \) and \( i \) is the complex unit, \( \alpha \) is the transform parameter defined as that in [6]. In addition, \( F_\alpha F_\beta(x(t)) = F_{\alpha + \beta}(x(t)) \). If \( \alpha = -\beta \), \( F_\alpha F_\beta(x(t)) = x(t) \), i.e., the inverse FRFT
\( x(t) = \int_{-\infty}^{\infty} X_\alpha(u) K_{-\alpha}(u,t) du \).

However, unlike the discrete FT, there are a few definitions for the DFRFT [32], but not only one. In this paper, we will employ the definition defined as follows [6] [32]:

\[
\hat{x}(k) = \sum_{n=1}^{N} \sqrt{\frac{1-i \cot \alpha}{N}} \cdot e^{-\frac{i \cot \alpha}{2}} e^{\frac{i \cot \alpha}{2}} e^{\frac{i \cot \alpha}{2}} \hat{x}(n) = \sum_{n=1}^{N} u_\alpha (k,n) \cdot \hat{x}(n), \quad 1 \leq n,k \leq N.
\]

Clearly, if \( \alpha = \pi/2 \), (2) reduces to the traditional discrete FT [6] [32]. Also, we can rewrite definition (2) as

\[
\hat{X}_\alpha = U_\alpha \hat{X},
\]

where \( U_\alpha = [u_\alpha (k,n)]_{N \times N}, \ \hat{X} = [\hat{x}(n)]_{N \times 1} \).

For DFRFT, we have the following property [5] [6] [32]:

\[
\|\hat{X}_\alpha\|_2 = \|U_\alpha \hat{X}\|_2 = 1.
\]

More details on DFRFT can be found in [6] and [32].

2.2. Frequency-Limiting Operators

**Definition 1**: Let \( x(t) \) be a complex-valued signal with \( \|x(t)\|_{L^2(R)} = 1 \) and its FRFT \( X_\alpha(u) \), if there is a function \( G_\alpha(u) \) vanishing outside \( W_\alpha \) (\( W_\alpha \) is a measurable set) such that

\[
\|X_\alpha(u) - G_\alpha(u)\|_{L^2(R)} \leq \varepsilon_\alpha, \quad (\varepsilon_\alpha \text{ is a small value with } 0 < \varepsilon_\alpha < 1),
\]

then \( X_\alpha(u) \) is \( \varepsilon_\alpha \)-concentrated.

Specially, if \( \alpha = 0 \), then definition 1 reduces to the case in time domain [23] [24]. If \( \alpha = \pi/2 \), then definition 1 reduces to the case in traditional frequency domain [23] [24]. The \( \varepsilon_\alpha \) can be calculated after the \( W_\alpha \) is
fixed because \( G_a(u) \subseteq X_a(u) \) and \( \|X_a(u)\|_{L^2(K)} = \|x(t)\|_{L^2(K)} = 1 \). Therefore, \( \|X_a(u) - G_a(u)\|_{L^2(K)} \leq \varepsilon_a \).

**Definition 2:** Generalized frequency-limiting operator \( P_{a,x} \) is defined as

\[
(P_{a,x} x)(t) = \int_{\mathbb{R}} X_a(u) K_{a,x}(u,t) \, du, \quad X_a(u) = F_a(x(t)).
\]  

If \( \alpha = 0 \), then definition 2 is the time-limiting operator [23] [24]. If \( \alpha = \pi/2 \), then definition 2 is the traditional frequency-limiting operator [23] [24]. Definitions 1 and 2 disclose the relation between \( \varepsilon_a \) and \( W' \).

**Definition 3:** Let \( \tilde{x}(n) \in l^2(\mathbb{R}) \) \( (n = 1, \cdots, N) \) with be a discrete sequence with \( \|\tilde{x}(n)\|_{l^2(K)} = 1 \) and its DFRFT \( \hat{x}(k) \), if there is a sequence \( \hat{x}'(k) \) satisfying \( \|\hat{x}'(k)\|_{l^2(K)} = N_a \) \( (N_a \leq N) \) such that \( \|\hat{x}(k) - \hat{x}'(k)\|_{l^2(K)} \leq \varepsilon_a \) \( (\varepsilon_a \) is a small value with \( 0 < \varepsilon_a < 1) \), then \( \hat{x}(k) \) is \( \varepsilon_a \)-concentrated.

Here, \( \|\| \) is the 0-norm operator that counts the non-zero elements.

**Definition 4:** Generalized discrete frequency-limiting operator \( P_{N_a} \) is defined as

\[
(P_{N_a} x)(n) = \sum_{k=1}^{N_a} \chi_{N_a}(k) \hat{x}(k) u_{-\alpha}(k,n) \quad \text{with} \quad \hat{x}(k) \text{ is the DFRFT of } \tilde{x}(n) \text{ and } \chi_{N_a} \text{ is the character function on } N_a \text{.}
\]

Clearly, definitions 3 and 4 are the discrete extensions of definitions 1 and 2. They have the similar physical meaning. These definitions are introduced for the first time, the traditional cases [23] [24] are only their special cases. Definition 3 and 4 disclose the relation between \( \varepsilon_a \) and \( N_a \).

**2.3. The Continuous Heisenberg Uncertainty Principles**

As shown in introduction, the existed continuous generalized uncertainty relations [9]-[21] are mainly for the infinite supports. Here, we discuss the case of finite support. First we introduce the following lemma.

**Lemma 1:** \( \|P_{a,x} P_{\beta,y}\|_{\text{F}} \) = \( \sqrt{W_a \|W_\beta\| \sin(\alpha - \beta)} \), where \( \|\|_{\text{F}} \) denotes the Frobenius norm operator.

**Proof:** From the definition of the operator \( P_{a,x} P_{\beta,y} \) in definition 2, we have

\[
(P_{a,x} P_{\beta,y} x)(t) = \int_{\mathbb{R}} K_{a,x}(v,t) \left( \int_{\mathbb{R}} K_{\alpha-x}(v,u) X_{\beta}(u) \, du \right) \, dv.
\]

Exchange the locations of the integral operators, we obtain

\[
(P_{a,x} P_{\beta,y} x)(t) = \int_{\mathbb{R}} K_{a-x}(v,t) \left( \int_{\mathbb{R}} K_{\alpha-y}(v,u) X_{\beta}(u) \, du \right) \, dv,
\]

so that

\[
(P_{a,x} P_{\beta,y} x)(t) = \int_{\mathbb{R}} \left( \int_{\mathbb{R}} K_{a-x}(v,t) K_{\alpha-y}(v,u) \, dv \right) X_{\beta}(u) \, du.
\]

Set \( q(u,t) = \int_{\mathbb{R}} K_{a-x}(v,t) K_{\alpha-y}(v,u) \, dv, \quad u \in W_\beta \), we have

\[
(P_{a,x} P_{\beta,y} x)(t) = \int_{W_\beta} q(u,t) X_{\beta}(u) \, du.
\]

Now, we know that [see the proof of (3.1) in 25]

\[
\|P_{a,x} P_{\beta,y}\|_{\text{F}} = \int_{W_\beta} \|q(u,t)\|^2 \, du.
\]
Let \( g_u(t) = q(u,t) \), then
\[
F_{\alpha}(g_u(t)) = \int_{-\infty}^{\infty} K_{\alpha}(u,t) \left( \int_{-\infty}^{\infty} K_{\alpha}(v,t) K_{\alpha}(v,u) \right) dv dt
\]
\[
= \int_{-\infty}^{\infty} K_{\alpha}(v,u) \left( \int_{-\infty}^{\infty} K_{\alpha}(v,t) K_{\alpha}(u,t) \right) dv
\]
where \( \chi_w \) is the character function of the set \( W \). Therefore, via Parseval’s theorem \([6]\) and the definition of FRFT in (1) we have
\[
\int_{-\infty}^{\infty} |g_u(t)|^2 dt = \int_{-\infty}^{\infty} |F_u(g_u(t))|^2 du = \int_{-\infty}^{\infty} \chi_w \cdot K_{\alpha}(v,u) \left( \frac{|W_w|}{\sin(\alpha - \beta)} \right)
\]
Hence, we obtain the final result
\[
\left\| P_{\alpha} P_{\beta} \right\|_{L^2} = \int_{-\infty}^{\infty} \left\| q(u,t) \right\|^2 du = \int_{-\infty}^{\infty} \left\| \chi_w \right\| \left( \frac{|W_w|}{\sin(\alpha - \beta)} \right)
\]
Now we give the first theorem.

**Theorem 1:** Let \( W_{\alpha} \) \( (W_{\beta}) \) be a measurable set and suppose \( X_{\alpha}(u) \) \( (X_{\beta}(u)) \) is the FRFT of \( x(t) \) for transform parameter \( \alpha \) \( (\beta) \), such that \( X_{\alpha}(u) \) \( (X_{\beta}(u)) \) is \( \text{e}_{W_{\alpha}} \) \( (\text{e}_{W_{\beta}}) \) concentrated on \( W_{\alpha} \) \( (W_{\beta}) \). Then
\[
|W_{\alpha}| |W_{\beta}| \geq \left( 1 - \text{e}_{W_{\alpha}} - \text{e}_{W_{\beta}} \right) |\sin(\alpha - \beta)|.
\]

**Proof:** Since \( \left\| P_{\alpha} \right\|_{L^2} = \sup_{f \in L^2} \left\| P_{\alpha} f \right\|_{L^2} \), therefore we can find such \( x(t) \) that makes
\[
\left\| P_{\alpha} \right\|_{L^2} = 1.
\]
Meanwhile, via triangle inequality and the definitions of concentration we have
\[
\left\| x(t) - P_{\alpha} x(t) \right\|_{L^2} \leq \left\| x(t) - P_{\alpha} x(t) + P_{\alpha} x(t) \right\|_{L^2} \leq \text{e}_{W_{\alpha}} + \text{e}_{W_{\beta}}.
\]
At the same time, we know
\[
\left\| x(t) - P_{\alpha} x(t) \right\|_{L^2} \geq \left\| x(t) \right\|_{L^2} - \left\| P_{\alpha} x(t) \right\|_{L^2} = 1 - \left\| P_{\alpha} x(t) \right\|_{L^2},
\]
so that
\[
\left\| P_{\alpha} x(t) \right\|_{L^2} \geq 1 - \left( \text{e}_{W_{\alpha}} + \text{e}_{W_{\beta}} \right).
\]
Use the above two results, we obtain
\[
\|P_a P_\beta\|_{L^2} = \sqrt{\frac{|W_a| |W_\beta|}{\sin(\alpha - \beta)}} \geq \|P_a P_\beta\|_{L^2(R)}.
\]

i.e., \(\sqrt{|W_a| |W_\beta|} \geq \|P_a P_\beta\|_{L^2(R)} \sqrt{\sin(\alpha - \beta)}\).

Hence, \(|W_a| |W_\beta| \geq \|P_a P_\beta\|_{L^2(R)}^2 \sin(\alpha - \beta) \geq (1 - \epsilon_{W_a} - \epsilon_{W_\beta}) \sin(\alpha - \beta)\). The special case \(\alpha - \beta = n\pi\) is trivial. Here, we find that when \(\alpha - \beta = \pi/2 + k\pi\), (4) reduce to the traditional case in Theorem 2 [(3.1), 25].

Obviously, this bound is different from that [20] of infinite case. In [20], the main involved objects are the variances of the signal in infinite supports. Here the measurable sets \((W_a, W_\beta)\) are involved, which is instructive for the discrete case in the next section. If \(\epsilon_{W_a} = \epsilon_{W_\beta} = 0\), what will happen? Clearly, it is impossible. From the conclusion [33], if \(\epsilon_{W_a} = 0\), then \(\epsilon_{W_\beta} \neq 0\), otherwise \(W_\beta = \infty\), which is in conflict with that \(W_\beta\) is measurable and limited. Therefore, in the continuous case, \(\epsilon_{W_a} = \epsilon_{W_\beta} = 0\) cannot hold true. However, what about the discrete case? The next section will answer.

3. The Discrete Heisenberg Uncertainty Principles

3.1. The Uncertainty Relation

First let us introduce a lemma.

**Lemma 3:** \(|P_a P_\beta| = \sqrt{N_a \cdot N_\beta \over N \sin(\alpha - \beta)}\), where \(|f|_F\) is the Frobenius matrix norm.

**Proof:** From the definition of the operator \(P_a P_\beta\) in definition 4, we have

\[
(P_a P_\beta \hat{x})(n) = \sum_{k=1}^N \sum_{v=1}^N \chi_a u_a(k,n) \sum_{v=1}^N \chi_\beta \hat{x}(v) u_\beta(k,v).
\]

Exchange the locations of the sum operators, we obtain

\[
(P_a P_\beta \hat{x})(n) = \sum_{k=1}^N \sum_{v=1}^N \chi_a \chi_\beta u_a(k,n) \hat{x}(v) u_\beta(k,v)
\]

\[
= \sum_{k=1}^N \sum_{v=1}^N \chi_a \chi_\beta \hat{x}(v) u_{a-\beta}(n,v).
\]

Hence, according to the definition of the Frobenius matrix norm [27] [34] and the definition of DFRFT, we have

\[
\|P_a P_\beta\|_F = \left(\sum_{k=1}^N \sum_{v=1}^N u_{a-\beta}(n,v) \right)^{1/2} \geq \sqrt{N_a \cdot N_\beta \over N \sin(\alpha - \beta)}.
\]

In the similar manner with the continuous case, we can obtain

\[
\|P_a P_\beta \hat{x}(n)\|_{L^2(R)} \geq \frac{1}{(\epsilon_a + \epsilon_\beta)}.
\]

Since

\[
\|P_a P_\beta\|_F \geq \sup_{x(n) \in L^2(R)} \|P_a P_\beta \hat{x}(n)\|_{L^2(R)} \geq \|P_a P_\beta \hat{x}(n)\|_{L^2(R)} \geq \frac{1}{(\epsilon_a + \epsilon_\beta)},
\]

thus, we get

\[
\sqrt{N_a \cdot N_\beta \over N \sin(\alpha - \beta)} = \|P_a P_\beta\|_F \geq \|P_a P_\beta \hat{x}(n)\|_{L^2(R)} \geq \frac{1}{(\epsilon_a + \epsilon_\beta)},
\]
Therefore, we can obtain the following theorem 2.

**Theorem 2:** Let $\hat{x}_a(k)$ ($\hat{x}_b(k)$) be the DFRFT of the time sequence $\hat{x}(n) \in l^2(R)$ ($n = 1, \cdots, N$) for transform parameter $\alpha$ ($\beta$), with $\hat{x}_a(k)$ ($\hat{x}_b(k)$) $\epsilon_a$ ($\epsilon_b$)-concentrated on index set $N$ ($\epsilon_a \epsilon_b \neq 0$). Let $N_a$ ($N_b$) be the numbers of nonzero entries in $\hat{x}_a(k)$ ($\hat{x}_b(k)$) respectively. Then

$$N_a \cdot N_b \geq N \cdot \left(1 - \epsilon_a - \epsilon_b\right)^2 \sin(\alpha - \beta).$$

Here, we find that when $\alpha - \beta = \pi/2 + k\pi$, (5) reduce to the traditional case in Theorem 3 [(3.9), 25].

### 3.2. The Extensions

Set $\epsilon_a = \epsilon_b = 0$ in theorem 2, we can obtain the following theorem 3 directly.

**Theorem 3:** Let $\hat{x}_a(k)$ ($\hat{x}_b(k)$) be the DFRFT of the time sequence $\hat{x}(n) \in l^2(R)$ ($n = 1, \cdots, N$) with length $N$. $N_a$ ($N_b$) counts the numbers of nonzero entries in $\hat{x}_a(k)$ ($\hat{x}_b(k)$) respectively. Then

$$N_a \cdot N_b \geq N \cdot \left|\sin(\alpha - \beta)\right|, \quad \alpha - \beta \neq n\pi$$

$$N_a \cdot N_b \geq 1, \quad \alpha - \beta = n\pi.$$  

Clearly, theorem 3 is a special case of theorem 2. Also, this theorem can be derived via theorem 1 in [26].

Differently, we obtain this result in a different way. Here we note that since $l^2(R)$ is band-limited, then $\hat{x}(n)$ is obtained through sampling $x(t)$. From the sequence length $N$ in the definition of DFRFT in (2), we know the sampling period defined as $T_s$; $T_s = 1$ ($\hat{x}(n) = \hat{x}(nT_s)$ implies this result). We assume there is no aliasing after sampling in the FRFT domain, then from the sampling Theorem, we know that all the energy of $\hat{x}_a(k)$ are limited within the scope $\left[-N\sin\alpha, N\sin\alpha\right]$, i.e., all the energy of $\hat{x}_a(k)$ must be within $\left[m+1,m+\left[N\sin\alpha\right]\right]$ $(0 \leq m, m+\left[N\sin\alpha\right] \leq N)$. Without loss of generality, we assume $m = 0$ based on the shifting property of FRFT [6] [32], i.e., all the energy of $\hat{x}_a(k)$ must be within $\left[1,N\sin\alpha\right]$. Let $n_1,n_2,\cdots,n_{N_1}$ be the sites where $\hat{x}(n)$ is nonzero, and $\hat{x}(n_l)$ ($l = 1, \cdots, N_1$) be the corresponding nonzero elements of $\hat{x}(n)$. Accordingly, from the definition of DFRFT [6] [32], we have

$$\hat{x}_a(k) = \sum_{l=1}^{N_1} \sqrt{1 - i\cot\alpha/N} \cdot e^{-i\frac{\pi}{2} \sin\alpha} \cdot e^{N\sin\alpha} \cdot \hat{x}(n_l), \quad k = 1, 2, \cdots, N\sin\alpha$$

We rewrite (8) in terms of matrices and vectors. Define the matrix $Z_{a} = \sqrt{1-i\cot\alpha/N} \cdot D_a(n_l,k)$, where

$$D_a(n_l,k) = e^{-i\frac{\pi}{2} \sin\alpha} \cdot e^{N\sin\alpha} \cdot e^{N\sin\alpha},$$

then we obtain

$$x(k) = \sum_{l=1}^{N_1} \sqrt{1 - i\cot\alpha/N} \cdot e^{-i\frac{\pi}{2} \sin\alpha} \cdot e^{N\sin\alpha} \cdot \hat{x}(n_l), \quad k = 1, 2, \cdots, N\sin\alpha.$$
$\hat{x}_w = Zb,$

where $\hat{x}_w = [\hat{x}_w(1), \ldots, X_w \left( N|\sin \alpha| \right)]^T,$ $Z = (Z_{kl})_{N|\sin \alpha| \times N_i}$ and $b = [\hat{x}(n_1), \ldots, \hat{x}(n_N)]^T.$

Clearly, $Z$ is a $N|\sin \alpha| \times N_i$ matrix, which includes $N|\sin \alpha| \times N_i$ matrixes with dimensions of $N_i \times N_i$ so that we can rewrite matrix $Z$ as $Z = [Z_1, Z_2, \ldots, Z_s]$ and $\hat{x}_w = [\hat{x}_{w,1}, \ldots, \hat{x}_{w,s}]^T,$ where $s = 1, 2, \ldots,$ $N|\sin \alpha|.$

From the definition of DFRFT, we know that the bases $\left( \cos \frac{2\pi}{L} \cos \alpha \right)^{\sum_{k=1}^{L} \frac{s}{\sin \alpha}} e^{-2\pi i n t_{kl}} e^{-2\pi i n t_{kl}} \frac{\sin \alpha}{N}$ (for different $k_s$ and $n_s$) are mutually orthogonal [6] [32]. Therefore, the different rows are not correlated so that $Z_s$ is nonsingular and $\hat{x}_{w,s} = Z_b$ can be rewritten as $(Z_s)^{-1} \hat{x}_{w,s} = b.$ Since every element in $b$ is not zero and $Z_s$ is nonsingular, then there must be a non-zero element in $\hat{x}_{w,s}$ at least. Otherwise, $b = 0,$ which is in conflict with $b \neq 0.$ Therefore, in every $\hat{x}_{w,s} \left( s = 1, 2, \ldots, \frac{N|\sin \alpha|}{N_i} \right)$ there is at least one non-zero element. Therefore, there are at least $N_a \geq \frac{N|\sin \alpha|}{N_i}$ non-zero elements in the DFRFT domain in total. Thus, theorem 3 is verified.

Furthermore, we can obtain the following more general uncertainty relation associated with DFRFT.

Clearly, if $|\sin \alpha| < 1$ and $|\sin (\alpha - \beta)| < 1,$ then the generalized uncertainty bounds are lower than the traditional cases. Therefore, the generalized uncertainty principles show that the resolution will be higher.

**Theorem 4:** Let $\hat{x}_w(k) \ (l = 1, 2, \ldots, L)$ be the DFRFT of the time sequence $\hat{x}(n) \in l^2(R) \ (n = 1, \ldots, N$ and $N > L)$ with length $N$ and $\|\hat{x}(n)\|_{l^2} = 1.$ $N_{a_l}$ counts the number of nonzero elements in $\hat{x}_w(k_l).$ Then

$$\frac{N_{a_1} + N_{a_2} + \cdots + N_{a_L}}{L} \geq \frac{N|\sin \alpha|}{N_i} \xi,$$

where $\xi = \inf_{\frac{l_{a_1}}{a_1}, \frac{l_{a_2}}{a_2}} \|a_1 - a_2\|_l.$

**Proof:** From the assumption and the definition of DFRFT [6] [32], we know

$$\hat{x}(n) = \sum_{k=1}^{N} u_{-a_1}(n, k_1) \hat{x}_w(k_1) = \sum_{k=1}^{N} u_{-a_1}^{\mu}(n, k_1) \hat{x}_w(k_1) \quad \text{for} \quad n = 1, 2, \ldots, N.$$

where $u_{-a_1}(n, k_1) = \sqrt{\frac{1}{N}} e^{-2\pi i n t_{a_1}} e^{-2\pi i n t_{a_2}} \frac{\sin \alpha}{N_i}, \ (l = 1, 2, \ldots, L).$

Therefore, let $\hat{X} = [\hat{x}(1) \ \hat{x}(2) \ \cdots \ \hat{x}(N)]^T,$ we have [26]

$$\hat{X}^T \hat{X} = [\hat{x}_{w_{a_1}}(1) \ \hat{x}_{w_{a_1}}(2) \ \cdots \ \hat{x}_{w_{a_1}}(N)] \begin{bmatrix} u_{-a_1}^T(1, :) & u_{-a_1}^T(2, :) & \cdots & u_{-a_1}^T(N, :) \end{bmatrix} = \begin{bmatrix} \hat{x}_{a_1}(1) \\ \hat{x}_{a_1}(2) \\ \vdots \\ \hat{x}_{a_1}(N) \end{bmatrix},$$

where $u_{-a_1}(n, :) = [u_{-a_1}(n, 1) u_{-a_1}(n, 2) \cdots u_{-a_1}(n, N)]$ and...
\[
\begin{align*}
  u_{-a_2}(n,:) = [u_{-a_2}(n,1) u_{-a_2}(n,2) \cdots u_{-a_2}(n,N)]^T \quad \text{with} \quad n = 1, 2, \cdots, N \quad \text{and} \quad l_1, l_2 = 1, 2, \cdots, L \quad \text{with} \quad l_1 \neq l_2.
\end{align*}
\]

Hence, we obtain
\[
\tilde{X}^\top \tilde{X} = \sum_{n=1}^{N} \sum_{k=1}^{N} \hat{x}_{a_0}(n) \left\{ u_{-a_2}(n,:) \cdot u_{-a_2}(k,:) \right\} \hat{x}_{a_2}(k).
\]

Set \( M_{(l_1,l_2)} = \sup_{n,k} \left\{ \left| \left( u_{-a_1}(n,:), u_{-a_2}(k,:) \right) \right| \right\} \), then
\[
\tilde{X}^\top \tilde{X} \leq \sum_{n=1}^{N} \sum_{k=1}^{N} \left| \hat{x}_{a_0}(n) \right| \left\{ u_{-a_2}(n,:) \cdot u_{-a_2}(k,:) \right\} \left| \hat{x}_{a_2}(k) \right|
\]
\[
\leq \sum_{n=1}^{N} \sum_{k=1}^{N} \left| \hat{x}_{a_0}(s_1) \right| \left| M_{(l_1,l_2)} \right| \left| \hat{x}_{a_2}(s_2) \right|
\]
\[
\leq M_{(l_1,l_2)} \sum_{n=1}^{N} \sum_{k=1}^{N} \left| \hat{x}_{a_0}(s_1) \right| \left| \hat{x}_{a_2}(s_2) \right|
\]

Using the triangle inequality, we have
\[
\left| \hat{x}_{a_0}(s_1) \right| \left| \hat{x}_{a_2}(s_2) \right| \leq \left| \hat{x}_{a_0}(s_1) \right|^2 + \left| \hat{x}_{a_2}(s_2) \right|^2,
\]

hence
\[
\tilde{X}^\top \tilde{X} \leq M_{(l_1,l_2)} \sum_{n=1}^{N} \sum_{k=1}^{N} \left| \hat{x}_{a_0}(s_1) \right|^2 + \frac{\left| \hat{x}_{a_2}(s_2) \right|^2}{2}
\]
\[
= M_{(l_1,l_2)} \left\{ \sum_{n=1}^{N} \sum_{k=1}^{N} \left| \hat{x}_{a_0}(s_1) \right|^2 + \sum_{n=1}^{N} \sum_{k=1}^{N} \left| \hat{x}_{a_2}(s_2) \right|^2 \right\}
\]
\[
= M_{(l_1,l_2)} \left\{ \sum_{s_1=1}^{N_{a_0}} \left( \sum_{n=1}^{N_0} \left| \hat{x}_{a_0}(s_1) \right|^2 \right) + \sum_{s_2=1}^{N_{a_2}} \left( \sum_{n=1}^{N_2} \left| \hat{x}_{a_2}(s_2) \right|^2 \right) \right\}.
\]

From \( \left\| \hat{x}(n) \right\|_2 = 1 \) and Parseval’s principle [6], we obtain
\[
\sum_{n=1}^{N_0} \frac{\left| \hat{x}_{a_0}(s_1) \right|^2}{2} = \sum_{s_2=1}^{N_{a_2}} \frac{\left| \hat{x}_{a_2}(s_2) \right|^2}{2} = \frac{1}{2}.
\]

Hence
\[
\tilde{X}^\top \tilde{X} \leq M_{(l_1,l_2)} \cdot \left( \frac{N_{a_0}}{2} + \frac{N_{a_2}}{2} \right) = M_{(l_1,l_2)} \cdot N_{a_0} + \frac{N_{a_2}}{2}.
\]

Therefore, we obtain
\[
\tilde{X}^\top \tilde{X} \leq M_{(1,2)} \cdot \frac{N_1 + N_2}{2},
\]
\[
\tilde{X}^\top \tilde{X} \leq M_{(1,3)} \cdot \frac{N_1 + N_3}{2},
\]
\[
\vdots
\]
\[
\tilde{X}^\top \tilde{X} \leq M_{(L-1,L)} \cdot \frac{N_{L-1} + N_L}{2}.
\]
Adding all the above inequalities, we have
\[
\Gamma_{L}^{2} \cdot \bar{X}^{T} \bar{X} \leq \sup_{1 \leq i, j \leq L \atop \frac{\alpha_i}{N}, \frac{\alpha_j}{N} \neq 0} \left\{ M_{(\frac{\alpha_i}{N}, \frac{\alpha_j}{N})} \right\} \frac{(L-1) \cdot (N_{1} + N_{2} + \cdots + N_{L})}{2} \quad \text{with} \quad \Gamma_{L}^{2} = \frac{L \cdot (L-1)}{2 \times 1}.
\]

Similarly, from \(\left\| \bar{X} \right\|_{2} = 1\) and Parseval’s principle [6], we obtain \(\bar{X}^{T} \bar{X} = 1\), hence
\[
\frac{(L-1) \cdot (N_{1} + N_{2} + \cdots + N_{L})}{2} \geq \sup_{1 \leq i, j \leq L \atop \frac{\alpha_i}{N}, \frac{\alpha_j}{N} \neq 0} \left\{ M_{(\frac{\alpha_i}{N}, \frac{\alpha_j}{N})} \right\}.
\]

From the definition and property of DFRFT [6] [32] we have
\[
\sup_{1 \leq i, j \leq L \atop \frac{\alpha_i}{N}, \frac{\alpha_j}{N} \neq 0} \left\{ M_{(\frac{\alpha_i}{N}, \frac{\alpha_j}{N})} \right\} = \sup_{1 \leq i, j \leq N} \left\{ K_{\alpha_i, \alpha_j}(s_1, s_2) \right\} = \sup_{1 \leq i, j \leq N} \left\{ \left| \frac{1}{\sqrt{N}} \cdot \sin \left( \alpha_i - \alpha_j \right) \right| \right\} = \frac{1}{\sqrt{N}} \cdot \sin \xi
\]
with \(\xi = \inf_{\frac{\alpha_i}{N}, \frac{\alpha_j}{N} \neq 0} \left| \alpha_i - \alpha_j \right|\).

Hence, we finally obtain the proof
\[
\frac{N_{1} + N_{2} + \cdots + N_{L}}{L} \geq \sqrt{N} \cdot \sin \xi \quad \text{with} \quad \xi = \inf_{\frac{\alpha_i}{N}, \frac{\alpha_j}{N} \neq 0} \left| \alpha_i - \alpha_j \right|.
\]

4. The Simulation

In this section we give an example to show that the data in FRFT domains may have much higher concentration than that in traditional time-frequency domains.

Now considering the chirp signal \(f(n)\) with \(n \in [1,1000]\) s and sampling period \(T_s = 1\) s,
\[
f(n) = \cos\left(0.001(n + 3n^2)\right) \quad \text{(see Figure 1(a)).}
\]

Clearly, we can obtain from Figure 1 that \(N_{\alpha} = 1000\), \(N_{\alpha s = 2} = 300\), \(N_{\alpha s = 0.061 s^2} = 1\). Therefore, we have \(N_{\alpha s = 2} \approx 300,000 > N_{\alpha s = 0.061 s^2} = 1000\). This verifies that the data in FRFT domains may have much higher concentration than that in traditional time-frequency domains. (Note here that if the transformed coefficient is less than 0.1, then we take it as zero value. See Figure 1(b) and Figure 1(c)).

5. Conclusion

In practice, we often process the data with limited lengths for both the continuous (\(\varepsilon\)-concentrated) and discrete signals. Especially for the discrete data, not only the supports are limited, but also they are sequences of data.
points whose number of non-zero elements is countable accurately. This paper discussed the generalized uncertainty relations on FRFT in term of data concentration. We show that the uncertainty bounds are related to the FRFT parameters and the support lengths. These uncertainty relations will enrich the ensemble of FRFT. Moreover, these uncertainty relations will help find the optimal filtering parameters [31] such as [6] [34] [36]. Our simulation also shows that the data in FRFT domains may have much higher concentration than that in traditional time-frequency domains.

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References


