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Complex Hilbert Transform Filter[*]

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ABSTRACT

Hilbert transform is a basic tool in constructing analytical signals for a various applications such as amplitude modulation, envelope and instantaneous frequency analysis, quadrature decoding, shift-invariant multi-rate signal processing and Hilbert-Huang decomposition. This work introduces a complex Hilbert transform (CHT) filter, where the real and imaginary parts are a Hilbert transform pair. The CHT filtered signal is analytic, i.e. its Fourier transform is zero in negative frequency range. The CHT filter is constructed by half-sample delay operators based on the B-spline transform interpolation and decimation procedure. The CHT filter has an ideal phase response and the magnitude response is maximally flat in the frequency range $0 \leq \omega \leq \pi$. The CHT filter has integer coefficients and the implementation in VLSI requires only summations and register shifts. We demonstrate the feasibility of the CHT filter in reconstruction of the sign modulated CMOS logic pulses in a fibre optic link.

Keywords: *Hilbert Transform, Analytic Signal, Fractional Delay Filters*

1. Introduction

Hilbert transform has an essential role in constructing analytical signals for a variety of signal processing applications, for example in envelope and instantaneous frequency analysis and in design of amplitude modulators and digital quadrature encoders. The recent applications include Hilbert-Huang decomposition [1], the shift-invariant wavelet transform algorithms [2-5], geophysical [6], seismic, ultrasonic radar and biomedical signal analyses [7-11]. The Hilbert transform theory is well established, but the computational methods are still under development. The frequently used methods are based on the fast Fourier transform (FFT) [7,12]. Also other methods have been proposed, such as the parametric modelling approach [13,14] and digital filtering [12,15].

In this work we describe a complex Hilbert transform (CHT) filter, where the real and imaginary parts are a Hilbert transform pair. The CHT yields analytic signals, whose Fourier transform is zero in negative frequency range. We construct the CHT filter by half-sample delay operators based on the B-spline transform. The phase response of the CHT filter is ideal and the magnitude re-

sponse is maximally flat in the frequency range $0 \leq \omega \leq \pi$.

2. Theoretical Considerations

2.1. Hilbert Transformer

Let us denote the frequency response of the *z*-transform filter $H(z)$ as

$$H(z) = \sum_n h_n z^{-n} \Rightarrow H(\omega) = \sum_n h_n \mathrm{e}^{-j\omega n} \quad (1)$$

where $h_n, n \in N$ is the impulse response of the filter. Correspondingly, we have the relation

$$H(-z) \Rightarrow H(\omega - \pi) \quad (2)$$

Our purpose is to design a Hilbert transform operator $\mathcal{H}$ for the discrete-time signal x_n, $n = 0,1,2,\cdots$ as

$$\mathcal{H}\{X(z)\} = \mathcal{H}(z) X(z) \quad (3)$$

where $\mathcal{H}(z)$ denotes the Hilbert transform filter.

The frequency response of the Hilbert transform operator is defined as

$$\mathcal{H}(\omega) = -j \operatorname{sgn}(\omega) \quad (4)$$

where the sign function is

$$\operatorname{sgn}(\omega) = \begin{cases} 1, & \text{for } \omega \geq 0 \\ -1, & \text{for } \omega < 0 \end{cases} \quad (5)$$

In this work we apply the Hilbert transform operator in the form

$$\mathcal{H}(\omega) = e^{-j\pi/2} \text{sgn}(\omega) \qquad (6)$$

2.2. Hilbert Transform Filter

We define the half-sample delay filter $D(z)$ by the infinite impulse response (IIR) structure

$$D(z) = z^{-1/2} = \frac{A(z)}{B(z)} \qquad (7)$$

where $A(z)$ and $B(z)$ are polynomials in z^{-1}. In frequency domain we have

$$B(\omega) = e^{j\omega/2} A(\omega) \qquad (8)$$

The corresponding quadrature mirror filters are $A(-z)$ and $B(-z)$. Due to (2) we have in frequency domain relation between them

$$B(\omega - \pi) = e^{j(\omega-\pi)/2} A(\omega - \pi) \qquad (9)$$

Now we may construct the Hilbert transform operator as

$$\mathcal{H}(\omega) = D(\omega) D^{-1}(\omega - \pi) = e^{-j\omega/2} e^{j(\omega-\pi)/2} = e^{-\pi/2} \quad (10)$$

In z transform domain we have the Hilbert transform filter

$$\mathcal{H}(z) = D(z) D^{-1}(-z) = \frac{A(z)B(-z)}{B(z)A(-z)} \qquad (11)$$

2.3. Design of the Half-Sample Delay Filter

Our approach is to construct a half-delay filter $D_p(z) = z^{-1/2}$, which has an exactly linear phase. We apply the fractional delay (FD) filter design method based on the B-spline transform interpolation and decimation procedure for implementation of the fractional delays $\tau = N/M$ ($N, M \in \mathbb{Z}$) [16]. The FD filter has the following representation

$$D_p(N, M, z) = \frac{1}{\beta_p(z)} \left[z^{-N} \beta_p(z) F(z) \right]_{\downarrow M} \quad (12)$$

where $\beta_p(z)$ is the p'th order discrete B-spline and $F(z)$ the polynomial

$$F(z) = \frac{1}{M^{p-1}} \left(\frac{1 - z^{-M}}{1 - z^{-1}} \right)^p = \frac{1}{M^{p-1}} \left(\sum_{k=o}^{M-1} z^{-k} \right)^p \quad (13)$$

The half-sample delay operator $D_p(z) = z^{-1/2}$ is yielded by inserting $N = 1$ and $M = 2$ in (12)

$$D_p(z) = D(1, 2, z) = \frac{Q_p(z)}{\beta_p(z)} \qquad (14)$$

where $Q(z)$ is a polynomial in z^{-1}. The phase of the

half-delay operator is exactly linear in the frequency range $-\pi \leq \omega \leq \pi$ independently of the B-spline order p.

Example. For the discrete B-spline order $p = 4$

$$\beta_4(z) = \frac{1 + 4z^{-1} + z^{-2}}{6} \qquad (15)$$

we have

$$Q_4(z) = \frac{1 + 23z^{-1} + 23z^{-2} + z^{-3}}{48} \qquad (16)$$

The phase spectrum of the Hilbert transform filter $\mathcal{H}(z) = Q_4(z)\beta_4(-z) / \beta_4(z)Q_4(-z)$ follows exactly the definition (4) in the frequency range $-\pi \leq \omega \leq \pi$. The IIR type Hilbert transform filter can be implemented by the inverse filtering procedure [16].

2.4. Adjustment of the Magnitude Response

The Hilbert transform filter (11) designed by the B-spline transform has an ideal phase response. However, for even p the half-sample delay operator $D_p(z)$ has zero magnitude response at $\omega = \pi$ due to the zero at $z = -1$. Therefore the magnitude response Hilbert transform filter intensifies at $\omega = 0$ and π. Usually the main demand for the Hilbert transformer is a maximally flat magnitude response centered at $\omega = \pi/2$. However, we may revisit the design by defining the Hilbert transform filter as

$$\mathcal{H}(z) = D_p(z) D_{p\pm1}^{-1}(-z) \qquad (17)$$

For even p the half delay operator $D_{p\pm1}(z)$ has a pole at $z = -1$ ($\omega = \pi$). Hence, the $D_{p\pm1}^{-1}(-z)$ operator has a zero at $z = 1$ ($\omega = 0$).

Example. For the B-spline of the order $p = 5$

$$\beta_5(z) = \frac{1 + 11z^{-1} + 11z^{-2} + z^{-3}}{24} \qquad (18)$$

we obtain

$$Q_5(z) = \frac{1 + 76z^{-1} + 230z^{-2} + 76z^{-3} + z^{-4}}{384} \qquad (19)$$

The magnitude response of the Hilbert transform filter $\mathcal{H}(z) = D_4(z) / D_5(-z)$ has a flat maximum at $\omega = \pi/2$ (**Figure 1**). The flatness of the magnitude response increases by using the B-spine of the higher order p, e.g. $\mathcal{H}(z) = D_6(z) / D_5(-z)$.

2.5. Complex Hilbert Transform Filter

In order to avoid the IIR type implementation of the Hilbert transform filter the key idea in this work is to write (11) as

$$\mathcal{H}(z) = \frac{D_{p\pm1}(-z)}{D_p(z)} = \frac{R_p(z)}{S_p(z)} \qquad (20)$$
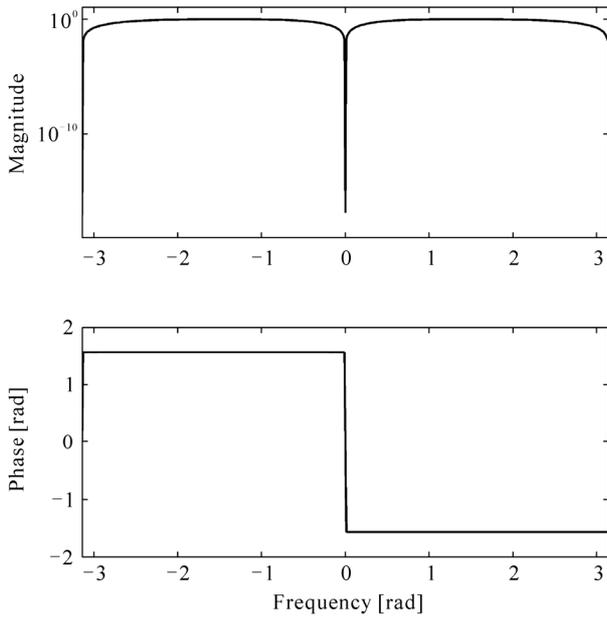
Figure 1. Magnitude and phase spectra of the Hilbert transform filter using the B-spline order $p = 4$.

where the polynomials

$$R_p(z) = \beta_p(z) Q_{p\pm1}(-z)$$
$$S_p(z) = \beta_{p\pm1}(-z) Q_p(z) \qquad (21)$$

work as a Hilbert transform pair and we may construct the complex Hilbert transform (CHT) filter as

$$\mathcal{H}(z) = S_p(z) + jR_p(z) \qquad (22)$$

The real and imaginary parts of the FIR type CHT filter form a Hilbert pair.

3. Experiments

The versatility of CHT filter was tested experimentally by transmitting the modulated logic pulses using the fibre optic link. The logic pulses (**Figure 2(a)**) were obtained from the CMOS gate and modulated by the waveform $M(n) = \sin(n\pi/2)$, $n = 0, 1, 2, \cdots$, which yields the sign modulation sequence [+1 0 –1 0 +1 0 –1 $\cdots$]. A diode laser was used to generate the modulated light pulses (**Figure 2(b)**), which were fed through a 10 m long optical fibre and measured with a PIN photo diode. The output signal was measured using a 16 bit analog-to-digital converter (ADC) and 100 kHz sampling rate (**Figure 2(b)**). The measured zero mean signal was fed to the CHT filter (22), which was constructed using the polynomials (21) with the B-spline order $p = 4$. **Figure 2(c)** shows the reconstructed logic signal, which equals the envelope (absolute value) of the output of the CHT filter. The correspondence with the original logic signal is excellent and only limitation is the time delay generated by

the CHT filter. The Fourier magnitude spectrum of the CHT filter output lacks almost totally the negative frequency components (**Figure 3**). It should be pointed out that the dc level of the modulated signal is removed before the CHT filtering. This can be done by inserting a small capacitor in front of the ADC or subtracting the mean value from the measured signal.

(a)

(b)

(c)
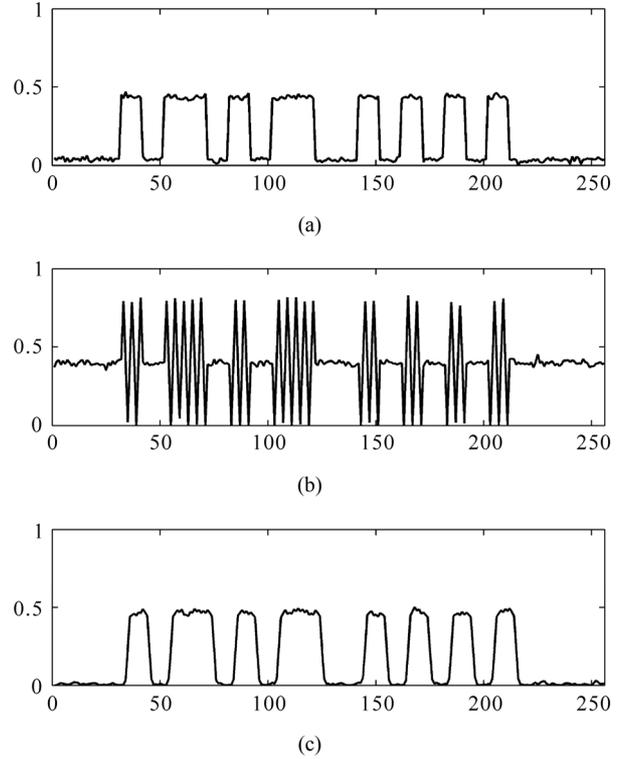
Figure 2. Reconstruction of the logic pulse sequence via the envelope of the CHT filtered signal. X-axis denotes the sampling number and Y-axis is in volts.
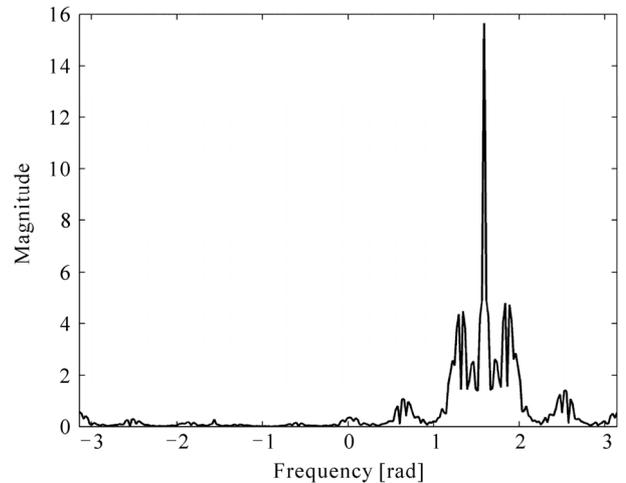
Figure 3. Fourier magnitude spectrum of the CHT filtered sign modulated logic pulse sequence.

4. Discussion

In this work a novel method for designing the Hilbert transform operator is introduced. The idea is based on the half-sample delay operator $D_p(z)$, which is constructed by the B-spline transform interpolation and decimation procedure [16]. The method yields a half-sample delay filter, which has a precisely linear phase. Some competing fractional delay design methods, such as Taylor series expansions of $D(\omega, \tau) = \exp(-j\omega\tau)$ and Thiran filters produce phase distortion [17-21]. However, the magnitude response of the Hilbert transform operator (11) is intensified at DC and at Nyquist frequency, which is not satisfying for many purposes. We solved the problem using even and odd order half-delay operators in cascade (17). The resulting Hilbert transform operator has a maximally flat frequency response in the range $0 \leq \omega \leq \pi$ (**Figure 1**). The flatness of the magnitude response can be increased by using the higher B-spline order p. It has been shown that the B-spline interpolation approaches asymptotically the sinc-interpolation with increasing p [22]. However, in practice a compromise has to be made between the length of the filter and the flatness requirements.

One of the advantages of the B-spline transform based operators is the integer valued filter coefficients. The CHT filter is feasible to implement in VLSI environment requiring only register shifts and summations. To avoid the implementation of the IIR-type filter, we divided the Hilbert transform filter into a complex FIR-type filter (21, 22). The real and imaginary parts of the resulting complex signal are a Hilbert pair. However, it should be pointed out that as in classical Hilbert transformers the real part of the Hilbert transformed signal equals the original signal, the CHT filter (22) results in a half-sample delayed version. The perfect reconstruction of the original signal requires the inverse filtering by $S_p^{-1}(z)$. However, in many communication systems this is not a restriction since the relative phase relations of the signals is of significance.

The CHT has a plenty of applications such as computation of the envelope and instantaneous frequency and the construction of the digital quadrature encoders and amplitude modulators. The FFT-based Hilbert transform algorithms [5,7] can be directly replaced by the CHT prefilter in the shift-invariant multi-scale analysis. We demonstrated the feasibility of the CHT filter in reconstruction of the sign modulated CMOS logic pulses traveling through a fibre optic link. Compared with the direct transmission of the light pulses, the sign modulation concentrates the power spectral density of the signal in the vicinity of the modulation frequency (**Figure 3**) and the DC-level variation in the transmitted light pulses

do not interfere the reconstruction. For example in robotics the mechanical vibrations in fibres may originate errors in direct transmission method.

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