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A Simple Generalisation of Kirk's Approximation for Multi-Asset Spread Options by the Lie-Trotter Operator Splitting Method

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Abstract

In this paper, by means of the Lie-Trotter operator splitting method, we have presented a new unified approach not only to rigorously derive Kirk's approximation but also to obtain a generalisation for multi-asset spread options in a straightforward manner. The derived price formula for the multi-asset spread option bears a great resemblance to Kirk's approximation in the two-asset case. More importantly, our approach is able to provide a new perspective on Kirk's approximation and the generalization; that is, they are simply equivalent to the Lie-Trotter operator splitting approximation to the Black-Scholes equation.

Keywords

Lognormal Random Variables, Black-Scholes Equation, Spread Options, Kirk's Approximation, Lie-Trotter Operator Splitting Method

1. Introduction

Spread options, whose payoff is contingent upon the price difference of two underlying assets form the simplest type of multi-asset options, are very popular in markets as diverse as interest rate markets, currency and foreign exchange markets, commodity markets, and energy markets nowadays [1]. Despite the rapid development of spread options, pricing spread options is a very challenging task and receives much attention in the literature. The major difficulty lies in the lack of knowledge about the distribution of the spread of two correlated

lognormal random variables. The simplest approach is to evaluate the expectation of the final payoff over the joint probability distribution of the two correlated lognormal underlyings by means of numerical integration. However, practitioners often prefer to use analytical approximations rather than numerical methods because of their computational ease. Among various analytical approximations, e.g. Carmona and Durrleman [1], Deng *et al.* [2], Bjerksund and Stensland [3], Venkatramana and Alexander [4], Kirk's approximation seems to be the most widely used and is the current market standard, especially in the energy markets [5]. It is well known that Kirk's approximation extends from Margrabe's exchange option formula with no rigorous derivation [6]. Recently, Lo [7] applied the idea of WKB method to provide a derivation of Kirk's approximation and discuss its validity. Nevertheless, it is not straightforward to provide a generalisation of Kirk's approximation for the case of multi-asset spread option via this approach.

Accordingly, it is the aim of this paper to present a simple unified approach, namely the Lie-Trotter operator splitting method [8], not only to rigorously derive Kirk's approximation but also to obtain a generalisation for the case of multi-asset spread option in a straightforward manner. The derived price formula for the multi-asset spread option bears a great resemblance to Kirk's approximation in the two-asset case. More importantly, the proposed approach is able to provide a new perspective on Kirk's approximation and the generalisation; that is, they are simply equivalent to the Lie-Trotter operator splitting approximation to the Black-Scholes equation. Illustrative numerical examples for the three-asset spread options are also shown to demonstrate both the accuracy and efficiency of the extended Kirk approximation. Furthermore, it should be emphasized that our approach is completely different from the extended Kirk approximation proposed by Li *et al.* [9]. Li *et al.* suggested to approximate the sum of $N-1$ lognormal assets as a single lognormal variable and then apply Kirk's approximation directly to this single lognormal variable and the remaining asset to price the N-asset spread option.

2. Two-Asset Spread Options

The price of a European call spread option obeys the two-dimensional Black-Scholes equation

$$\left\{\frac{1}{2}\sigma_1^2 F_1^2 \frac{\partial^2}{\partial F_1^2} + \rho\sigma_1\sigma_2 F_1 F_2 \frac{\partial^2}{\partial F_1 \partial F_2} + \frac{1}{2}\sigma_2^2 F_2^2 \frac{\partial^2}{\partial F_2^2} - r - \frac{\partial}{\partial \tau}\right\} P(F_1, F_2, \tau) = 0 \tag{1}$$

with the final payoff condition

$$P(F_1, F_2, 0) = \max(F_1 - F_2 - K, 0). \tag{2}$$

Here F_1 and F_2 are the future prices of the two lognormal underlying assets, σ_1 and σ_2 are the volatilities, ρ is the correlation between the two assets, K is the strike price, r is the risk-free interest rate, and τ denotes the time-to-maturity. It is well known that no analytical solution is available in closed form and one needs to resort to numerical methods. In the following, we propose a simple derivation of the well-known price formula of Kirk's approximation by means of the Lie-Trotter operator splitting method [8].

Proposition 1:

The price of the two-asset spread option can be approximated by

$$P(F_1, F_2, \tau) \approx e^{-r\tau}\left\{F_1 N(\xi_1) - (F_2 + K)N(\xi_2)\right\} \tag{3}$$

where

$$\xi_1 = \frac{1}{\tilde{\sigma}_-\sqrt{\tau}}\left\{\ln\left(\frac{F_1}{F_2 + K}\right) + \frac{1}{2}\tilde{\sigma}_-^2\tau\right\} \tag{4}$$

$$\xi_2 = \xi_1 - \tilde{\sigma}_-\sqrt{\tau} \tag{5}$$

$$\tilde{\sigma}_- = \sqrt{\sigma_1^2 - 2\rho\sigma_1\tilde{\sigma}_2 + \tilde{\sigma}_2^2} \tag{6}$$

$$\tilde{\sigma}_2 = \sigma_2\left(\frac{F_2}{F_2 + K}\right). \tag{7}$$

Proof:

In terms of the two new variables

$$R_1 = \frac{F_1}{F_2 + K} \quad \text{and} \quad R_2 = F_2 + K, \tag{8}$$

Equation (1) can be rewritten as follows:

$$\left\{ \hat{L}_0 + \hat{L}_1 - r - \frac{\partial}{\partial \tau} \right\} P(R_1, R_2, \tau) = 0, \tag{9}$$

where

$$\hat{L}_0 = \frac{1}{2} \tilde{\sigma}_-^2 R_1^2 \frac{\partial^2}{\partial R_1^2} \tag{10}$$

$$\hat{L}_1 = \frac{1}{2} \tilde{\sigma}_2^2 R_2^2 \frac{\partial^2}{\partial R_2^2} + (\rho \sigma_1 - \tilde{\sigma}_2) \tilde{\sigma}_2 R_1 R_2 \frac{\partial^2}{\partial R_1 \partial R_2} - (\rho \sigma_1 - \tilde{\sigma}_2) \tilde{\sigma}_2 R_1 \frac{\partial}{\partial R_1}. \tag{11}$$

The final payoff condition now becomes

$$P(R_1, R_2, 0) = R_2 \max(R_1 - 1, 0). \tag{12}$$

Accordingly, the formal solution of Equation (9) is given by

$$P(R_1, R_2, \tau) = e^{-r\tau} \exp\left\{ \tau(\hat{L}_0 + \hat{L}_1) \right\} R_2 \max(R_1 - 1, 0). \tag{13}$$

Since the exponential operator $\exp\left\{ \tau(\hat{L}_0 + \hat{L}_1) \right\}$ is difficult to evaluate, the Lie-Trotter operator splitting method [8] can be applied to approximate the operator by (see the Appendix)

$$\hat{O}^{LT} = \exp\left\{ \tau \hat{L}_0 \right\} \exp\left\{ \tau \hat{L}_1 \right\} \tag{14}$$

and obtain an approximation to the formal solution $P(F_1, F_2, \tau)$, namely

$$P^{LT}(R_1, R_2, \tau) = \hat{O}^{LT} R_2 \max(R_1 - 1, 0) = R_2 \exp\left\{ \tau \hat{L}_0 \right\} \max(R_1 - 1, 0) \equiv R_2 C(R_1, \tau), \tag{15}$$

for

$$\hat{L}_1 R_2 \max(R_1 - 1, 0) = 0 \Rightarrow \exp\left\{ \tau \hat{L}_1 \right\} R_2 \max(R_1 - 1, 0) = R_2 \max(R_1 - 1, 0). \tag{16}$$

It is not difficult to recognise that $C(R_1, \tau)$ satisfies the partial differential equation

$$\left\{ \frac{1}{2} \tilde{\sigma}_-^2 R_1^2 \frac{\partial^2}{\partial R_1^2} - r - \frac{\partial}{\partial \tau} \right\} C(R_1, \tau) = 0 \tag{17}$$

with the initial condition: $C(R_1, 0) = \max(R_1 - 1, 0)$, and that the admissible solution is given by

$$C(R_1, \tau) = e^{-r\tau} \left\{ R_1 N(\xi_1) - N(\xi_2) \right\}, \tag{18}$$

where $N(\cdot)$ is the cumulative normal distribution function and

$$\xi_1 = \frac{1}{\tilde{\sigma}_- \sqrt{\tau}} \left\{ \ln R_1 + \frac{1}{2} \tilde{\sigma}_-^2 \tau \right\} \tag{19}$$

$$\xi_2 = \xi_1 - \tilde{\sigma}_- \sqrt{\tau}. \tag{20}$$

As a result, we obtain

$$P^{LT}(R_1, R_2, \tau) = e^{-r\tau} R_2 \left\{ R_1 N(\xi_1) - N(\xi_2) \right\} \tag{21}$$

which is exactly the approximate price formula given in Equation (3). (Q.E.D.)

In terms of the spot asset prices, namely $S_i \equiv F_i \exp(-r\tau)$, the price formula given in Equation (3) turns out

to be identical to the price formula of Kirk's approximation

$$P_{Kirk}\left(S_1, S_2, \tau\right) = S_1 N\left(d_1\right) - \left(S_2 + Ke^{-r\tau}\right)N\left(d_2\right) \tag{22}$$

where

$$d_1 = \frac{\ln S_1 - \ln\left(S_2 + Ke^{-r\tau}\right)}{\bar{\sigma}_-\sqrt{\tau}} + \frac{1}{2}\bar{\sigma}_-\sqrt{\tau} \tag{23}$$

$$d_2 = d_1 - \bar{\sigma}_-\sqrt{\tau} \tag{24}$$

$$\bar{\sigma}_- = \sqrt{\sigma_1^2 - 2\rho\sigma_1\bar{\sigma}_2 + \bar{\sigma}_2^2} \tag{25}$$

$$\bar{\sigma}_2 = \sigma_2\left(\frac{S_2}{S_2 + Ke^{-r\tau}}\right). \tag{26}$$

It should be noted that for the Lie-Trotter operator splitting approximation to be valid, $\bar{\sigma}_-^2\tau$ needs to be sufficiently small, namely $\bar{\sigma}_-^2\tau \ll 1$. In accordance with Equation (25), this implies that Kirk's approximation is more favourable for positive correlation ρ between the two assets. Furthermore, for $K = 0$, the operators $\hat{L}_0$ and $\hat{L}_1$ commute so that the Lie-Trotter splitting approximation becomes exact and Margrabe's formula is recovered.

3. Multi-Asset Spread Options

To price a European N-asset call spread option, we need to solve the N-dimensional Black-Scholes equation

$$\left\{\sum_{i,j=1}^{N}\frac{1}{2}\rho_{ij}\sigma_i\sigma_j F_i F_j \frac{\partial^2}{\partial F_i \partial F_j} - r - \frac{\partial}{\partial \tau}\right\}P\left(F_1, F_2, \cdots, F_N, \tau\right) = 0 \tag{27}$$

subject to the final payoff condition

$$P\left(F_1, F_2, \cdots, F_N, 0\right) = \max\left(F_N - \sum_{j=1}^{N-1}F_j - K, 0\right), \tag{28}$$

where F_i is the future price of the lognormal underlying asset i with the volatility σ_i, ρ_{ij} is the correlation between the assets i and j, K is the strike price, r is the risk-free interest rate, and τ is the time-to-maturity. Unfortunately, this is a very formidable task. In the following, we apply the Lie-Trotter operator splitting method to derive a closed-form approximate price formula for the multi-asset spread option, which bears a great resemblance to the price formula of Kirk's approximation in the two-asset case.

Proposition 2:
    The price of the N-asset spread option can be approximated by

$$P\left(F_1, F_2, \cdots, F_N, \tau\right) \approx e^{-r\tau}\left\{F_N N\left(\theta_1\right) - \left(\sum_{j=1}^{N-1}F_j + K\right)N\left(\theta_2\right)\right\} \tag{29}$$

where

$$\theta_1 = \frac{1}{\bar{\sigma}_-\sqrt{\tau}}\left\{\ln\left(\frac{F_N}{\sum_{j=1}^{N-1}F_j + K}\right) + \frac{1}{2}\bar{\sigma}_-^2\tau\right\} \tag{30}$$

$$\theta_2 = \theta_1 - \bar{\sigma}_-\sqrt{\tau} \tag{31}$$

$$\bar{\sigma}_- = \sqrt{\sigma_N^2 - 2\tilde{\rho}\sigma_N\bar{\sigma}_+ + \bar{\sigma}_+^2} \tag{32}$$

$$\bar{\sigma}_+ = \tilde{\sigma}_+\left(\frac{\sum_{j=1}^{N-1}F_j}{\sum_{j=1}^{N-1}F_j + K}\right) \tag{33}$$

$$\tilde{\sigma}_+ = \frac{\sqrt{\sum_{j,k=1}^{N-1} \rho_{jk} \sigma_j \sigma_k F_j F_k}}{\sum_{j=1}^{N-1} F_j} \tag{34}$$

$$\tilde{\rho} = \frac{1}{\tilde{\sigma}_+} \left(\frac{\sum_{j=1}^{N-1} \rho_{jN} \sigma_j F_j}{\sum_{j=1}^{N-1} F_j} \right). \tag{35}$$

Proof:

Introducing the new variables: $f_i \equiv F_i - F_+$ for $i = 1, 2, \cdots, N-2$ and $F_+ \equiv \sum_{j=1}^{N-1} F_j$, Equation (27) can be cast in the form

$$\left\{ \hat{L}_N + \hat{L}_+ + \hat{L}_{N+} + \hat{L}_R - r - \frac{\partial}{\partial \tau} \right\} P(F_1, F_2, \cdots, F_N, \tau) = 0, \tag{36}$$

where

$$\hat{L}_N = \frac{1}{2} \sigma_N^2 F_N^2 \frac{\partial^2}{\partial F_N^2} \tag{37}$$

$$\hat{L}_+ = \frac{1}{2} \tilde{\sigma}_+^2 F_+^2 \frac{\partial^2}{\partial F_+^2} \tag{38}$$

$$\hat{L}_{N+} = \tilde{\rho} \sigma_N \tilde{\sigma}_+ F_+ F_N \frac{\partial^2}{\partial F_+ \partial F_N} \tag{39}$$

$$\hat{L}_R = \frac{1}{2} \sum_{k=1}^{N-2} \sum_{n=1}^{N-1} \left(\rho_{kn} \sigma_k F_k - \sum_{m=1}^{N-1} \rho_{mn} \sigma_m F_m \right) \sigma_n F_n \frac{\partial^2}{\partial F_+ \partial f_k}$$
$$+ \frac{1}{2} \sum_{k=1}^{N-2} \sum_{n=1}^{N-2} \left\{ \left(\rho_{kn} \sigma_k F_k - 2 \sum_{m=1}^{N-1} \rho_{mn} \sigma_m F_m \right) \sigma_n F_n + \sum_{p=1}^{N-1} \sum_{q=1}^{N-1} \rho_{pq} \sigma_p \sigma_q F_p F_q \right\} \frac{\partial^2}{\partial f_n \partial f_k}. \tag{40}$$

The formal solution of Equation (36) is given by

$$P(F_1, F_2, \cdots, F_N, \tau) = e^{-r\tau} \exp \left\{ \tau \left(\hat{L}_N + \hat{L}_+ + \hat{L}_{N+} + \hat{L}_R \right) \right\} \times \max(F_N - F_+ - K, 0). \tag{41}$$

Then we apply the Lie-Trotter operator splitting method [10] to obtain an approximation to the formal solution $P(F_1, F_2, \cdots, F_N, \tau)$, namely (see the Appendix)

$$P^{LT}(F_1, F_2, \cdots, F_N, \tau)$$
$$= e^{-r\tau} \exp \left\{ \tau \left(\hat{L}_N + \hat{L}_+ + \hat{L}_{N+} \right) \right\} \exp \left\{ \tau \hat{L}_R \right\} \max(F_N - F_+ - K, 0) \tag{42}$$
$$= e^{-r\tau} \exp \left\{ \tau \left(\hat{L}_N + \hat{L}_+ + \hat{L}_{N+} \right) \right\} \max(F_N - F_+ - K, 0),$$

where the relation

$$\hat{L}_R \max(F_N - F_+ - K, 0) = 0 \Rightarrow \exp \left\{ \tau \hat{L}_R \right\} \max(F_N - F_+ - K, 0) = \max(F_N - F_+ - K, 0). \tag{43}$$

is utilized.

Next, in terms of the two new variables

$$R_1 = \frac{F_N}{F_+ + K} \quad \text{and} \quad R_2 = F_+ + K, \tag{44}$$

we rewrite $P^{LT}(F_1, F_2, \cdots, F_N, \tau)$ as follows:

$$P^{LT}\left(F_1, F_2, \cdots, F_N, \tau\right) = \mathrm{e}^{-r\tau} \exp\left\{\tau\left(\hat{L}_A + \hat{L}_B\right)\right\} R_2 \max\left(R_1 - 1, 0\right), \tag{45}$$

where

$$\hat{L}_A = \frac{1}{2}\bar{\sigma}_-^2 R_1^2 \frac{\partial^2}{\partial R_1^2} \tag{46}$$

$$\hat{L}_B = \frac{1}{2}\bar{\sigma}_+^2 R_2^2 \frac{\partial^2}{\partial R_2^2} + \left(\tilde{\rho}\sigma_N - \bar{\sigma}_+\right)\bar{\sigma}_+ R_1 R_2 \frac{\partial^2}{\partial R_1 \partial R_2} - \left(\tilde{\rho}\sigma_N - \bar{\sigma}_+\right)\bar{\sigma}_+ R_1 \frac{\partial}{\partial R_1} \tag{47}$$

$$\bar{\sigma}_- = \sqrt{\sigma_N^2 - 2\tilde{\rho}\sigma_N\bar{\sigma}_+ + \bar{\sigma}_+^2} \tag{48}$$

$$\bar{\sigma}_+ = \tilde{\sigma}_+\left(\frac{F_+}{F_+ + K}\right). \tag{49}$$

It is clear that the exponential operator $\exp\left\{\tau\left(\hat{L}_A + \hat{L}_B\right)\right\}$ is difficult to evaluate, so we need to apply the Lie-Trotter operator splitting method [10] again to approximate the operator by $\exp\left\{\tau\hat{L}_A\right\}\exp\left\{\tau\hat{L}_B\right\}$. As a result, we obtain

$$P^{LT}\left(F_1, F_2, \cdots, F_N, \tau\right) \approx \mathrm{e}^{-r\tau} \exp\left\{\tau\hat{L}_A\right\}\exp\left\{\tau\hat{L}_B\right\} R_2 \max\left(R_1 - 1, 0\right)$$
$$\approx \mathrm{e}^{-r\tau} R_2 \exp\left\{\tau\hat{L}_A\right\} \max\left(R_1 - 1, 0\right) \tag{50}$$

for

$$\hat{L}_B R_2 \max\left(R_1 - 1, 0\right) = 0 \Rightarrow \exp\left\{\tau\hat{L}_B\right\} R_2 \max\left(R_1 - 1, 0\right) = R_2 \max\left(R_1 - 1, 0\right). \tag{51}$$

Since $\bar{\sigma}_-$ is independent of R_1, the standard workhorse of the Black-Scholes model can be used to evaluate $\exp\left\{\tau\hat{L}_A\right\} \max\left(R_1 - 1, 0\right)$ such that

$$P^{LT}\left(F_1, F_2, \cdots, F_N, \tau\right) \approx \mathrm{e}^{-r\tau} R_2 \left\{R_1 N\left(\theta_1\right) - N\left(\theta_2\right)\right\} \tag{52}$$

where

$$\theta_1 = \frac{1}{\bar{\sigma}_-\sqrt{\tau}}\left\{\ln\left(\frac{F_N}{F_+ + K}\right) + \frac{1}{2}\bar{\sigma}_-^2\tau\right\} \tag{53}$$

$$\theta_2 = \theta_1 - \bar{\sigma}_-\sqrt{\tau}. \tag{54}$$

It is obvious that this approximate solution is identical to the approximate price formula given in Equation (29). (Q.E.D.)

In terms of the spot asset prices, namely $S_i \equiv F_i \exp\left(-r\tau\right)$, the price formula given in Equation (29) becomes

$$P_{Kirk}\left(S_1, S_2, \cdots, S_N, \tau\right) = S_N N\left(d_1\right) - \left(\sum_{j=1}^{N-1} S_j + K\mathrm{e}^{-r\tau}\right) N\left(d_2\right) \tag{55}$$

where

$$d_1 = \frac{\ln S_N - \ln\left(\sum_{j=1}^{N-1} S_j + K\mathrm{e}^{-r\tau}\right)}{\hat{\sigma}_-\sqrt{\tau}} + \frac{1}{2}\hat{\sigma}_-\sqrt{\tau} \tag{56}$$

$$d_2 = d_1 - \hat{\sigma}_-\sqrt{\tau} \tag{57}$$

$$\hat{\sigma}_- = \sqrt{\sigma_N^2 - 2\bar{\rho}\sigma_N\hat{\sigma}_+ + \hat{\sigma}_+^2} \tag{58}$$

$$\hat{\sigma}_+ = \frac{\sqrt{\sum_{j,k=1}^{N-1}\rho_{jk}\sigma_j\sigma_k S_j S_k}}{\sum_{j=1}^{N-1} S_j + K\mathrm{e}^{-r\tau}} \tag{59}$$

$$\bar{\rho} = \frac{1}{\hat{\sigma}_+} \left(\frac{\sum_{j=1}^{N-1} \rho_{jN} \sigma_j S_j}{\sum_{j=1}^{N-1} S_j} \right). \tag{60}$$

Obviously, this approximate price formula resembles the price formula of Kirk's approximation in the two-asset case very closely. In fact, by setting $N = 2$ we can recover Kirk's approximation readily. Moreover, for the Lie-Trotter splitting approximation to be valid, we need to require $\hat{\sigma}_-^2 \tau$ to be sufficiently small, namely $\bar{\sigma}_-^2 \tau \ll 1$. In accordance with Equation (58), this implies that the extended Kirk approximation is more favourable for those cases with positive effective correlation $\bar{\rho}$.

4. Illustrative Numerical Examples for the Three-Asset Spread Options

In this section, illustrative numerical examples are presented to demonstrate the accuracy of the extended Kirk approximation for the three-asset spread options. Although most spread options involve two assets only, yet there is a growing demand for three-asset spread options which can be found in the models for power plants or their financial equivalents—tolling contracts. We examine a simple three-asset spread option with the final payoff $\max(S_3 - S_1 - S_2 - K, 0)$. **Table 1** tabulates the approximate option prices estimated by the extended Kirk approximation for different values of the strike price K and time-to-maturity T. Other input model parameters are set as follows: $r = 0.05$, $\sigma_1 = \sigma_2 = \sigma_3 = 0.3$, $\rho_{12} = 0.4$, $\rho_{23} = 0.2$, $\rho_{13} = 0.8$, $S_1 = 50$, $S_2 = 60$ and $S_3 = 150$. Monte Carlo estimates and the corresponding standard deviations are also presented for comparison. It is observed that the computed errors of the approximate option prices are capped at 1% (in magnitude). In fact, most of them are less than 0.5%. Then, in **Table 2** the effect of increasing the three volatilities (from 0.3 to 0.6) upon the approximate estimation of the option prices is investigated. Obviously only a slight increase occurs in the computed errors, and these errors are still less than 1% (in magnitude). Finally, we study a case in which all the three volatilities are different, namely $\sigma_1 = 0.5$, $\sigma_2 = 0.4$ and $\sigma_3 = 0.3$, while the other parameters remain the same. According to **Table 3**, the computed errors generally increase a little bit in this case but they do not exceed 1.3% (in magnitude). As a result, it can be concluded that the extended Kirk approximation for the three-asset spread option is found to be very accurate and efficient.

5. Conclusion

In this paper, we have proved that Kirk's approximation for two-asset spread options can be rigorously derived

Table 1. Prices of a European three-asset call spread option. Other input parameters are: r = 0.05, $\sigma_1 = \sigma_2 = \sigma_3 = 0.3$, $\rho_{12} = 0.4$, $\rho_{23} = 0.2$, $\rho_{13} = 0.8$, $S_1 = 50$, $S_2 = 60$ and $S_3 = 150$. Here "EK" refers to the extended Kirk approximation while "MC" denotes the Monte Carlo estimates with 900,000,000 replications. The relative errors of the "EK" option prices with respect to the "MC" estimates are also presented.

K\T	0.25	0.5	1	2	
30	13.5410	16.4210	20.7761	27.1974	EK
	−0.3%	−0.3%	−0.3%	−0.3%	error
	13.5763 ± 0.0089	16.4735 ± 0.0142	20.8471 ± 0.0185	27.2841 ± 0.0264	MC
35	10.3383	13.5024	18.1191	24.8196	EK
	−0.2%	−0.2%	−0.2%	−0.2%	error
	10.3576 ± 0.0086	13.5286 ± 0.0124	18.1541 ± 0.0176	24.8573 ± 0.0276	MC
40	7.6613	10.9586	15.7231	22.6176	EK
	0.0%	0.0%	0.0%	0.1%	error
	7.6608 ± 0.0068	10.9572 ± 0.0109	15.7197 ± 0.0161	22.6072 ± 0.0244	MC
45	5.5097	8.7824	13.5805	20.5856	EK
	0.4%	0.3%	0.3%	0.3%	error
	5.4903 ± 0.0051	8.7565 ± 0.0106	13.5421 ± 0.0141	20.5302 ± 0.0260	MC
50	3.8470	6.9540	11.6795	18.7162	EK
	0.9%	0.7%	0.6%	0.5%	error
	3.8140 ± 0.0040	6.9058 ± 0.0086	11.6109 ± 0.0136	18.6195 ± 0.0245	MC

Table 2. Prices of a European three-asset call spread option. Other input parameters are: r = 0.05, $\sigma_1 = \sigma_2 = \sigma_3 = 0.6$, $\rho_{12} = 0.4$, $\rho_{23} = 0.2$, $\rho_{13} = 0.8$, $S_1 = 50$, $S_2 = 60$ and $S_3 = 150$. Here "EK" refers to the extended Kirk approximation while "MC" denotes the Monte Carlo estimates with 900,000,000 replications. The relative errors of the "EK" option prices with respect to the "MC" estimates are also presented.

K\T	0.25	0.5	1	2	
30	20.1436	26.0640	34.5186	46.3820	EK
	−0.3%	−0.3%	−0.1%	0.3%	error
	20.2066 ± 0.0168	26.1269 ± 0.0278	34.5402 ± 0.0425	46.2242 ± 0.0716	MC
35	17.4529	23.5976	32.2944	44.4495	EK
	−0.1%	0.0%	0.1%	0.5%	error
	17.4778 ± 0.0172	23.6076 ± 0.0268	32.2508 ± 0.0390	44.2275 ± 0.0787	MC
40	15.0417	21.3320	30.2150	42.6221	EK
	0.1%	0.2%	0.4%	0.7%	error
	15.0290 ± 0.0171	21.2938 ± 0.0266	30.1079 ± 0.0362	42.3425 ± 0.0656	MC
45	12.9002	19.2587	28.2733	40.8938	EK
	0.4%	0.4%	0.6%	0.9%	error
	12.8527 ± 0.0182	19.1753 ± 0.0234	28.1113 ± 0.0381	40.5466 ± 0.0628	MC
50	11.0139	17.3676	26.4620	39.2590	EK
	0.7%	0.7%	0.8%	1.0%	error
	10.9347 ± 0.0151	17.2410 ± 0.0236	26.2513 ± 0.0345	38.8658 ±0.0637	MC

Table 3. Prices of a European three-asset call spread option. Other input parameters are: r = 0.05, $\sigma_1 = 0.5$, $\sigma_2 = 0.4$, $\sigma_3 = 0.3$, $\rho_{12} = 0.4$, $\rho_{23} = 0.2$, $\rho_{13} = 0.8$, $S_1 = 50$, $S_2 = 60$ and $S_3 = 150$. Here "EK" refers to the extended Kirk approximation while "MC" denotes the Monte Carlo estimates with 900,000,000 replications. The relative errors of the "EK" option prices with respect to the "MC" estimates are also presented.

K\T	0.25	0.5	1	2	
30	13.8987	16.9731	21.6091	28.4620	EK
	−0.5%	−0.6%	−0.8%	−1.2%	error
	13.9635 ± 0.0092	17.0801 ± 0.0119	21.7922 ± 0.0191	28.8014 ± 0.0242	MC
35	10.6503	13.9613	18.8011	25.8630	EK
	−0.4%	−0.5%	−0.7%	−1.1%	error
	10.6905 ± 0.0078	14.0301 ± 0.0118	18.9316 ± 0.0163	26.1380 ± 0.0238	MC
40	7.9011	11.3085	16.2480	23.4424	EK
	−0.1%	−0.2%	−0.4%	−0.9%	error
	7.9101 ± 0.0074	11.3342 ± 0.0104	16.3206 ± 0.0172	23.6452 ± 0.0223	MC
45	5.6664	9.0191	13.9501	21.1989	EK
	0.4%	0.2%	−0.1%	−0.6%	error
	5.6426 ± 0.0064	9.0004 ± 0.0092	13.9627 ± 0.0145	21.3286 ± 0.0210	MC
50	3.9253	7.0838	11.9023	19.1291	EK
	1.3%	0.8%	0.4%	−0.3%	error
	3.8758 ± 0.0058	7.0242 ± 0.0081	11.8587 ± 0.0147	19.1882 ± 0.0216	MC

by applying the Lie-Trotter operator splitting approximation to the Black-Scholes equation, and that for cases with vanishing strike prices, the Lie-Trotter splitting approximation becomes exact and Margrabe's formula is recovered. Our derivation also shows that Kirk's approximation is more favourable for those cases with two positively correlated assets. Moreover, we apply the Lie-Trotter operator splitting method to obtain a generalisation of Kirk's approximation for multi-asset spread options in a straightforward manner. The derived price formula for the multi-asset spread option closely resembles Kirk's approximation in the two-asset case. By setting the total number of assets to be two, we can recover Kirk's approximation readily. Thus, the generalisation possesses the same nice features as Kirk's approximation. For instance, as shown by the illustrative

examples, the generalization is found to be very accurate and efficient in pricing the multi-asset spread options. All in all, our approach is able to provide a new perspective on Kirk's approximation and the generalisation; that is, they are simply equivalent to the Lie-Trotter operator splitting approximation to the Black-Scholes equation.

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Appendix: Lie-Trotter Operator Splitting Method

Suppose that one needs to exponentiate an operator $\hat{C}$ which can be split into two different parts, namely $\hat{A}$ and $\hat{B}$. For simplicity, let us assume that $\hat{C} = \hat{A} + \hat{B}$, where the exponential operator $\exp\left(\hat{C}\right)$ is difficult to evaluate but $\exp\left(\hat{A}\right)$ and $\exp\left(\hat{B}\right)$ are either solvable or easy to deal with. Under such circumstances, the exponential operator $\exp\left(\varepsilon\hat{C}\right)$, with ε being a small parameter, can be approximated by the Lie-Trotter operator splitting formula:

$$\exp\left(\varepsilon\hat{C}\right) = \exp\left(\varepsilon\hat{A}\right)\exp\left(\varepsilon\hat{B}\right) + O\left(\varepsilon^2\right). \tag{A.1}$$

This can be seen as the approximation to the solution at $t = \varepsilon$ of the equation $d\hat{Y}/dt = \left(\hat{A} + \hat{B}\right)\hat{Y}$ by a composition of the exact solutions of the equations $d\hat{Y}/dt = \hat{A}\hat{Y}$ and $d\hat{Y}/dt = \hat{B}\hat{Y}$ at time $t = \varepsilon$. Details of the Lie-Trotter splitting approximation can be found in Trotter [8] [10], Suzuki [11], Drozdov and Brey [12], Hatano and Suzuki [13], and Blanes *et al*. [14]. The Lie-Trotter splitting approximation is particularly useful for studying the short-time behaviour of the solutions of evolutionary partial differential equations of parabolic type because for this class of problems it is sensible to split the spatial differential operator into several parts each of which corresponds to a different physical contribution (e.g., reaction and diffusion).