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Existence and Uniqueness of Solution to Semilinear Fractional Elliptic Equation

Shangjian Liu

College of Science, University of Shanghai for Science and Technology, Shanghai, China
Email: lxjfyt@163.com

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Abstract

In this work, we study the following problem.
$\begin{cases}(-\Delta)^s u+g(u)=f(x), & x\in\Omega,\\ u=0, & x\in\mathbb{R}^N\setminus\Omega,\end{cases}$, where $(-\Delta)^s$ is the fractional Laplacian and Ω is a bounded domain in $\mathbb{R}^N$ with Lipschitz boundary. $g:\mathbb{R}\to\mathbb{R}$ is an increasing locally Lipschitz continuous function. and $f\in L^m(\Omega)$, $m\geq\frac{2N}{N+2s}$. We use Stampacchia's theorem to study existence of the solution *u*, and we prove the uniqueness of *u* by contradiction.

Keywords

Sobolev Embedding Theorem, Stampacchia's Theorem, Existence, Uniquness

1. Introduction

In recent years, many people pay attention to the fractional Laplacian. One of the reasons for this comes from the fact that this operator naturally arises in several phenomena like flames propagation and geophysical fluid dynamics, or in mathematical finance. About the Fractional Sobolev space we can refer [1] [2]. In this work, we consider the problem

$$\begin{cases}(-\Delta)^s u+g(u)=f(x), & x\in\Omega,\\ u=0, & x\in\mathbb{R}^N\setminus\Omega.\end{cases} \tag{1.1}$$

where $s\in(0,1)$, $N>2s$, $\Omega\in\mathbb{R}^N$ is a bounded domain with Lipschitz boundary. $(-\Delta)^s$ as the fractional Laplacian, which defined as

$$(-\Delta)^s u(x)=a_{N,s}P.V.\int_{\mathbb{R}^N}\frac{u(x)-u(y)}{|x-y|^{N+2s}}\mathrm{d}y, \tag{1.2}$$

where

$$a_{N,s}=\left(\int_{\mathbb{R}^N}\frac{1-\cos(\xi_1)}{|\xi_1|^{N+2s}}\right)^{-1}=2^{2s-1}\pi^{-\frac{N}{2}}\frac{\Gamma\left(\frac{N+2s}{2}\right)}{|\Gamma(-s)|}\tag{1.3}$$

It is worthy to point out that

$$\lim_{s\to 0}a_{N,s}=0=\lim_{s\to 1}a_{N,s},\tag{1.4}$$

we can refer [3].

For $s\in(0,1)$, we can also define the fractional Laplacian $(-\Delta)^s$ as the operator given by the Fourier multiplier $|\xi|^{2s}$,that is, for $u\in S(\mathbb{R}^N)$

$$\mathcal{F}\left((-\Delta)^s u\right)(\xi)=|\xi|^{2s}\mathcal{F}(u)(\xi),\tag{1.5}$$

where we denote by $S(\mathbb{R}^N)$ the class of all Schwartz functions in $\mathbb{R}^N$.

We introduce the Sobolev space

$$H^s(\mathbb{R}^N)=\left\{u\in L^2(\mathbb{R}^N):|\xi|^s\mathcal{F}(u)(\xi)\in L^2(\mathbb{R}^N)\right\},\tag{1.6}$$

and the space $H_0^s(\Omega)$

$$H_0^s(\Omega)=\left\{u\in H^s(\Omega),u=0\ a.e.\ x\in\mathbb{R}^N\setminus\Omega\right\},\tag{1.7}$$

endowed with the norm

$$\|u\|_{H_0^s(\Omega)}=\left(\iint_{D_\Omega}\frac{|u(x)-u(y)|^2}{|x-y|^{N+2s}}\mathrm{d}x\mathrm{d}y\right)^{\frac{1}{2}},\tag{1.8}$$

where $D_\Omega=\mathbb{R}^N\times\mathbb{R}^N\setminus\mathcal{C}\Omega\times\mathcal{C}\Omega$, $\mathcal{C}\Omega=\mathbb{R}^N\setminus\Omega$. This space allows us to deal with the problems proposed in a bounded domain Ω, as we need. The pair $\left(H_0^s(\Omega),\|\cdot\|_{H_0^s(\Omega)}\right)$ yields a Hilbert space [4]. Moreover, it can be seen that

$$(-\Delta)^s:H_0^s(\Omega)\to H^{-s}(\Omega)\tag{1.9}$$

is a continuous operator.

Theorem 1.1. Let $g:\mathbb{R}\to\mathbb{R}$ be an increasing locally Lipschitz continuous function. Let $f\in L^m(\Omega),m\geq\frac{2N}{N+2s}$. Then (1.1) have a unique solution $u\in H_0^s(\Omega)$. Moreover,

$$g(u)\in L^1(\Omega).$$

2. Preliminaries

In this section, we give some basic results of fractional Sobolev space $H_0^s(\Omega)$ that will be used in the next section.

Definition 2.1 We say that $u\in H_0^s(\Omega)$ is a weak solution to (1.1) if we have

$$\iint_{D_\Omega}\frac{(u(x)-u(y))(\varphi(x)-\varphi(y))}{|x-y|^{N+2s}}\mathrm{d}x\mathrm{d}y+\int_\Omega g(u)\varphi\mathrm{d}x=\int_\Omega f\varphi\mathrm{d}x,\tag{2.10}$$

for any $\varphi \in H_0^s(\Omega) \cap L^\infty(\Omega)$.

Lemma 2.1. [5] Let $N \geq 1$ and $s \in (0,1)$.Then for all $u \in H^s(\Omega)$ we have

$$\int_{\mathbb{R}^N} |\xi|^{2s} |\mathcal{F}(u)(\xi)|^2 \mathrm{d}\xi = a_{N,s} \iint_{\mathbb{R}^N \times \mathbb{R}^N} \frac{|u(x)-u(y)|^2}{|x-y|^{N+2s}} \mathrm{d}x\mathrm{d}y, \tag{2.11}$$

where $a_{N,s}$ is the constant defined in (1.3).

Proof. Fixed y we change coordinates $z = x - y$ and apply Plancherel.

Recalling that $(u(\cdot + z))^\wedge(\xi) = \mathrm{e}^{i\xi\cdot z}\hat{u}(\xi)$ we obtain

$$\iint \frac{|u(x)-u(y)|^2}{|x-y|^{N+2s}} \mathrm{d}x\mathrm{d}y = \int \left(\int |z|^{-(N+2s)} \left|\mathrm{e}^{i\xi\cdot z} - 1\right|^2 \mathrm{d}z \right) |\hat{u}(\xi)|^2 \mathrm{d}\xi \tag{2.12}$$

The integral in brackets is of the form $c_{N,s}|\xi|^{2s}$, with

$$\begin{aligned} c_{N,s} &:= \int_0^\infty \int_{\mathcal{S}^{N-1}} \left|\mathrm{e}^{ir\omega\cdot\theta} - 1\right|^2 \mathrm{d}\theta r^{-2s-1} \mathrm{d}r \\ &= 2\int_0^\infty \left(\left|\mathcal{S}^{N-1}\right| - (2\pi)^{N/2} r^{-(N-2)/2} J_{(N-2)/2}(r) \right) r^{-(2s+1)} \mathrm{d}r \end{aligned} \tag{2.13}$$

where $J_{(N-2)/2}$ is the Bessel function of the first kind of order $(N-2)/2$, we can refer [6].

Recall that $\left|\mathcal{S}^{N-1}\right| = 2\pi^{N/2}/\Gamma(N/2)$. The formula (1.3) for $c_{N,s} = a_{N,s}^{-1}$ now follows from

$$\int_0^\infty r^{-z} \left(J_{(N-2)/2}(r) - 2^{-(N-2)/2} \Gamma(N/2)^{-1} r^{(N-2)/2} \right) \mathrm{d}r = 2^{-z} \frac{\Gamma((N-2z)/4)}{\Gamma((N+2z)/4)}, \tag{2.14}$$

for $N/2 < \mathrm{Re}\, z < (N+4)/2$, we can see[5].

Lemma 2.2. [7] For $s \in (0,1)$, $N \geq 2s$, there exists a positive constant $C = C(N,s)$, for any $u \in D^s(\mathbb{R}^N)$,we have

$$\|u\|_{L^{2_s^*}(\mathbb{R}^N)}^2 \leq C \iint_{\mathbb{R}^N \times \mathbb{R}^N} \frac{|u(x)-u(y)|^2}{|x-y|^{N+2s}} \mathrm{d}x\mathrm{d}y, \tag{2.15}$$

where $2_s^* = \dfrac{2N}{N-2s}$ is called fractional critical Sobolev exponent. In particular, if $u \in H_0^s(\Omega)$ then

$$\|u\|_{L^{2_s^*}(\Omega)}^2 \leq C \iint_{D_\Omega} \frac{|u(x)-u(y)|^2}{|x-y|^{N+2s}} \mathrm{d}x\mathrm{d}y. \tag{2.16}$$

Lemma 2.3. (Egorov's theorem) [8] Let f_n be a sequence of functions and f be a function defined on E, with $meas(E) < +\infty$. Assume that $f_n \to f$ a.e. in E. Then for every $\varepsilon > 0$ there exists a measurable subset A of E such that $meas(E \setminus A) < \varepsilon$ and $f_n \to f$ uniformly on A,as $n \to \infty$.

Lemma 2.4. (Vitali) [9] Let f_n be a sequence of functions and f be a function in $L^p(\Omega)$. Assume that

1) $f_n \to f$ a.e. in Ω;

2) if E is a measurable subset of Ω, and we have

$$\lim_{meas(E)\to 0} \int_E |f_n|^p = 0, \tag{2.17}$$

uniformly with respect n. Where $meas(E)$ means measure representing E. Then $f_n \to f$ in $L^p(\Omega)$.

Proof. Fixed $\varepsilon > 0$, let $E \subset \Omega$ be a measurable set, we have

$$\int_\Omega |f_n - f|^p \le \int_{\mathbb{R}^N \setminus \Omega} |f_n - f|^p + 2^{p-1} \int_E \left(|f_n|^p + |f|^p\right). \tag{2.18}$$

Using assumption (2), we know that there exists $\delta_1(\varepsilon) > 0$ such that, if $meas(E) < \delta_1(\varepsilon)$, then for any $n \in \mathbb{N}$ we have

$$\int_E |f_n|^p < \varepsilon. \tag{2.19}$$

Since $f \in L^p(\Omega)$ there exists $\delta_2(\varepsilon) > 0$ such that if $meas(E) < \delta_2(\varepsilon)$, then

$$\int_E |f|^p < \varepsilon. \tag{2.20}$$

In conclusion the second term of the right-hand side of (2.18) is less than $2^p \varepsilon$. Let us study the first one. We set $\delta = \min\{\delta_1(\varepsilon), \delta_2(\varepsilon)\}$, and use Egorov's theorem, there exist $v_\varepsilon \in \mathbb{N}$ and a measurable set $E_0 \subset \Omega$ such that $meas(E_0) < \delta$, and

$$\int_{\Omega \setminus E_0} |f_n - f|^p < \varepsilon, \tag{2.21}$$

for any $n > v_\varepsilon$.Choosing $E = E_0$ in(2.18), we get the result.

Lemma 2.5. (Stampacchia) [10] Let H be a Hilbert space, $a: H \times H \to \mathbb{R}$ is a continuous and linear form in the second variable such that

1) for $\beta \in \mathbb{R}^+$,any $\psi_1, \psi_2, w \in H$, we have

$$|a(\psi_1, w) - a(\psi_2, w)| \le \beta \|\psi_1 - \psi_2\| \|w\|, \tag{2.22}$$

2) for a positive constant C, any $\psi_1, \psi_2 \in H$ we have

$$a(\psi_1, \psi_1 - \psi_2) - a(\psi_2, \psi_1 - \psi_2) \ge C \|\psi_1 - \psi_2\|^2. \tag{2.23}$$

Lemma 2.6. (Hölder inequality) [11] Let p and q are dual indicators, stisfies

$$1/p + 1/q = 1,$$

where $1 \le p \le \infty$, if $f \in L^p(\Omega)$, and $g \in L^q(\Omega)$, then the product of $(fg)(x) = f(x)g(x)$ the defined function belongs to $L^1(\Omega)$, and we have

$$\left|\int_\Omega fg\,\mathrm{d}x\right| \le \int_\Omega |f||g|\,\mathrm{d}x \le \|f\|_p \|g\|_q. \tag{2.24}$$

If and only if there is a real constant m that makes the following formula hold

$$fg = \mathrm{e}^{im} |f||g|. \tag{2.25}$$

The first unequal sign of (2.24) is established. If f not constant equals 0,then the second unequal sign of (2.24) is established, if and only if there exists a constant $\eta \in \mathbb{R}$, such that

1) if $1 < p < \infty$, then $|g(x)| = \eta |f(x)|^{p-1}$ μ a.e. $\in \Omega$.

2) if $p = 1$, then $|g(x)| \le \eta$ μ a.e. $\in \Omega$, and when $f(x) \ne 0$, we have

$|g(x)| = \eta$.

3) if $p = \infty$, then $|f(x)| \le \eta \;\; \mu$ a.e. $\in \Omega$, and when $g(x) \ne 0$, we have $|f(x)| = \eta$.

3. Proof of Theorem 1.1

Theorem 3.1. Let $g : \mathbb{R} \to \mathbb{R}$ be an increasing function, and g is Lipschitz continuous, that is, there exists a positive constant μ such that for any $s, t \in \mathbb{R}$ we have

$$|g(s) - g(t)| \le \mu |s - t|, \tag{3.1}$$

Let $f \in L^m(\Omega), m \ge \frac{2N}{N+2s}$. Then (1.1) exists a unique solution $u \in H_0^s(\Omega)$.

Proof. We define the following form on $H_0^s(\Omega) \times H_0^s(\Omega)$:

$$a(u, w) = \iint_{D_\Omega} \frac{(u(x) - u(y))(w(x) - w(y))}{|x - y|^{N+2s}} \mathrm{d}x\mathrm{d}y + \int_\Omega g(u) w \mathrm{d}x. \tag{3.2}$$

Using Hölder inequality and (3.1) we have

$$|a(u, w)| \le \iint_{D_\Omega} \frac{u(x) - u(y)}{|x - y|^{\frac{N+2s}{2}}} \cdot \frac{w(x) - w(y)}{|x - y|^{\frac{N+2s}{2}}} \mathrm{d}x\mathrm{d}y + \int_\Omega \left[\mu |u| + g(0)\right] |w| \mathrm{d}x, \tag{3.3}$$

that is, *a* is well defined. By the definition of *a*, we know that *a* is continuous and linear in the second variable. If $w_n \to w$ in $H_0^s(\Omega)$, then

$$\begin{aligned} &\iint_{D_\Omega} \frac{(u(x) - u(y))(w_n(x) - w_n(y))}{|x - y|^{N+2s}} \mathrm{d}x\mathrm{d}y \\ &\to \iint_{D_\Omega} \frac{(u(x) - u(y))(w(x) - w(y))}{|x - y|^{N+2s}} \mathrm{d}x\mathrm{d}y \end{aligned}, \tag{3.4}$$

$$\int_\Omega g(u) w_n \mathrm{d}x \to \int_\Omega g(u) w \mathrm{d}x. \tag{3.5}$$

Since

$$\begin{aligned} &|a(u_1, w) - a(u_2, w)| \\ &= \left| \iint_{D_\Omega} \frac{(u_1(x) - u_1(y))(w(x) - w(y))}{|x - y|^{N+2s}} \mathrm{d}x\mathrm{d}y \right. \\ &\left. - \iint_{D_\Omega} \frac{(u_2(x) - u_2(y))(w(x) - w(y))}{|x - y|^{N+2s}} \mathrm{d}x\mathrm{d}y \right| + \left| \int_\Omega \left[g(u_1) - g(u_2)\right] w \mathrm{d}x \right| \\ &= \left| \iint_{D_\Omega} \frac{(u_1 - u_2)(x) - (u_1 - u_2)(y)}{|x - y|^{\frac{N+2s}{2}}} \cdot \frac{(w_1)(x) - (w_2)(y)}{|x - y|^{\frac{N+2s}{2}}} \right| + \left| \int_\Omega \left[g(u_1) - g(u_2)\right] w \mathrm{d}x \right| \\ &\le \|u_1 - u_2\|_{H_0^s(\Omega)} \|w\|_{H_0^s(\Omega)} + \mu \|u_1 - u_2\|_{L^2(\Omega)} \|w\|_{L^2(\Omega)}, \end{aligned} \tag{3.6}$$

the last inequality following from Hölder inequality and (3.1), by lemma 2.2

$$|a(u_1, w) - a(u_2, w)| \le \left(1 + \mu C^2\right) \|u_1 - u_2\|_{H_0^s(\Omega)} \|w\|_{H_0^s(\Omega)}. \tag{3.7}$$

Since

$$\begin{aligned} & a(u_1,u_1-u_2)-a(u_2,u_1-u_2) \\ & = \iint_{D_\Omega} \frac{\left|(u_1-u_2)(x)-(u_1-u_2)(y)\right|^2}{|x-y|^{N+2s}} \mathrm{d}x\mathrm{d}y + \int_\Omega \left[g(u_1)-g(u_2)\right](u_1-u_2)\mathrm{d}x \end{aligned} \tag{3.8}$$

by (3.1)

$$\int_\Omega \left[g(u_1)-g(u_2)\right](u_1-u_2)\mathrm{d}x \ge 0, \tag{3.9}$$

then

$$a(u_1,u_1-u_2)-a(u_2,u_1-u_2) \ge \|u_1-u_2\|^2_{H_0^s(\Omega)}. \tag{3.10}$$

We know that a satisfies lemma 2.4 from (3.2) and (3.10), the result follows from lemma 2.4.

We define the following function, for $k>0$:

$$T_k(s) = \begin{cases} -k, & s \le -k, \\ s, & |s| \le k, \\ k, & s \ge k. \end{cases} \tag{3.11}$$

Proof of theorem 1.1: First, we proof the existence of a solution by approximation. Let $g_n(t) = T_n(g(t))$, By theorem 3.1 we know that there exists $u_n \in H_0^s(\Omega)$ be the solution to problems

$$\begin{cases} (-\Delta)^s u_n + g_n(u_n) = f(x), & x \in \Omega, \\ u_n = 0, & x \in \mathbb{R}^N \setminus \Omega. \end{cases} \tag{3.12}$$

We use u_n as a test function in (3.12), we get

$$\|u_n\|^2_{H_0^s(\Omega)} + \int_\Omega u_n g_n(u_n)\mathrm{d}x = \int_\Omega f u_n \mathrm{d}x. \tag{3.13}$$

Then use Hölder inequality on the right-hand side implies

$$\begin{aligned} & \|u_n\|^2_{H_0^s(\Omega)} + \int_\Omega u_n g_n(u_n)\mathrm{d}x \\ & \le \left(\int_\Omega f^{\frac{2N}{N+2s}}\mathrm{d}x\right)^{\frac{N+2s}{2N}} \left(\int_\Omega u_n^{2_s^*}\mathrm{d}x\right)^{\frac{1}{2_s^*}} = \|f\|_{L^{\frac{2N}{N+2s}}(\Omega)} \|u_n\|_{L^{2_s^*}(\Omega)}. \end{aligned} \tag{3.14}$$

Because g is increasing, then $\|u_n\|^2_{H_0^s(\Omega)} \le \|f\|_{L^{\frac{2N}{N+2s}}(\Omega)} \|u_n\|_{L^{2_s^*}(\Omega)}$. This means $\|u_n\|_{H_0^s(\Omega)}$ is uniformly bounded. We can deduce there exists $u_n \to u$ weakly in $H_0^s(\Omega)$ and a.e., since $\|u_n\|^2_{H_0^s(\Omega)} \ge 0$, by (3.13) there exists a positive constant C such that

$$\int_\Omega u_n g_n(u_n)\mathrm{d}x \le C, \tag{3.14}$$

for every n.

Now we prove $g_n(u_n) \to g(u)$ in $L^1(\Omega)$. Since g is continuous in Ω then it is clear that $g_n(u_n) \to g(u)$ a.e. in Ω. If E is a subset of Ω, for $t \in \mathbb{R}^+$ have

$$\begin{aligned}\int_E |g_n(u_n)| &= \int_{\{x\in E:|u_n(x)|\le t|\}} |g_n(u_n)| + \int_{\{x\in E:|u_n(x)>t|\}} |g_n(u_n)| \\ &\le \int_E |g_n(t)| + \frac{1}{t}\int_{\{x\in E:|u_n(x)>t|\}} u_n g(u_n) \\ &\le |g(t)| meas(E) + \frac{C}{t},\end{aligned} \tag{3.15}$$

combining (3.14), for $t\in\mathbb{R}^+$ we have

$$\lim_{meas(E)\to 0}\int_E |g_n(u_n)| \le \frac{C}{t}. \tag{3.16}$$

Using lemma 2.4, we know that $g_n(u_n)\to g(u)$ in $L^1(\Omega)$. Then for any $\phi\in H_0^s(\Omega)\cap L^\infty(\Omega)$
we from

$$\iint_{D_\Omega}\frac{(u_n(x)-u_n(y))(\phi(x)-\phi(y))}{|x-y|^{N+2s}}\mathrm{d}x\mathrm{d}y+\int_\Omega g_n(u_n)\phi\mathrm{d}x=\int_\Omega f\phi\mathrm{d}x \tag{3.17}$$

get

$$\iint_{D_\Omega}\frac{(u(x)-u(y))(\phi(x)-\phi(y))}{|x-y|^{N+2s}}\mathrm{d}x\mathrm{d}y+\int_\Omega g(u)\phi\mathrm{d}x=\int_\Omega f\phi\mathrm{d}x. \tag{3.18}$$

Finally we prove the solution of problem (1.1) is unique. We assume u_1 and u_2 are two solutions, $u_1\ne u_2$, we take u_1-u_2 as a test function

$$\begin{aligned}&\iint_{D_\Omega}\frac{(u_1(x)-u_1(y))[(u_1-u_2)(x)-(u_1-u_2)(y)]}{|x-y|^{N+2s}}\mathrm{d}x\mathrm{d}y+\int_\Omega g(u_1)(u_1-u_2)\mathrm{d}x \\ &=\int_\Omega f(u_1-u_2)\mathrm{d}x,\end{aligned} \tag{3.19}$$

$$\begin{aligned}&\iint_{D_\Omega}\frac{(u_2(x)-u_2(y))[(u_1-u_2)(x)-(u_1-u_2)(y)]}{|x-y|^{N+2s}}\mathrm{d}x\mathrm{d}y+\int_\Omega g(u_2)(u_1-u_2)\mathrm{d}x \\ &=\int_\Omega f(u_1-u_2)\mathrm{d}x.\end{aligned} \tag{3.20}$$

We can deduce from (3.19) and (3.20)

$$\begin{aligned}&\iint_{D_\Omega}\frac{[u_1(x)-u_1(y)-u_2(x)+u_2(y)][(u_1-u_2)(x)-(u_1-u_2)(y)]}{|x-y|^{N+2s}}\mathrm{d}x\mathrm{d}y \\ &=\int_\Omega (g(u_1)-g(u_2))(u_1-u_2)\mathrm{d}x.\end{aligned} \tag{3.21}$$

This means

$$\|u_1-u_2\|^2_{H_0^s(\Omega)}=\int_\Omega (g(u_1)-g(u_2))(u_1-u_2)\mathrm{d}x. \tag{3.22}$$

By the monotonicity of g we know

$$\int_\Omega (g(u_1)-g(u_2))(u_1-u_2)\mathrm{d}x\le 0. \tag{3.23}$$

Combining (3.22) and (3.23) we know $u_1=u_2$ a.e. in Ω.

Conflicts of Interest

The author declares no conflicts of interest regarding the publication of this paper.

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