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An Elementary Study of Chaotic Behaviors in 1-D Maps

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Abstract

In this article, we have discussed basic concepts of one-dimensional maps like Cubic map, Sine map and analyzed their chaotic behaviors in several senses in the unit interval. We have mainly focused on Orbit Analysis, Time Series Analysis, Lyapunov Exponent Analysis, Sensitivity to Initial Conditions, Bifurcation Diagram, Cobweb Diagram, Histogram, Mathematical Analysis by Newton's Iteration, Trajectories and Sensitivity to Numerical Inaccuracies of the said maps. We have tried to make decision about these mentioned maps whether chaotic or not on a unique interval of parameter value. We have performed numerical calculations and graphical representations for all parameter values on that interval and have tried to find if there is any single value of parameter for which those maps are chaotic. In our calculations we have found there are many values for which those maps are chaotic. We have showed numerical calculations and graphical representations for single value of the parameter only in this paper which gives a clear visualization of chaotic dynamics. We performed all graphical activities by using Mathematica and MATLAB.

Keywords

Sensitivity, Trajectory, Numerical Inaccuracies, Orbit, Bifurcation, Cobweb Diagram

1. Introduction and Background

During the last few decades, dynamical system [1] has made long strides. Dynamical system is the study of the long-term behavior of an evolving system. It is observed that in various models of economics, biology and various other sciences of the chaotic nonlinear dynamical system has made its presence felt.

The phenomenon of chaos [2] has been studied extensively and it has attracted increasing interests from mathematicians, physicists, engineers, and so on. Since chaotic systems not only admit abundant complex and interesting dynamical behaviors but also have many potential practical applications, great efforts have been devoted to an investigation related to these systems.

Chaos is an interdisciplinary theory stating that within the apparent randomness of chaotic complex systems, there are underlying patterns, constant feedback loops, repetition, self-similarity, fractals, self-organization, and reliance on programming at the initial point known as sensitive dependence on initial conditions. The butterfly effect describes how a small change in one state a deterministic nonlinear system can result in large differences in a later state.

Chaos theory concerns deterministic systems whose behavior can in principle be predicted. Chaotic systems are predictable for a while and then "appear" to become random [3]. The amount of time that the behavior of a chaotic system can be effectively predicted depends on three things: How much uncertainty can be tolerated in the forecast, how accurately its current state can be measured, and a time scale depending on the dynamics of the system, called the Lyapunov time. Some examples of Lyapunov times are: chaotic electrical circuits. In chaotic systems, the uncertainty in a forecast increases exponentially with elapsed time. Hence, mathematically, doubling the forecast time more than squares the proportional uncertainty in the forecast. This means, in practice, a meaningful prediction cannot be made over an interval of more than two or three times the Lyapunov time. When meaningful predictions cannot be made, the system appears random. The most important aspects of chaotic behavior should appear in systems of lowest dimension. Thus, we would like in the first step to reduce as much as possible the dimension of state space. However, this quickly conflicts with the requirement of invertibility. On the one hand, it can be shown that maps based on a one-dimensional homeomorphism can only display stationary or periodic regimes, and hence cannot be chaotic. On the other hand, if we sacrifice invertibility temporarily, thereby introducing singularities, one-dimensional chaotic systems can easily be found. In Mathematics, researchers deal with various maps to study the different qualitative features related to it. It is also seen that map with one critical point, is not too difficult to study. But a map with two critical points in symmetrical case which was first investigated by May [4] [5] who was motivated by a problem in genetics involving one locus with two alleles is little difficult. After the investigation, he concluded that cubic map could describe the population dynamics of certain genetic groups. Also various researchers contributed in the study of the cubic map [6]. Perhaps the two most frequently mentioned are the logistic map and the tent map. It is shown to be "universal" for a large family of maps. It is also shown in [7] that unimodal maps such as the sine map with negative Schwarzian derivative are chaotic (in many definitions on the word). The use of symbolic dynamics in analysis of maps on the unit interval can be seen in [8].

In the literature, many chaotic properties numerically and graphically have been developed in order to understand the dynamics of one-dimensional maps and have been applied to different maps. Griffin [9] discussed the bifurcation and entropy of one dimensional sine map, Hemanta [10] represents various bifurcations in a cubic map, Hemanta and Sarmah [11] showed Lyapunov exponents and time series analysis of a chaotic cubic map. Hidayet and Mustafa [12] represented the application of sine map in image cryptosystem; Xiuping [13] showed the application of cubic map in image encryption. Ruman [14] discussed the dynamical behaviors of one-dimensional logistic map. Zakir [15] represents the dynamical behaviors of one-dimensional doubling map. The study aimed at finding whether the considered maps represent randomness in dynamics on a unique interval.

Yet despite these tremendous accomplishments and other remarkable advances in our understanding of chaotic dynamics, there is still no clear discussion about the dynamics of sine and cubic maps in different chaotic approaches. These two maps are neglected yet. Such gaps in our understanding thrives us to find the dynamics of these two maps by analyzing more chaotic properties.

2. Basic Preliminaries

Dynamical Systems: Dynamical Systems is a branch of mathematics that attempts to understand processes in motion. The world's weather is another system that changes in time as is the stock market.

Iteration: Iterate means to repeat a process over and over again. To iterate a functions means to evaluate the function over and over, using the output of the previous application as the input for the next.

Orbit: Given $x_0 \in R$ (x_0 is called the seed or initial value of the orbit), we define the orbit of x_0 under *f* to be the sequence
$\{x_0, x_1 = f(x_0), x_2 = f^2(x_0), \cdots, x_n = f^n(x_0), \cdots\}$.

Fixed Points: Let $f : \mathbf{R} \to \mathbf{R}$ be a map. The point x_0 is called fixed point if $f(x_0) = x_0$. Note that $f^2(x_0) = f(f(x_0)) = f(x_0) = x_0$, and in general $f^n(x_0) = x_0$.

Periodic Orbit or Cycle: The point x_0 is called periodic if $f^n(x_0) = x_0$ for some $n > 0$, where n is called the prime period of the orbit.

Chaotic Orbits: Over the last twenty five years, one of the major developments in mathematics is that many simple functions such as quadratic function of real variable exhibits orbits of incredible complexity called "sensitivity to initial conditions" and also called chaotic behavior.

Sensitivity on Initial Conditions: Mathematically, A continuous map $f : X \to X$ has sensitive dependence on initial conditions if there exists $\delta > 0$ such that, for any $x \in X$ and any neighborhood $N(x)$ of *x*, there exist $y \in N(x)$, $n \geq 0$ such that $d(f^n(x), f^n(y)) > \delta$, where (X, d) is a compact metric space.

Devaney's Definition of Chaos (*R. L. Devaney* 1989):

Let X be a metric space. A continuous function $f : X \to X$ is said to be *chaotic* on X if f has the following three properties:

(C-1) Periodic points are dense in the space X.

(C-2) f is topologically transitive.

(C-3) f *has* sensitive dependence on initial conditions.

Lyapunov Definition of Chaos: Consider the continuous and differentiable map $f:\mathbf{R}\to\mathbf{R}$. Then f is said to be chaotic according to Lyapunov or L-chaotic if:

1) f is topologically transitive.

2) f has a positive Lyapunov exponent.

Trajectory: In dynamical systems, a trajectory is the set of points in state space that are the future states resulting from a given initial state. In a discrete dynamical system, a trajectory is a set of isolated points in state space. In a continuous dynamical system, a trajectory is a curve in state space.

Cubic Map: The Cubic map $f:\mathbf{R}\to\mathbf{R}$ is defined by $f(x)=x^3-rx$. Where r is the parameter, which is one of the simplest polynomial maps of the desired type. If r is restricted to the range $0\le r\le 3$ then f maps the interval $x\in[-1,1]$ into itself and we will study the family $f=f_r$ for these parameter values.

Sine Map: The Sine map is defined by $f(x)=\lambda\sin(\pi x)$; $x\in[0,1]$, $\lambda\in[0,1]$, where λ is the parameter value which lies between 0 and 1. This map is similar to the logistic map on the unit interval.

3. Theorems

3.1. Proposition

Show that the Cubic map $f(x)=x^3-rx$ is chaotic in the interval $0\le r\le 3$.

Proof: Here we discuss the Lyapunov exponent for the Cubic map $f(x)=x^3-rx$.

Consider two iterations of the Cubic map starting from two values of x which are close together. Let the two starting values be x_0 and $x_0+\delta x_0$. These map to x_1 and $x_1+\delta x_1,\cdots,x_n+\delta x_n$. Expanding $f(x)$ about x_n we have $\delta x_n=f'(x_{n-1})\delta x_{n-1}$. Assuming that δx_n is sufficiently small. Hence the separation of two trajectories after n steps, δx_n is related to their initial separation δx_0 by $\left|\frac{\delta x_n}{\delta x_0}\right|=\prod_{i=0}^{n-1}\left|f'(x_i)\right|$. We expect that this will vary exponentially at large n like $\left|\frac{\delta x_n}{\delta x_0}\right|=e^{r_L}$ (Large n).

And so we define the Lyapunov exponent r_L by $r_L=\lim_{n\to\infty}\frac{1}{n}\sum_{i=1}^{n}\ln\left|f'(x_i)\right|$. If $r_L>0$ neighboring trajectories diverge from each other at large n this corresponds to chaos. However if the trajectories converge to a fixed point or limit cycle they will get closer together, which corresponds to $r_L<0$.

Hence we can determine whether or not the system is chaotic by the sign of the Lyapunov exponent.

Below we calculate the Lyapunov exponent for some values of parameter r (not to be confused with the Lyapunov exponent r_L). For $r = 2.56$ we get $r_L = 0.571265$. The positive value indicates that $r = 2.56$ is in a region of chaos. By contrast if we specify $r = 2.116$ we get $r_L = -1.54957$ which is a negative value, indicating that the trajectory of points $x_i\left(i = 0,1,2,\cdots\right)$ converges to an attractor, which in this case we already know is a length of 2 limit cycle. Next lets plot the Lyapunov exponent for a range of values of r.

The main significance of Figure 1 is that one can easily distinguish the regions which are chaotic $\left(r_L > 0\right)$ from the regions which tend to a fixed point or limit cycle $\left(r_L < 0\right)$. We see several points (the first is at $r = 1$) where the Lyapunov exponent hits 0 and then goes negative again. These are the period doubling bifurcations. Precisely at the period doubling point the system is at the limit of chaos, but then becomes non-chaotic when the period doubles. However, at the end of the period doubling regime, at r about 2.3024, r_L crosses the axis and the system enters a chaotic regime.

Note that for r in the range greater than the point where r_L first goes positive, there are many regions where r_L is negative, is known as "island of stability" where the behavior is fixed point or limit cycle [16]. Notice that there are fine details no matter how much one expands the scale. We see that chaos emerges (*i.e.* $r_L > 0$) for r between 2.3024 and 2.8999. It is again become non chaotic for the values of r between 2.8999 and 2.985 then it enters into the chaotic region again.

3.2. Proposition

Show that the Sine map $f\left(x\right) = \lambda \sin\left(\pi x\right)$ is chaotic in the interval $0 \le \lambda \le 1$.

Proof: Here we discuss the Lyapunov exponent for the Sine map $f\left(x\right) = \lambda \sin\left(\pi x\right)$.

Consider two iterations of the Sine map starting from two values of x which are close together. Let the two starting values be x_0 and $x_0 + \delta x_0$. These map

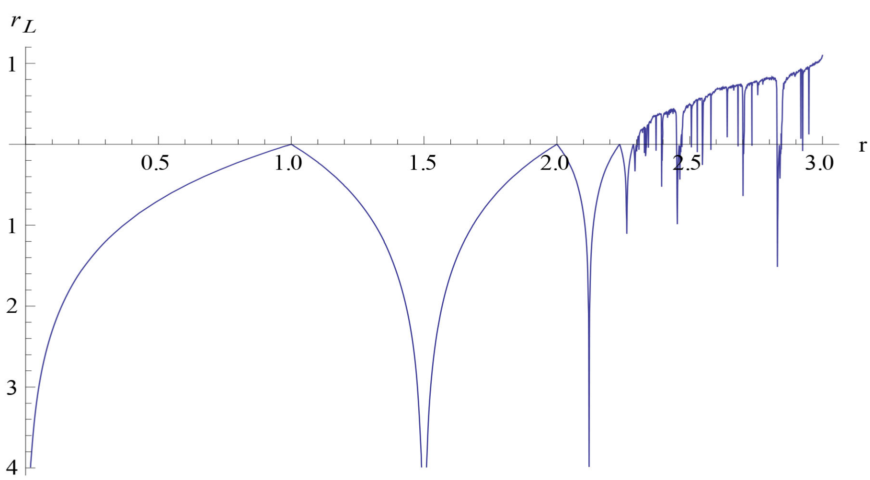

Figure 1. Lyapunov exponent for $0 \le r \le 3$.

to x_1 and $x_1+\delta x_1,\cdots,x_n+\delta x_n$. Expanding $f(x)$ about x_n we have $\delta x_n = f'(x_{n-1})\delta x_{n-1}$. Assuming that δx_n is sufficiently small. Hence the separation of two trajectories after n steps, δx_n is related to their initial separation δx_0 by $\left|\frac{\delta x_n}{\delta x_0}\right| = \prod_{i=0}^{n-1}\left|f'(x_i)\right|$. We expect that this will vary exponentially at large n like $\left|\frac{\delta x_n}{\delta x_0}\right| = e^{\lambda_L}$.

(Large n) And so we define the Lyapunov exponent λ_L by $\lambda_L = \lim_{n\to\infty}\frac{1}{n}\sum_{i=1}^{n}\ln\left|f'(x_i)\right|$. If $\lambda_L > 0$ neighboring trajectories diverge from each other at large n this corresponds to chaos. However if the trajectories converge to a fixed point or limit cycle they will get closer together, which corresponds to $\lambda_L < 0$. Hence we can determine whether or not the system is chaotic by the sign of the Lyapunov exponent. Below we calculate the lyapunov exponent for some values of parameter λ (not to be confused with the Lyapunov exponent λ_L). For $\lambda = 0.9$ we get $\lambda_L = 0.354839$. The positive value indicates that $\lambda = 0.9$ is in a region of chaos. By contrast if we specify $\lambda = 0.55$ we get $\lambda_L = -1.42149$ which is a negative value, indicating that the trajectory of points $x_i\,(i = 0,1,2,\cdots)$ converges to an attractor, which in this case we already know is a length of 2 limit cycle. Next lets plot the Lyapunov exponent for a range of values of λ.

One can easily distinguish from **Figure 2** the regions which are chaotic $(\lambda_L > 0)$ from the regions which tend to a fixed point or limit cycle $(\lambda_L < 0)$. We see several points (the first is at $a = 0.31849$) where the Lyapunov exponent hits 0 and then goes negative again. These are the period doubling bifurcations. Precisely at the period doubling point the system is at the limit of chaos, but then becomes non-chaotic when the period doubles [16]. However, at the end of the period doubling regime, at λ about 0.865, λ_L crosses the axis and the system enters a chaotic regime. Notice that there is fine detail no matter how

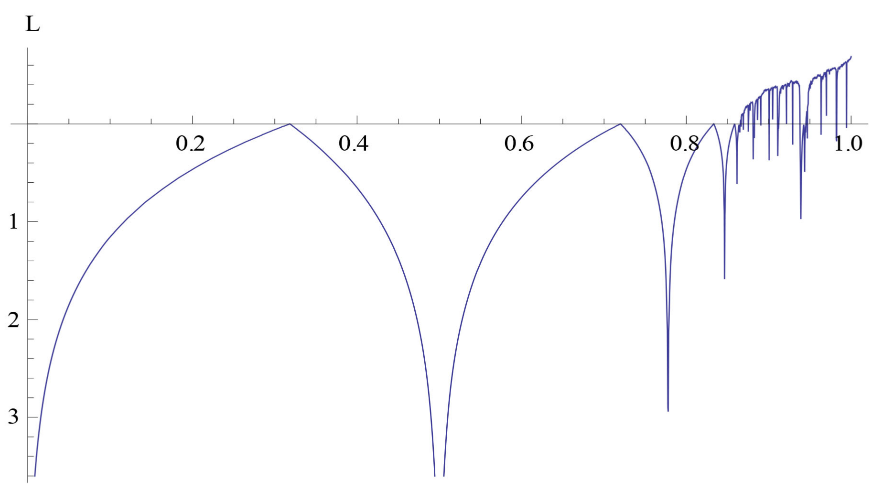

Figure 2. Lyapunov exponent for $0 \le \lambda \le 1$.

much one expands the scale. We see that chaos emerges (*i.e.* $\lambda_L > 0$) for λ between 0.8655 and 0.8660. It is again become non chaotic for the values of λ between 0.8810 and 0.8825 then it enters into the chaotic region finally.

4. Theoretical Foundation

In this article we have not established new theorem but we have analyzed new chaotic properties for two maps named Sine map, Cubic map graphically and numerically based on the chaotic properties established by renowned mathematicians R. L. Devaney, Henry Poincare, Edward Lorentz and some of modern researchers. These strong properties show that these maps are chaotic clearly.

5. Result Discussion

5.1. Behavior of the Maps Taking Initial Seeds

Cubic map: In this section we iterate the cubic map $f(x) = x^3 - rx$ for the following r and x_0 values (initial seeds) and investigate the dynamical behavior of the given function $f(x)$ considering the following case: $r = 2.9$

1) For $x_0 = 0.30$, The orbit is:

$\{0.30, -0.84, 1.84, 0.93, -1.89, -1.30, 1.57, -0.68, 1.66, \cdots\}$

2) For $x_0 = 0.35$, The orbit is:

$\{0.35, -0.97, 1.90, 1.35, -1.44, 1.16, -1.79, -0.59, 1.51, \cdots\}$

3) For $x_0 = 0.45$, The orbit is:

$\{0.45, -1.21, 1.73, 0.17, -0.48, 1.30, -1.56, 0.68, -1.67, \cdots\}$

4) For $x_0 = 0.50$, The orbit is:

$\{0.50, -1.32, 1.51, -0.91, 1.88, 1.23, -1.69, 0.05, -0.17, \cdots\}$

Do you see any pattern? Obviously not. There is no initial seed for which the orbit forms any cycle and forms any kind of patterns. The orbits approach randomly. The initial seeds 0.30, 0.35, 0.45 and 0.50 are neither fixed/periodic points nor eventually fixed/periodic points of $f(x)$. We conclude that the dynamical behavior [17] of the given function for $r = 2.9$ is chaotic.

Sine map: Now we want to investigate the behavior of sine map for $\lambda \in [0,1]$. For this we iterate the Sine map $f(x) = \lambda \sin(\pi x)$ for the following λ and x_0 values (initial seed) and we investigate the dynamical behavior of the given function $f(x)$ considering the following case: Taking $\lambda = 0.95$;

1) For $x_0 = 0.30$, The orbit is:

$\{0.30, 0.77, 0.63, 0.87, 0.37, 0.88, 0.35, 0.84, 0.44, \cdots\}$

2) For $x_0 = 0.35$, The orbit is:

$\{0.35, 0.84, 0.44, 0.93, 0.19, 0.55, 0.94, 0.18, 0.52, \cdots\}$

3) For $x_0 = 0.45$, The orbit is:

$\{0.45, 0.94, 0.18, 0.51, 0.95, 0.15, 0.44, 0.93, 0.20, \cdots\}$

4) For $x_0 = 0.50$, The orbit is:

$\{0.50, 0.95, 0.15, 0.92, 0.22, 0.60, 0.89, 0.30, \cdots\}$

We see that there is no initial seed for which the orbit forms any cycle and forms any kind of patterns. The orbits approach randomly. The initial seeds 0.30, 0.35, 0.45 and 0.50 are neither fixed/periodic points nor eventually fixed/periodic points of $f(x)$. We conclude that the dynamical behavior of the given function is chaotic [18] for $\lambda = 0.95$.

5.2. Orbit Analysis of the Maps

Cubic map: Here we want to observe closely the behavior of the orbit graphically for given cubic map $f(x) = x^3 - rx$. Consider $r = 2.6$ to make a decision about the dynamical behavior of the orbits graphically for the following initial seeds.

Taking different initial seeds graphically we see from **Figure 3** and **Figure 4** that the orbit of given dynamical system changes its nature randomly. So we can conclude that the given Cubic map is chaotic for some values of $r \in [0,3]$.

Sine map: Now we want to perform same analysis for Sine map to do this we observe closely the behavior of the orbit graphically for given Sine map $f(x) = \lambda \sin(\pi x)$ considering different x_0 values (initial seeds). Take $\lambda = 0.95$ and observe the dynamical behavior of the orbits graphically for the following initial seeds.

Taking different initial seeds from **Figure 5** and **Figure 6** graphically we see that the orbit of given dynamical system changes its nature randomly. So the given Sine map is chaotic for some values of $\lambda \in [0,1]$.

5.3. Sensitivity Analysis of the Maps

We want to check the difference of the orbit by taking two neighbouring initial seeds. We first define the function governing the system and then calculate the distance between two orbits for the considered neighbouring initial seeds. Here we will consider 100 iteration and calculate the distance between two orbits. After that we will analyze whether the function is chaotic or not.

Cubic map: In this passage we want to analyze the sensitivity of Cubic map $f(x) = x^3 - rx$. Taking $r = 2.6$ and two neighboring initial seeds $x_0 = 0.30$ and $x_0 = 0.31$ we get the following table.

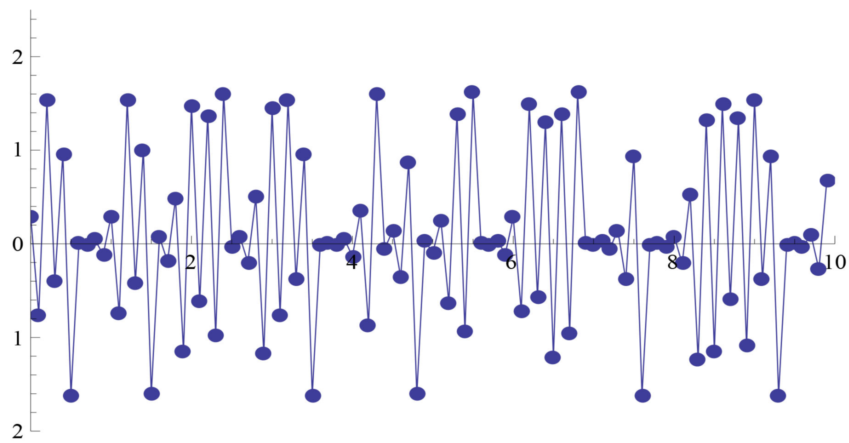

Figure 3. Consider initial seed $x_0 = 0.30$.

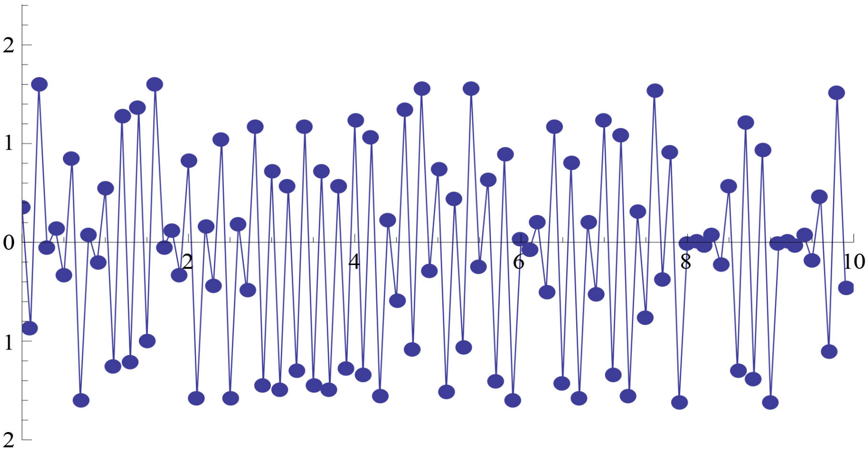

Figure 4. Consider initial seed $x_0 = 0.35$.

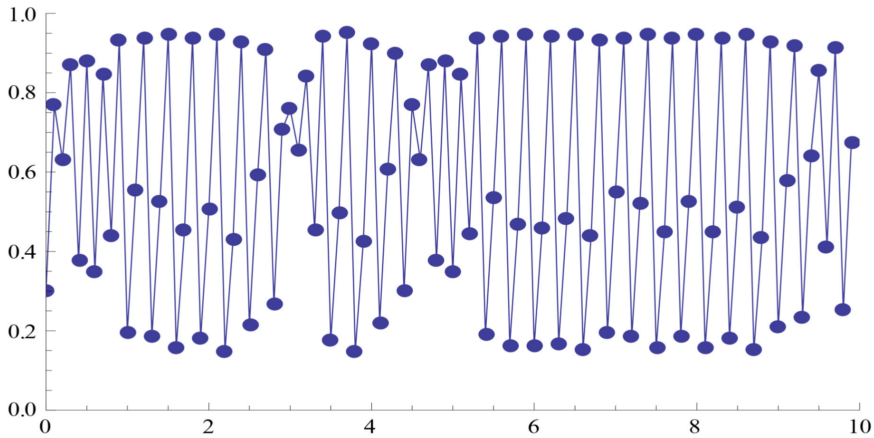

Figure 5. Consider initial seed $x_0 = 0.30$.

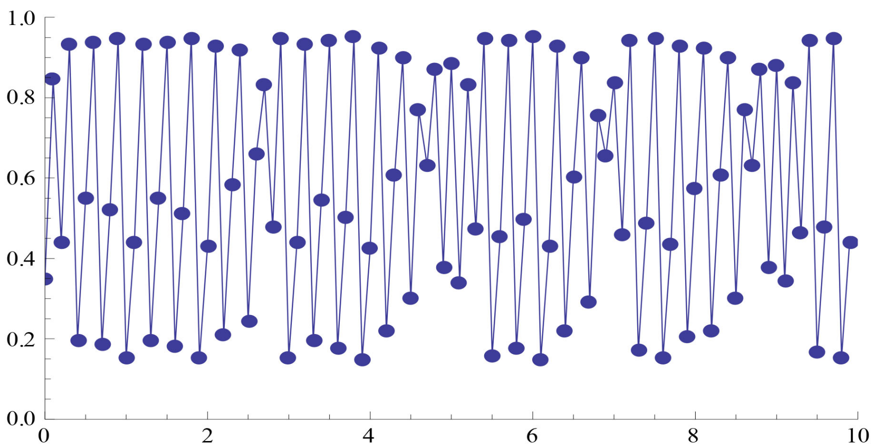

Figure 6. Consider initial seed $x_0 = 0.35$.

Table 1 tells us the "story" of the two orbits from the 1st to 100th positions. We see that the distance between the two orbits is bouncing between 0 and 2.5 in an apparent erratic manner. This type of behavior tells us the system is chaotic.

Table 1. Sensitivity analysis of cubic map.

(Iteration) n	$x_0 = 0.30$	$x_0 = 0.31$	Distance	$x_0 = 0.6$	$x_0 = 0.6000001$	Distance
1	−0.75	−0.77	0.02	−1.58	−1.58	0
2	1.53	1.55	0.02	0.77	0.77	0
3	−0.39	−0.30	0.09	−1.86	−1.86	0
4	0.96	0.76	0.20	−0.87	−0.87	0
5	−1.61	−1.54	0.07	1.95	1.95	0
.	.	.	.	.	.	.
45	0.14	−1.31	1.45	−0.58	1.83	2.41
46	−0.35	1.16	1.51	1.54	0.67	0.87
47	0.87	−1.45	2.33	−0.96	−1.72	0.76
48	−1.60	0.70	2.30	1.99	0.04	1.95
49	0.04	−1.48	1.52	1.97	−0.13	2.10
50	−0.09	0.62	0.71	1.71	0.40	1.31

We can say that the given Cubic map is chaotic for $r = 2.6$ according to the sensitivity to initial condition. Here we have only taken the values up to 2 decimal places and analyzed the orbits from the 1st to 100th positions. From this analysis we have already reached to our goal which we sat before. We found that given function is chaotic for some values of $r \in [0,3]$. The graphical representation of the above sensitivity analysis is shown below.

Is there any clear message for us in Figure 7 and Figure 8? Yes. In Figure 7 the orbits for two considered initial seeds are same up to 7th iteration than go far from each other. In Figure 8 the orbits for two considered initial seeds are same up to 18th iteration than separate. The orbits of the given Cubic map for two sets of neighbouring initial seeds do not coincide with each other. Two orbits are scattered. They show randomness in distance from each other and go far from each other after a large number of iteration. In above Figures red line represents the orbit of $x_0 = 0.30$ and 0.6 and green line represents the orbit of $x_0 = 0.31$ and 0.6000001.

Sine map: In this passage we want to analyze the sensitivity of Sine map $f(x) = \lambda \sin(\pi x)$. Taking $\lambda = 0.95$ and two neighbouring initial seeds $x_0 = 0.600$ and $x_0 = 0.601$ we get the following table.

We see from Table 2 that the distance between the two orbits is bouncing between 0 and 1 in an apparent erratic manner. This type of behavior tells us the system is chaotic. We can conclude that the given Sine map is chaotic [19] for $\lambda = 0.95$ according to the sensitivity to initial condition.

Here we have only taken the values up to 2 decimal places and analyzed the orbits from the 1st to 100th positions considering neighbouring initial seeds $x_0 = 0.3$, $x_0 = 0.301$ and $x_0 = 0.6$, $x_0 = 0.6000001$. From this analysis we can make decision that if we take any other neighbouring initial seeds our results will be same. The graphical representation of the above analysis is shown below.

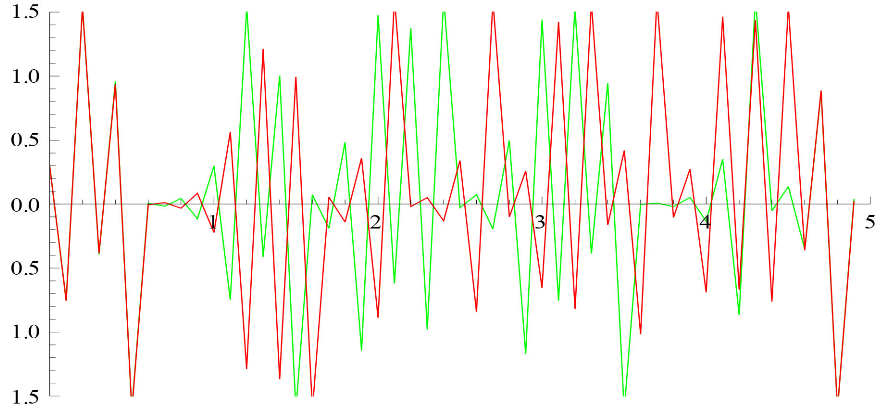

Figure 7. Two nearby seeds $x_0 = 0.30, 0.31$.

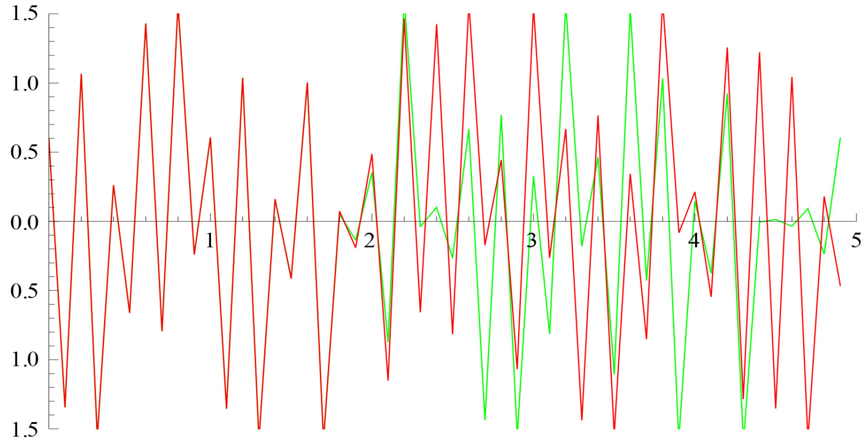

Figure 8. Two nearby seeds $x_0 = 0.6, 0.6000001$.

Table 2. Sensitivity analysis of Sine map.

(Iteration) N	$x_0 = 0.3$	$x_0 = 0.301$	Distance	$x_0 = 0.6$	$x_0 = 0.6000001$	Distance
1	0.77	0.77	0	0.90	0.90	0
2	0.63	0.63	0	0.28	0.28	0
3	0.87	0.87	0	0.74	0.74	0
4	0.37	0.36	0.01	0.69	0.69	0
5	0.88	0.86	0.02	0.78	0.78	0
.	.	.	.	.	.	.
45	0.77	0.44	0.33	0.15	0.16	0.01
46	0.63	0.93	0.30	0.43	0.47	0.04
47	0.87	0.19	0.68	0.93	0.95	0.02
48	0.38	0.53	0.15	0.21	0.15	0.06
49	0.88	0.94	0.06	0.59	0.45	0.14
50	0.35	0.16	0.19	0.91	0.94	0.03

From **Figure 9** we see that the orbits for two considered initial seeds are same up to 4^{th} iteration than go far from each other. In **Figure 10** the orbits for two considered initial seeds are same up to 22^{th} iteration than separate. The orbits of the given Sine map for two sets of neighbouring initial seeds do not coincide with each other. They show randomness in distance from each other and go far from each other after a large number of iteration.

In above Figures red line represents the orbit of $x_0 = 0.3$ and 0.6 and green line represents the orbit of $x_0 = 0.301$ and 0.6000001.

5.4. Time Series Analysis Graphically

Cubic map: The orbits listed in the numerical iteration part considering different initial seeds seem to be wandering around the interval $-1 \le x \le 1$ rather aimlessly. Let's see if we can detect a pattern from the time series for one of these orbits. Here is the time series graph for the seed $x_0 = 0.2, 0.2001$ with iteration 100 respectively.

Just when we think we are beginning to see a pattern in the above **Figure 11** and **Figure 12**, the time series graphs begins to do something else and a new pattern emerges after some iteration we observe that there is no pattern in the above picture. This is called the unpredictability [20] which is another meaning of Chaos.

Sine map: The orbits of given Sine map seem to be wandering around the interval $-1 \le x \le 1$ rather aimlessly. Let's see if we can detect a pattern from the time series for one of these orbits. Here is the time series graph for the seed $x_0 = 0.34, 0.2$ with iteration 50, 100 respectively.

From the above time series graph we see that there is no pattern and when we think for a pattern the graph in the **Figure 13** and **Figure 14** display some new shape. This graph approaches with erratic manner which we call as chaos.

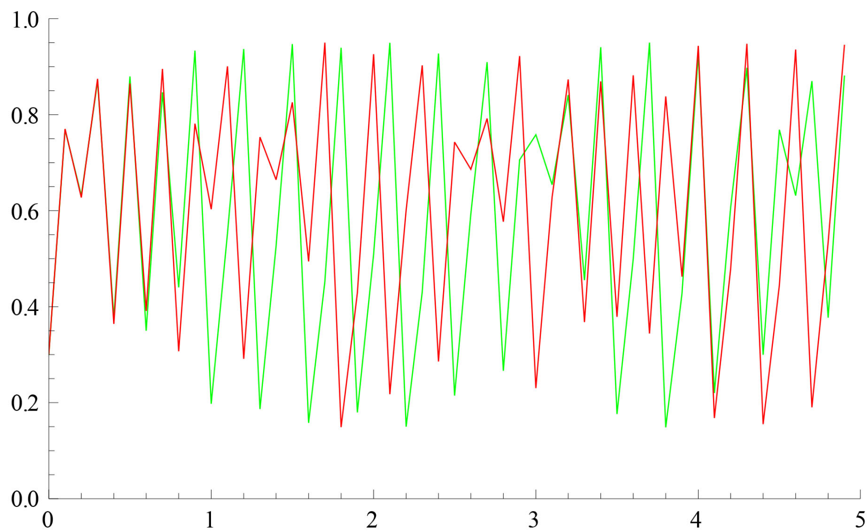

Figure 9. For initial seeds $x_0 = 0.3, 0.301$.

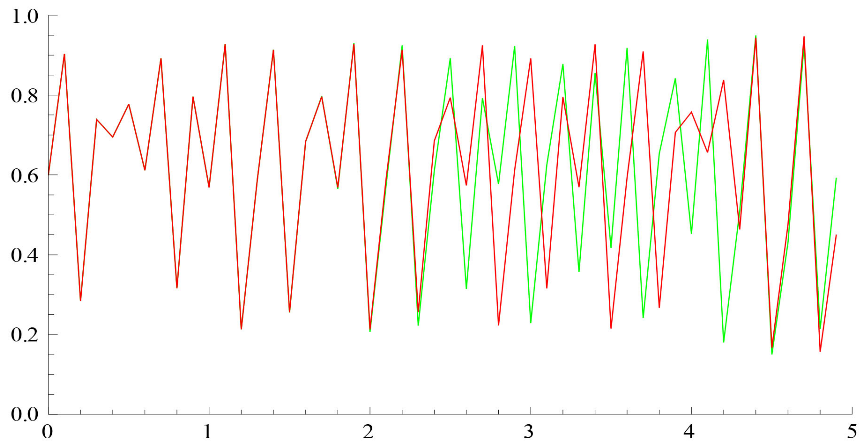

Figure 10. For initial seeds $x_0 = 0.6, x_0 = 0.6000001$.

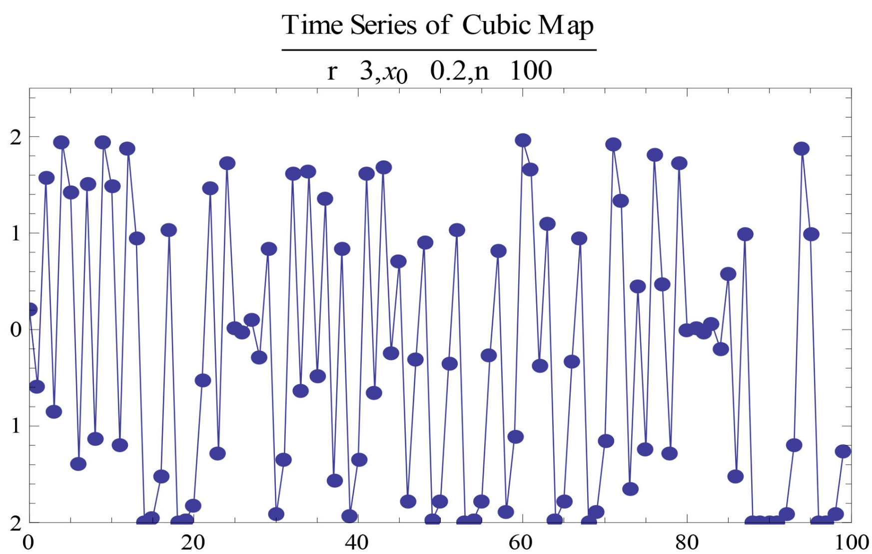

Figure 11. For $r = 3, x_0 = 0.2, n = 100$.

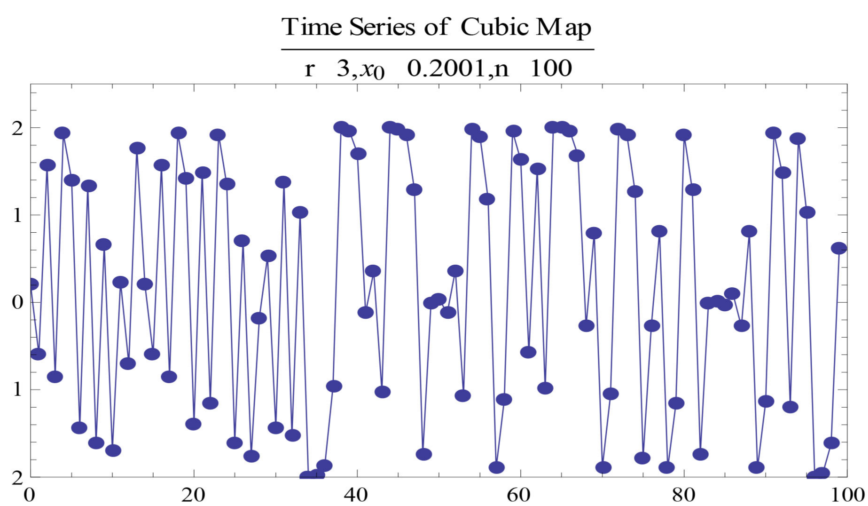

Figure 12. For $r = 3, x_0 = 0.2001, n = 100$.

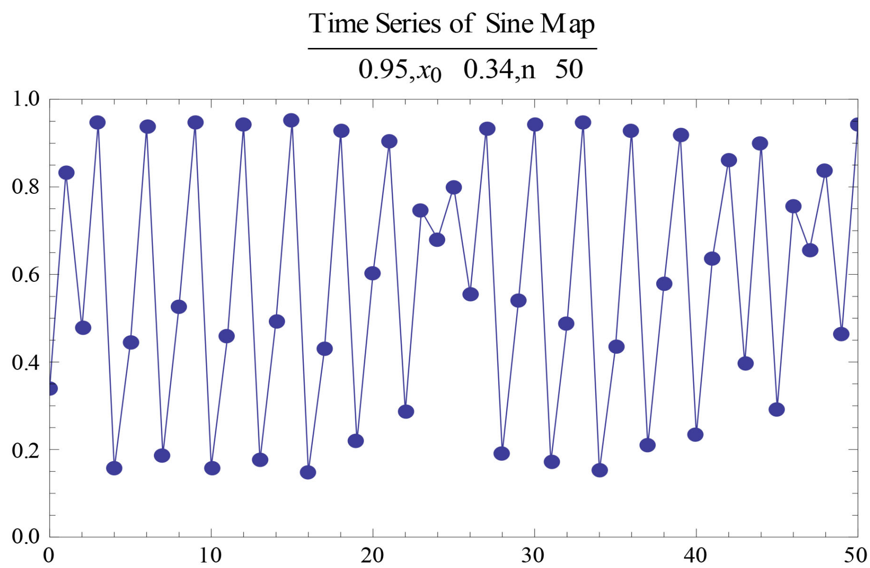

Figure 13. For $\lambda = 0.95, x_0 = 0.34, n = 50$.

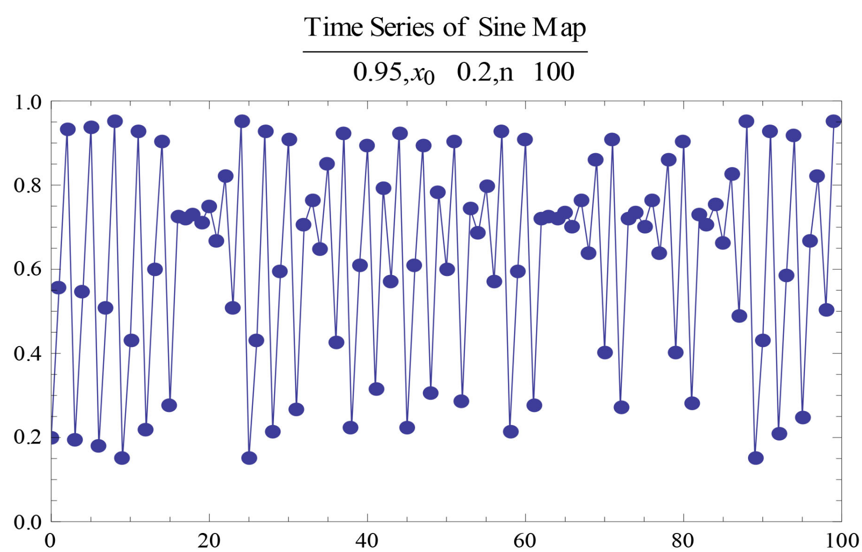

Figure 14. For $\lambda = 0.95, x_0 = 0.2, n = 100$.

5.5. Cobweb Diagram

Analysis: While all of this vocabulary is helpful, a visual presentation of orbits helps solidify the concept. We call these diagram *cobweb plots* [18] and construct them as follows.

Let x_0 be the seed of our orbit. In our plot we graph both our function $f(x)$ and the line $g(x) = x$, in red. With these guidelines, we first trace a line, in black, from (x_0, x_0) to $(x_0, f(x_0))$, then from $(x_0, f(x_0))$ to $(f(x_0), f(x_0))$ (this is where plotting $g(x) = x$ is useful). From there we can trace a line to $(f(x_0), f^2(x_0))$, then to $(f^2(x_0), f^2(x_0))$ and so on. With these plots, we can find $f^n(x)$ for any n and perhaps more importantly, see how the orbit of x got to $f^n(x)$.

Cubic map: With a basic understanding of cobweb plots, we can start to visual the behavior of cubic map $f(x)=x^3-rx$ for larger values of n.

We see that the orbit of 0.1 continues to hit new points. Figures 15-17 reveal that the orbit of $x_0=0.1$ under f seems to travel all over the interval $[-1.5,1.5]$. This phenomenon is called chaotic behavior. So we can say that the given Cubic map is chaotic for $r=2.5$.

Sine map: Now we can start to analyze the behavior of sine map $f(x)=\lambda\sin(\pi x)$ from cobweb plot for larger values of *n.*

The orbits still covering new ground. Figures 18-20 reveal that the orbit of $x_0=0.1$ under *f* seems to travel all over the interval $[0.25,0.95]$. This phenomenon is called chaotic behavior. So we can say that the given Sine map is chaotic for $\lambda=0.9$.

5.6. Histogram Analysis

Cubic map: Here we consider the cubic map $f(x)=x^3-rx$ and investigate the dynamical behavior of the given function by analyzing the histogram image.

From above histogram in Figure 21 we see that the variable values of the given function fall into different bin or bucket [20] and each bin touches each other. The variable of the given function is scattered and hence the function is chaotic for $r=2.6$.

Sine map: Now we plot the histogram image of Sine map $f(x)=\lambda\sin(\pi x)$ and investigate the dynamical behavior of the given function by analyzing the histogram image with the idea mentioned above.

From above histogram in Figure 22 again we see that the variable values of the given function fall into different bin or bucket [20] and each bin touches each other. The variable of the given Sine map is scattered and hence the function is chaotic for some $\lambda\in[0,1]$.

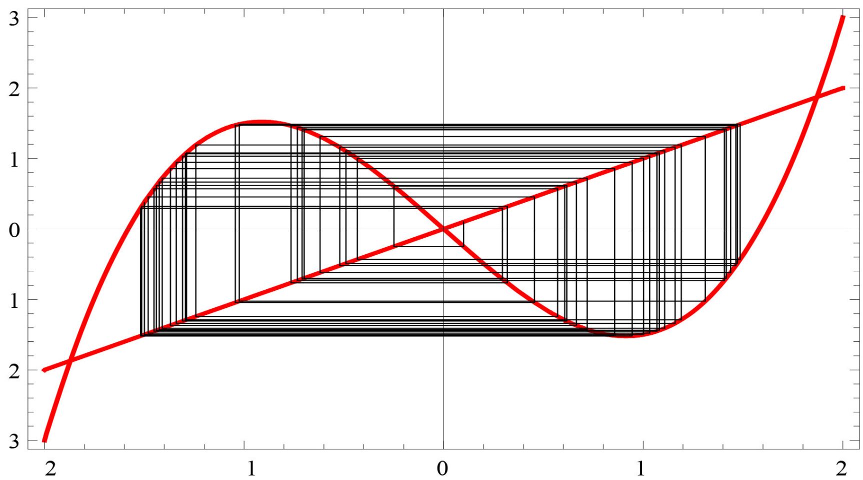

Figure 15. Represents the cobweb plot of $x_0=0.1$ under the map $f(x)=x^3-rx$ with $r=2.5$ up to 50 iterations. Clearly the orbit of 0.1 is covering a fair amount of the interval $[-1.5,1.5]$.

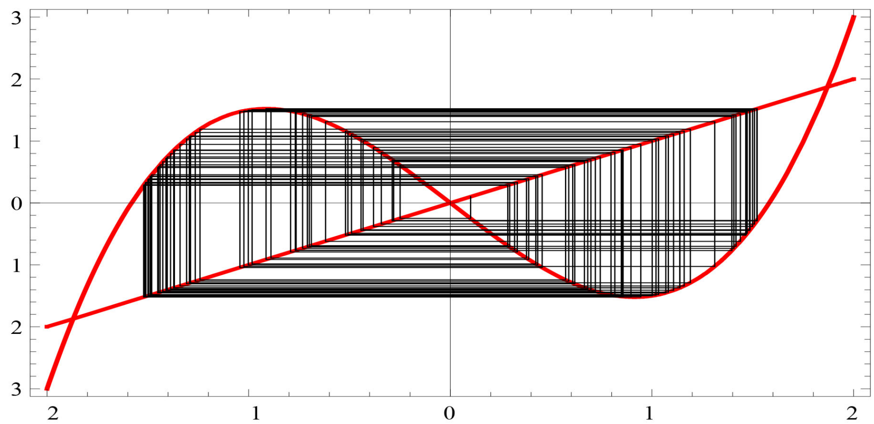

Figure 16. Represents the cobweb plot of $x_0 = 0.1$ under the map $f(x) = x^3 - rx$ with $r = 2.5$ up to 100 iterations.

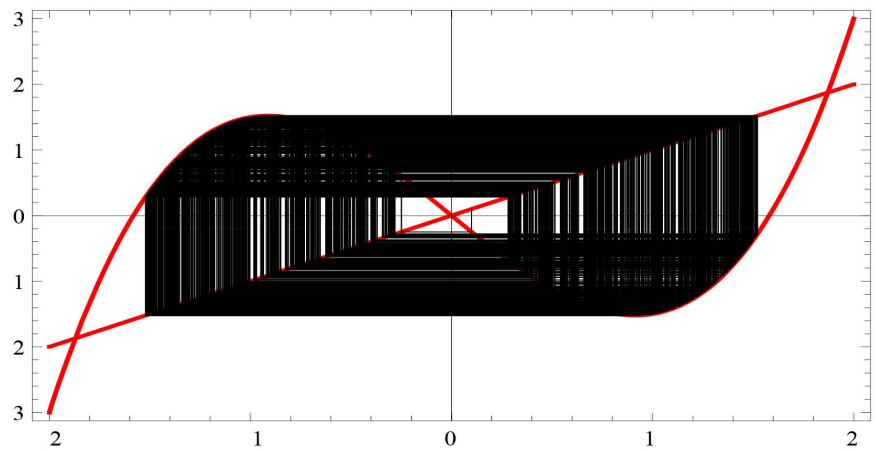

Figure 17. The cobweb plot of $x_0 = 0.1$ under the map $f(x) = x^3 - rx$ with $r = 2.5$ up to 100 iterations.

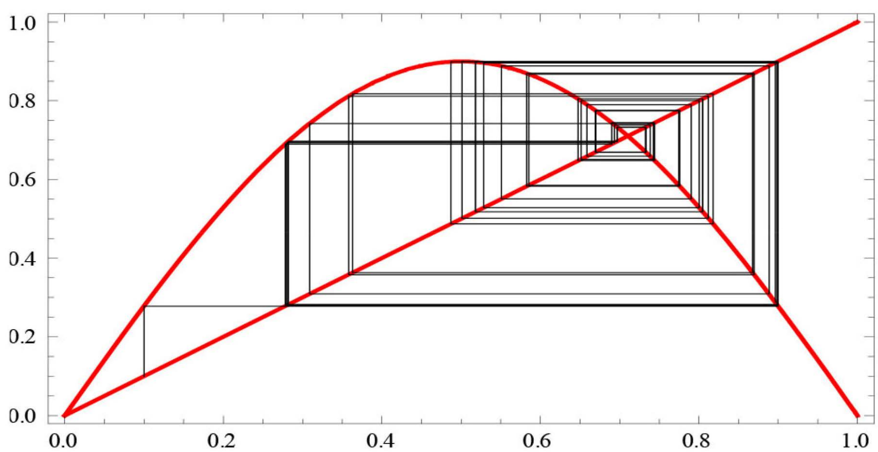

Figure 18. Represents the cobweb plot of $x_0 = 0.1$ under the map $f(x) = \lambda \sin(\pi x)$ with $\lambda = 0.9$ up to 50 iterations. Clearly the orbit of 0.1 is covering a fair amount of the interval $[0.25, 0.95]$.

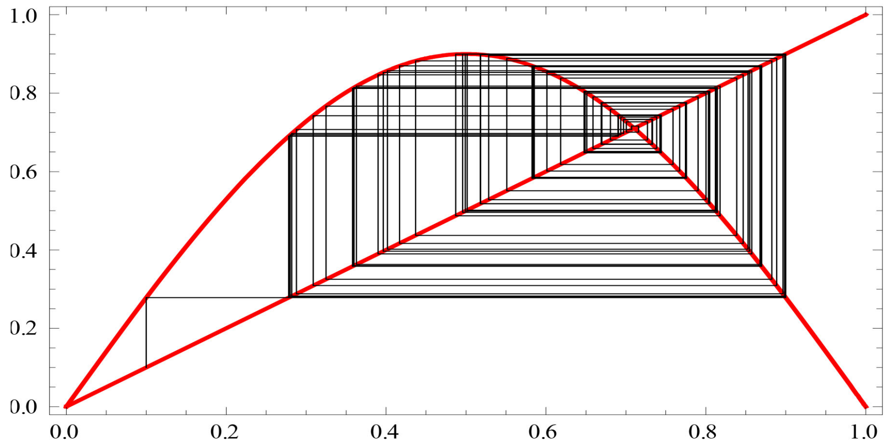

Figure 19. Represents the cobweb plot of $x_0 = 0.1$ under the map $f(x) = \lambda \sin(\pi x)$ with $\lambda = 0.9$ up to 100 iterations.

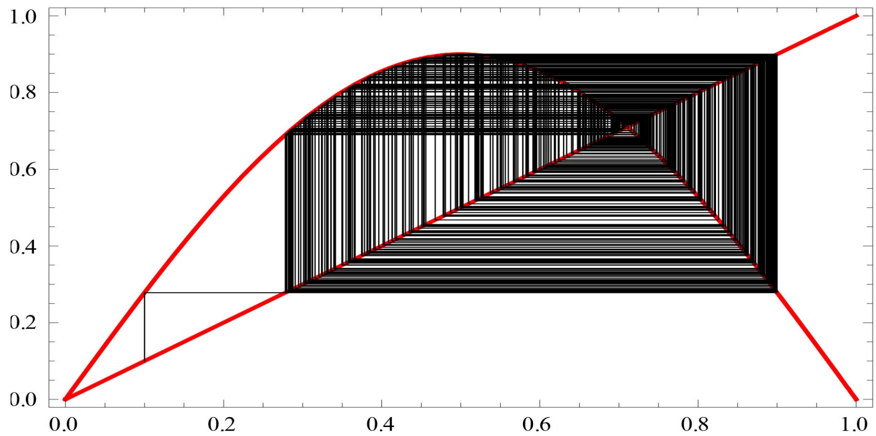

Figure 20. The cobweb plot of $x_0 = 0.1$ under the map $f(x) = \lambda \sin(\pi x)$ with $\lambda = 0.9$ up to 500 iterations.

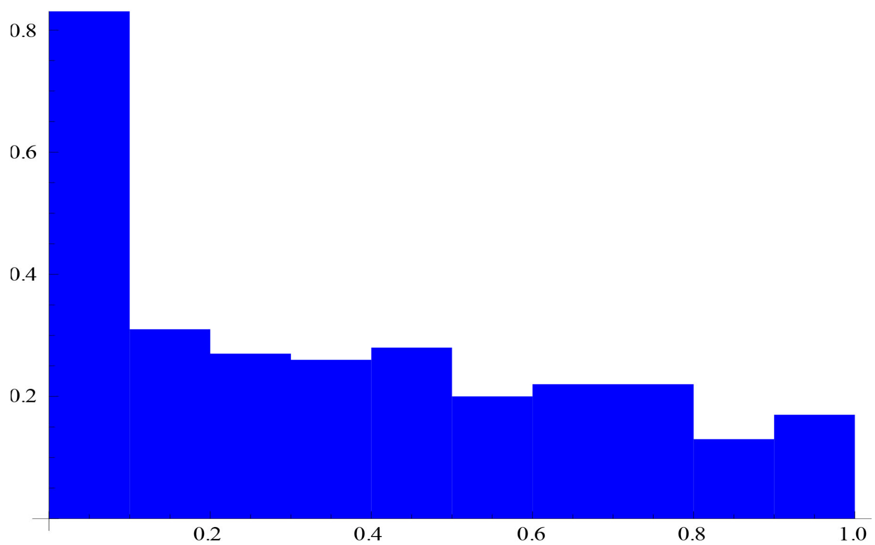

Figure 21. Histogram of cubic map for $r = 2.6$.

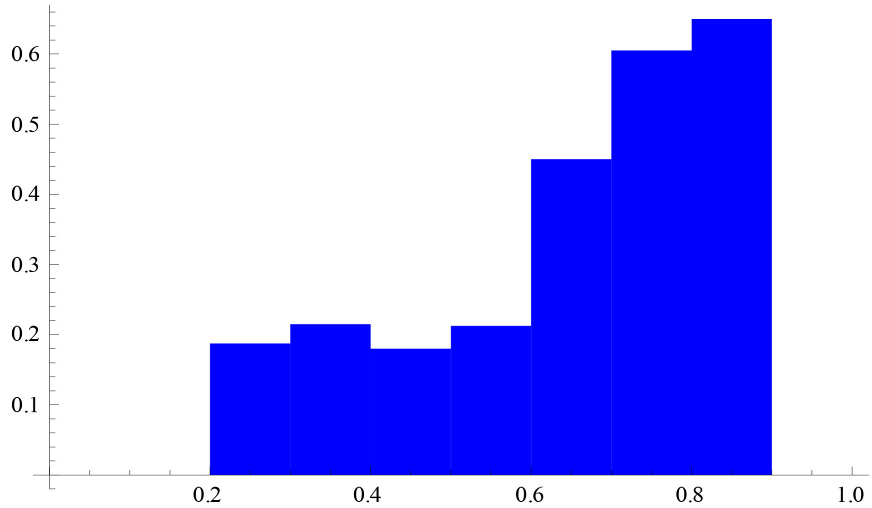

Figure 22. Histogram of sine map for $\lambda = 0.90$.

5.7. Mathematical Analysis by Newton's Iteration

Cubic map: Consider the cubic map $f(x) = x^3 - rx$ and choose $r = 2.6$. The Newton's iteration function [21] associated with $f(x)$ is defined by

$$N(x) = x - \frac{f(x)}{f'(x)} = x - \frac{x^3 - 2.6x}{3x^2 - 2.6} = \frac{3x^3 - 2.6x - x^3 + 2.6x}{3x^2 - 2.6} = \frac{2x^3}{3x^2 - 2.6}$$

Using the Mathematica program, we get the following orbits for the points $x = -3.90, -3.92, -3.94, -3.96, -3.98, 3.90, 3.92, 3.94, 3.96$ and 3.98 under *N*.

$3.90 \to 2.13 \to 1.49 \to 1.79 \to 1.50 \to 1.76 \to 1.51 \to 1.75 \to 1.52 \to 1.73 \to 1.52 \to 1.72 \to 1.53$

$3.92 \to 2.14 \to 1.49 \to 1.79 \to 1.50 \to 1.76 \to 1.51 \to 1.74 \to 1.52 \to 1.73 \to 1.53 \to 1.72 \to 1.53$

$3.94 \to 2.15 \to 1.49 \to 1.78 \to 1.51 \to 1.76 \to 1.51 \to 1.74 \to 1.52 \to 1.73 \to 1.53 \to 1.72 \to 1.53$

$3.96 \to 2.16 \to 1.50 \to 1.78 \to 1.51 \to 1.76 \to 1.51 \to 1.74 \to 1.52 \to 1.73 \to 1.53 \to 1.72 \to 1.53$

$3.98 \to 2.17 \to 1.50 \to 1.78 \to 1.51 \to 1.76 \to 1.52 \to 1.74 \to 1.52 \to 1.73 \to 1.53 \to 1.72 \to 1.53$

Thus we see that orbit of any positive or negative real point under *N* makes a cycle of period-2 for all real value of *r*.

Sine map: Now we consider the Sine map $f(x) = \lambda \sin(\pi x)$ and choose $\lambda = 0.9$. The Newton's iteration function associated with $f(x)$ is defined by

$$N(x) = x - \frac{f(x)}{f'(x)} = x - \frac{\lambda \sin(\pi x)}{\lambda \cos(\pi x)} = x - \frac{\sin(\pi x)}{\cos(\pi x)} = x - \tan(\pi x)$$

Using similar program we get the following orbits for the points $x = -3.90, -3.92, -3.94, -3.96$ under *N*.

$0.6 \to 1.58 \to 2.82 \to 3.02 \to 2.99 \to 3 \to 3 \to 3 \to 3 \to 3$

$1.2 \to 0.97 \to 1 \to 1 \to 1 \to 1 \to 1 \to 1$

$1.8 \to 2.03 \to 1.99 \to 2 \to 2 \to 2 \to 2 \to 2$

$2.4 \to 1.42 \to 0.17 \to -0.02 \to 0 \to 0 \to 0 \to 0 \to 0$

Thus we see that orbit of any positive or negative real point under *N* converges to a fixed point. Here in this analysis we notice that there are infinitely many fixed points for all real value of λ. For different initial values we get new fixed points whether we increase or decrease the initial value in any scale [21].

5.8. Sensitivity to Numerical Inaccuracies

Cubic Map: For some values of the parameter *r*, the Cubic model $x_{n+1} = x_n^3 - rx_n$ is very sensitive to numerical inaccuracies. To see this, we calculate 100 values from the model with *r* = 3, first by using normal decimal numbers and then by using high-precision numbers. In the latter case, we start with numbers that have a precision of 65 digits:

$$\text{values1} = \text{NestList}\left[\left(\#^3 - 3\#\right)\&, 0.02, 40\right];$$

$$\text{values2} = \text{NestList}\left[\left(\#^3 - 3\#\right)\&, 0.02\text{`}55, 40\right]$$

Values corresponding to values 2 are thick. From approximately iteration 37 on, the values differ greatly. In calculating values 2, we started with numbers having 55 digits of precision. During the calculation, many digits were lost so that the last value −0.7512329 only has a precision of approximately 7.25962. Look at some elements of values 2.

Thus, we know that all the digits of values2 are correct. This means that the values in values1 are incorrect from approximately iteration 37 on. This demonstrates the sensitivity to numerical inaccuracies of the Cubic map for some values of the parameter *r*. Thus, if we calculate long sequences from the Cubic function, it is important to use a high enough precision during the calculation. From the plot of values2 in **Figure 23**, we see that the series behaves quite chaotically. It is known that *chaotic* models are very sensitive to numerical inaccuracies.

Sine map: For some values of the parameter λ, the Sine map $x_{n+1} = \lambda \sin\left(\pi x_n\right)$ is very sensitive to numerical inaccuracies. To see this, we calculate 90 values from the model with $\lambda = 1$, first by using normal decimal numbers and then by using high-precision numbers. In the latter case, we start with numbers that have a precision of 35 digits:

$$\text{val1} = \text{NestList}\left[\left(\sin\left(\pi\#\right)\right)\&, 0.01, 90\right]; \text{val2} = \text{NestList}\left[\left(\sin\left(\pi\#\right)\right)\&, 0.01\text{`}35, 90\right];$$

Values corresponding to val2 are thick. From approximately iteration 53 on, the values differ greatly. In calculating val2, we started with numbers having 35 digits of precision. During the calculation, many digits were lost so that the last value 0.1282 only has a precision of approximately 9.68. Look at some elements of val2.

Thus, we know that all the digits of val2 are correct. This means that the values in val1 are incorrect from approximately iteration 53 on. This demonstrates the sensitivity to numerical inaccuracies of the Sine map for some values of the parameter λ. To calculate long sequences from the Sine function, it is important to use a high enough precision during the calculation. From the plot of val2

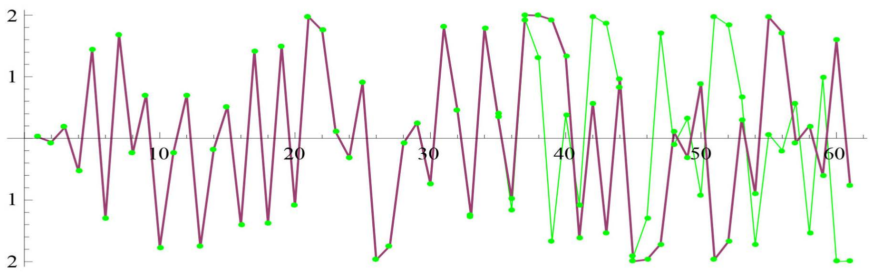

Figure 23. Sensitivity to numerical inaccuracies of the cubic map.

in **Figure 24**, again we see that the series behaves quite chaotically. It is known that chaotic models are very sensitive to numerical inaccuracies.

5.9. Trajectories of the Maps

Cubic map: Write the equation in the form $y_{n+1} - y_n = \left(y_n^3 - r y_n\right) - y_n$ and draw the trajectories for different $r \in [0,3]$. We first calculate a solution set by starting from various points and iterating the equation *n* times. The starting points are chosen between $y_0 1$ and $y_0 2$ in steps of dy_0. When *r* = 1.5, we get the following trajectories.

The trajectory in **Figure 25** seems to form a cycle of two points. In **Figure 26** the trajectory appears to be chaotic. Thus from the trajectory [22] of Cubic map we can make decision that it is chaotic for some $r \in [0,3]$.

Sine map: Now we want to perform same analysis for sine map which we performed above for cubic map. To do this we first write the equation in the form $x_{n+1} - x_n = \lambda \sin(\pi x_n) - x_n$ and plot the trajectories for different value of $\lambda \in [0,1]$. We first calculate a solution set by starting from various points and iterating the equation *n* times. The starting points are chosen between $x_0 1$ and $x_0 2$ in steps of dx_0. When $\lambda = 0.6$, we get the following trajectories.

The trajectory in **Figure 27** seems to approach to a fixed point. In **Figure 28** the trajectory appears to be chaotic. From the above trajectories of Sine map we can find that it is chaotic for some $\lambda \in [0,1]$.

5.10. Bifurcation Diagram

Cubic map: A bifurcation diagram is a visual summary of the succession of period-doubling produced as *r* increases. The next figure shows the bifurcation diagram of the Cubic map, *r* along the *x*-axis. For each value of the system is first allowed to settle down and then the successive values of *x* are plotted for a few hundred iterations.

We observe from below **Figure 29** that for *r* lees than one, all the points are plotted at zero. Zero is the one point attractor for *r* less than one. Bifurcations occur at *r* = 1, *r* = 2, 2.23, 2.28, 2.29... (approximately), etc., until just beyond 2.30, where the system is chaotic. However, the system is not chaotic for all values of *r* greater than 2.30. From the above figure, we can see that we are getting nice clean bifurcation, and we can see some nice details in the chaotic parts of

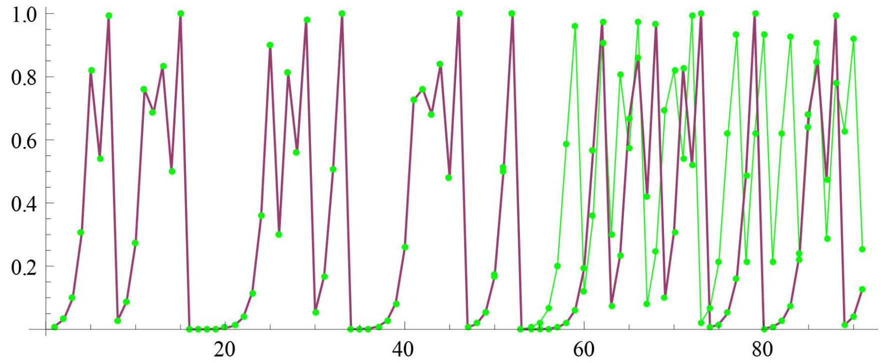

Figure 24. Sensitivity to numerical inaccuracies of the sine map.

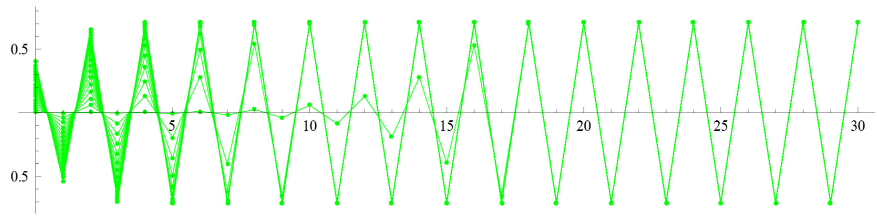

Figure 25. Trajectory for $r = 1.5$.

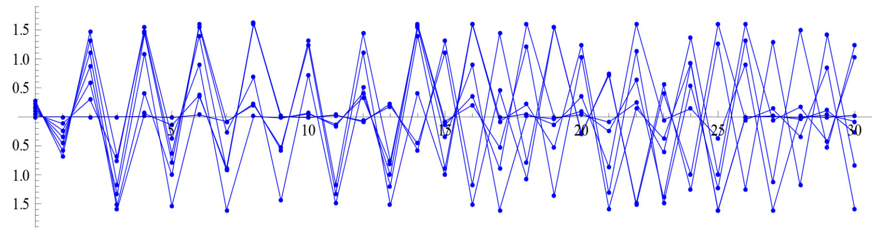

Figure 26. Trajectory for $r = 2.6$.

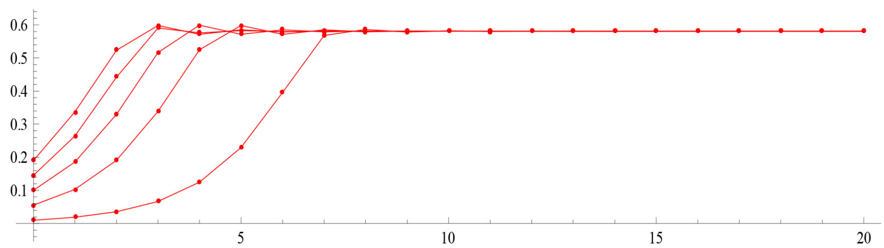

Figure 27. Trajectory for $\lambda = 0.6$.

the diagram. The bifurcation diagram is a fractal because if we zoom in on any of the bifurcation point and focus on one arm up to $r = 2.30$, the situation nearby looks like a shrunk and slightly distorted version of the whole diagram. The same is true for all other non-chaotic points. The period two (at about $r = 2$), period four (at about $r = 2.23$), and period eight (at about $r = 2.28$) are clearly visible in the above diagram. Notice that at several values of r, greater than 2.30, a small number of x values are visited. These regions produce the "white space" in the diagram. In fact, between 2.30 and 3, there is a rich interleaving of chaos and order. A small change in r can make a stable system chaotic [23], and vice versa.

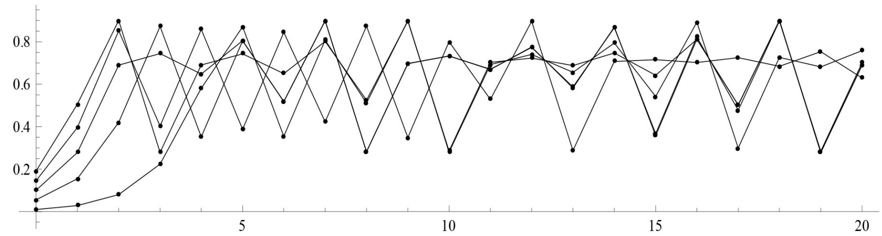

Figure 28. Trajectory for $\lambda = 0.9$.

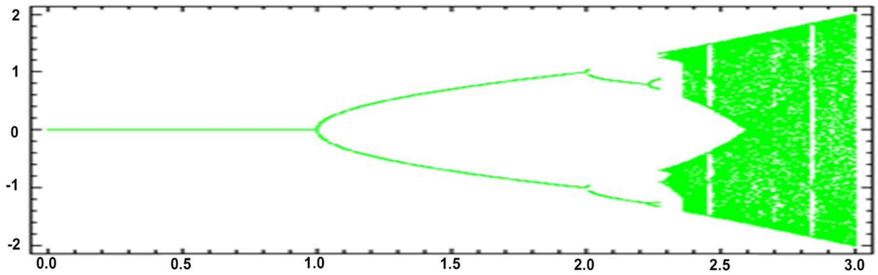

Figure 29. Bifurcation diagram of cubic map for $0 \le r \le 3$.

Sine map: The next figure shows the bifurcation diagram of the Sine map, λ along the *x*-axis. For each value of the system is first allowed to settle down and then the successive values of *x* are plotted for a few hundred iterations.

We observe that for λ less than 0.3, all the points are plotted at zero in below **Figure 30**. Zero is the one point attractor for λ less than 0.3. For λ between 0.3 and 0.72 (approximately), we still have one-point attractors, but the "attracted" value of *x* increases as λ increases, at least to $\lambda = 0.72$. Bifurcations occur at $\lambda = 0.72$, $\lambda = 0.72, 0.83, 0.85, 0.86, \cdots$ (approximately), etc., until just beyond 0.94, where the system is chaotic. However, the system is not chaotic for all values of *r* greater than 0.94. Here we can see some new lines appear. For the non-chaotic parts of the diagram, these lines trace the values that *x* visits before settling into an oscillation. The windows of period three (at about $\lambda = 0.94$), period four (at about $\lambda = 0.85$), and period eight (at about $\lambda = 0.86$) are clearly visible in the above diagram. The appearance and behavior of the bifurcation diagram is very similar to that of the logistic map, albeit with different parameter values. There is a good reason for this.

6. Socio-Economic Importance

Chaos theory was born from observing weather patterns, but it has become applicable to a variety of other situations. Some areas benefiting from chaos theory today are geology, mathematics, microbiology, biology, computer science, economics, engineering, finance, algorithmic trading, meteorology, philosophy, physics, politics, population dynamics, psychology, and robotics. Chaos theory is not new to computer science and has been used for many years in cryptography. Another type of computing, DNA computing, when paired with chaos theory, offers a more efficient way to encrypt images and other information. Robotics

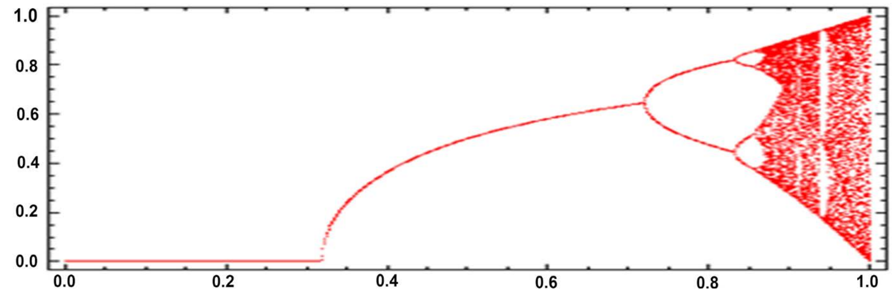

Figure 30. Bifurcation diagram of Sine map for $0 \le \lambda \le 1$.

is another area that has recently benefited from chaos theory. For over a hundred years, biologists have been keeping track of populations of different species with population models. Another biological application is found in cardiography. Fetal surveillance is a delicate balance of obtaining accurate information while being as non-invasive as possible.

In this paper we have worked on two real maps namely Sine and Cubic maps. These two maps have many uses in socio-economic sectors such as Sine map can be used in electricity, digital signals, sound systems, image encryptions etc. and cubic map can be used in traffic systems, in robotics, in computer science etc. To apply any map in any sector it is sufficient to know the nature or dynamics of that map otherwise the predictions will be wrong and the ultimate goal will not been obtained. We have tried to analyze the main dynamical properties of chaos which are essential for the new mathematics researchers. Then they will be able to apply the chaos in real life and will bring new revolutionary changes in the research fields of chaos which will open the new way of mathematical research.

7. Limitations

In this paper we have found a major limitation which is software problem while preparing Orbit analysis, Cobweb and bifurcations of Sine & Cubic map. First we have done these properties graphically using Mathematica 7.0 but it works very slowly and takes much time to show those graphs. The graphs are not clearly visualized for the software problem. Therefore Software version problem can be treated as our major limitation of our paper. Of course high speed super computer is very much needed to run those related programs quickly to show the graphs clearly.

8. Conclusion

In this paper, we have tried to discuss the basic chaotic properties for one-dimensional mentioned maps. We have shown all properties numerically and graphically. To perform such kind of activities we have used software named Mathematica and MATLAB. In the near future, we will try to make a relation among these properties. We will also try to establish new theorems taking these dynamical behaviors which will change the world.

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Conflicts of Interest

The authors declare no conflicts of interest regarding the publication of this paper.

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