Extremal Problems Related to Dual Gauss-John Position

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Abstract

In this paper, the extremal problem, \( \min \{ (\tilde{I}_p(\phi K) : o \in \phi K \subseteq L, \phi \in GL(n) \} \), of two convex bodies \( K \) and \( L \) in \( \mathbb{R}^n \) is considered. For \( K \) to be in extremal position in terms of a decomposition of the identity, give necessary conditions together with the optimization theorem of John. Besides, we also consider the weaker optimization problem:
\[
\min \{ (\tilde{I}_p(\phi K))^{1/p} : \phi K \subseteq B_2^n, \phi K \cap S^{n-1} \neq \emptyset, \phi \in GL(n) \}.
\]
As an application, we give the geometric distance between the unit ball \( B_2^n \) and a centrally symmetric convex body \( K \).

Keywords

Dual Gauss-John Position, Optimization Theorem of John, Dual \( \tilde{I}_p \)-Norm, Contact Pair

1. Introduction

Let \( \gamma_n \) be the classical Gaussian probability measure with density \( \frac{1}{(\sqrt{2\pi})^n} e^{-\frac{|x|^2}{2}} \), and \( \| \cdot \|_K \) is the Minkowski functional of a convex body \( K \subset \mathbb{R}^n \). An important quantity on local theory of Banach space is the associated \( l \)-norm:
\[
l(K) = \int_{\mathbb{R}^n} \| x \|_K d\gamma_n(x).
\]
The minimum of the functional
\[
\int_{\mathbb{R}^n} \| x \|_K d\gamma_n(x)
\]
under the constraint \( \phi K \subseteq B_2^n \) is attained for \( \phi = I_n \), then a convex body \( K \) is in the Gauss-John position, where \( \phi \in GL(n) \), \( B_2^n \) is the Euclidean unit ball.
and \( I_n \) is the identity mapping from \( \mathbb{R}^n \) to \( \mathbb{R}^n \).

For \( x \in \mathbb{R}^n \setminus \{0\} \), the map \( x \otimes x : \mathbb{R}^n \to \mathbb{R}^n \) is the rank 1 linear operator \( y \mapsto (x,y)x \).

Giannopoulos et al. in [1] showed that if \( K \) is in the Gauss-John position, then there exist \( m \leq n(n+1)/2 \) contact points \( x_1, x_2, \ldots, x_m \in \partial K \cap S^{n-1} \), and constants \( c_1, c_2, \ldots, c_m > 0 \) such that \( \sum_{i=1}^{m} c_i = 1 \) and

\[
\int_{\mathbb{R}^n} (x \otimes x - I_n) \|x\|_K^p\, d\gamma_n(x) = \int_{\mathbb{R}^n} \|x\|_K^p \, d\gamma_n(x) \left( \sum_{i=1}^{m} c_i x_i \otimes x_i \right). 
\]

Note that the Gauss-John position is not equivalent to the classical John position. Giannopoulos et al. [1] pointed out that, when \( K \) is in the Gauss-John position, the distance between the unit ball \( B^n_2 \) and the John ellipsoid is of order \( \sqrt{n/\log n} \).

Notice that the study of the classical John theorem went back to John [2]. It states that each convex body \( K \) contains a unique ellipsoid of maximal volume, and when \( B^n_2 \) is the maximal ellipsoid in \( K \), it can be characterized by points of contact between the boundary of \( K \) and that of \( B^n_2 \). John’s theorem also holds for arbitrary centrally symmetric convex bodies, which was proved by Lewis [3] and Milman [4]. It was provided in [5] that a generalization of John’s theorem for the maximal volume position of two arbitrary smooth convex bodies. Bastero and Romance [6] proved another version of John’s representation removing smoothness condition but with assumptions of connectedness. For more information about the study of its extensions and applications, please see [7]-[13].

Recall that a convex body \( \tilde{K} \) is a position of \( K \) if \( \tilde{K} = \phi K + a \), for some non-degenerate linear mapping \( \phi \in \text{GL}(n) \) and some \( a \in \mathbb{R}^n \). We say that \( K \) is in a position of maximal volume in \( L \) if \( K \subseteq L \) and for any position \( \tilde{K} \) of \( K \) such that \( \tilde{K} \subseteq L \) we have \( \text{vol}_n(\tilde{K}) \leq \text{vol}_n(K) \), where \( \text{vol}_n(\cdot) \) denotes the volume of appropriate dimension.

Recently, Li and Leng in [14] generalized the Gauss-John position to a general situation. For \( p \geq 1 \), denote \( L_p \)-norm by

\[
I_p(K) = \left( \int_{\mathbb{R}^n} \|x\|_K^p \, d\gamma_n(x) \right)^{\frac{1}{p}}.
\]

They consider the following extremal problem:

\[
\min \left\{ I_p(\phi K) : \phi \in \text{GL}(n), \phi \in L, \phi \subseteq K \subseteq L \right\},
\]

where \( L \) is a given convex body in \( \mathbb{R}^n \) and \( K \) is a convex body containing the origin \( o \) such that \( o \in K \subseteq L \).

Li and Leng [14] showed that let \( L \) be a given convex body in \( \mathbb{R}^n \) and \( K \) be a convex body such that \( o \in K \subseteq L \). If \( K \) is in extremal position of (1.2), then there exist \( m \leq n^2 \) contact pairs \( (x_i, y_i)_{i \leq m} \) of \( (K, L) \), and constants \( c_1, c_2, \ldots, c_m > 0 \) such that

\[
I_n = \int_{\mathbb{R}^n} (x \otimes x) \, d\mu(x) - p \sum_{i=1}^{m} c_i x_i \otimes y_i, \quad \sum_{i=1}^{m} c_i = 1,
\]
where \( d\mu(x) \) is the probability measure on \( \mathbb{R}^n \) with normalized density
\[
d\mu(x) = \|x\|_\infty^p d\gamma_s(x)/(l_p(K))^p.
\]

In this paper, we first present a dual concept of \( l_p \)-norm \( l_p(K) \). The generalizations of John’s theorem and Li and Leng [14] play a critical role. It would be impossible to overstate our reliance on their work.

For \( p \geq 1 \), we define the dual \( \tilde{l}_p \)-norm of convex body \( K \) by
\[
\tilde{l}_p(K) = \left( \int_{\mathbb{R}^n} \rho_K(x)^p d\gamma_s(x) \right)^{\frac{1}{p}},
\]
where \( \rho_K \) is the radial function of the star body \( K \) about the origin.

Now, we consider the extremal problem:
\[
\min \left\{ \tilde{l}_p(\phi K) : o \in \phi K \subseteq L, \phi \in \text{GL}(n) \right\},
\]
where \( L \) is a given convex body in \( \mathbb{R}^n \) and \( K \) is a convex body containing the origin \( o \) such that \( o \in K \subseteq L \).

Then we prove that the necessary conditions for \( K \) to be in extremal position in terms of a decomposition of the identity.

**Theorem 1.1.** Let \( L \) be a given convex body in \( \mathbb{R}^n \) and \( K \) be a convex body such that \( o \in K \subseteq L \). If \( K \) is in extremal position of (1.4), then there exist \( m \leq n^2 \) contact pairs \( (x_i, y_i) \) of \( (K, L) \), and \( c_1, c_2, \ldots, c_m > 0 \) such that
\[
I_n = \int_{\mathbb{R}^n} (x \otimes x) d\tilde{\mu}(x) - p \sum_{i=1}^{m} c_i x_i \otimes y_i, \quad \sum_{i=1}^{m} c_i = 1,
\]
where \( d\tilde{\mu}(x) \) is the probability measure on \( \mathbb{R}^n \) with normalized density
\[
d\tilde{\mu}(x) = \|x\|_\infty^p d\gamma_s(x)/(\tilde{l}_p(K))^p.
\]

Next the following result is obtained, which is an restriction that is weaker than the extremal problem (1.4):
\[
\min \left\{ \left( \tilde{l}_p(\phi K) \right)^p : \phi K \subseteq B^*_n, \phi \in \text{GL}(n) \right\}.
\]

**Theorem 1.2.** Let \( K \) be a given convex body in \( \mathbb{R}^n \) and \( I_n \) is the solution of the extremal problem (1.5), then there exist contact points \( u, u' \) of \( K \) and \( B^*_n \) such that
\[
\langle u', \theta \rangle^2 \leq \langle \tilde{l}_p(K), \theta \rangle^p \int_{\mathbb{R}^n} \|x\|_\infty^{p-1} \langle \nabla h_{K_n}(x), \theta \rangle \langle x, \theta \rangle d\gamma_s(x) \leq \langle u, \theta \rangle^2,
\]
for every \( \theta \in S^{n-1} \).

The rest of this paper is organized as follows: In Section 2, some basic notation and preliminaries are provided. We prove Theorem 1.1 and Theorem 1.2 in Section 3. In particular, as an application of the extremal problem of
\[
\min \left\{ \left( \tilde{l}_p(\phi K) \right)^p : o \in \phi K \subseteq B^*_n, \phi \in \text{GL}(n) \right\},
\]
Section 3 shows the geometric distance between the unit ball \( B^*_n \) and a centrally symmetric convex body \( K \).
2. Notation and Preliminaries

In this section, we present some basic concepts and various facts that are needed in our investigations. We shall work in \( \mathbb{R}^n \) equipped with the canonical Euclidean scalar product \( \langle \cdot, \cdot \rangle \) and write \( |\cdot| \) for the corresponding Euclidean norm. We denote the unit sphere by \( S^{n-1} \).

Let \( K \) be a convex body (compact, convex sets with non-empty interiors) in \( \mathbb{R}^n \). The support function of \( K \) is defined by

\[
h_K(x) = \max\{\langle x, y \rangle : y \in K\}, \quad x \in \mathbb{R}^n.
\]

Obviously, \( h_{\phi x}(x) = h_K(\phi^t x) \) for \( \phi \in \text{GL}(n) \), where \( \phi^t \) denotes the transpose of \( \phi \).

A set \( K \subset \mathbb{R}^n \) is said to be a star body about the origin, if the line segment from the origin to any point \( x \in K \) is contained in \( K \) and \( K \) has continuous and positive radial function \( \rho_K(\cdot) \). Here, the radial function of \( K \), \( \rho_K : S^{n-1} \to [0, \infty) \), is defined by

\[
\rho_K(u) = \max\{\lambda : \lambda u \in K\}.
\]

Note that if \( K \) be a star body (about the origin) in \( \mathbb{R}^n \), then \( K \) can be uniquely determined by its radial function \( \rho_K(\cdot) \) and vice versa. If \( \alpha > 0 \), we have

\[
\rho_K(\alpha x) = \alpha^{-1} \rho_K(x) \quad \text{and} \quad \rho_{\alpha K}(x) = \alpha \rho_K(x).
\]

More generally, from the definition of the radial function it follows immediately that for \( \phi \in \text{GL}(n) \) the radial function of the image \( \phi K = \{\phi y : y \in K\} \) of star body \( K \) is given by \( \rho_{\phi K}(x) = \rho_K(\phi^{-1} x) \), for all \( x \in \mathbb{R}^n \).

If \( K, L \in S^n_+ \) and \( \lambda, \mu \geq 0 \) (not both zero), then for \( p > 0 \), the \( p \)-radial combination, \( \lambda K +_p \mu L \in S^n_+ \), is defined by (see [15])

\[
\rho((\lambda K +_p \mu L)^p) = \lambda \rho(K)^p + \mu \rho(L)^p.
\]

If a star body \( K \) contains the origin \( o \) as its interior point, then the Minkowski functional \( \|\cdot\|_K \) of \( K \) is defined by

\[
\|x\|_K = \min\{\lambda > 0 : x \in \lambda K\}.
\]

In this case,

\[
\|x\|_K = \rho_K^{-1}(x) = h_K(x),
\]

where \( K^* \) denotes the polar set of \( K \), which is defined by

\[
K^* = \{x \in \mathbb{R}^n : \langle x, y \rangle \leq 1 \text{ for all } y \in K\}.
\]

It is easy to verify that for \( \phi \in \text{GL}(n) \),

\[
(\phi K)^* = \phi^{-t} K^*,
\]

where \( \phi^{-t} \) denotes the reverse of the transpose of \( \phi \). Obviously, \( (K^*)^* = K \) (see [13] for details).

Let \( K \) and \( L \) be two convex bodies in \( \mathbb{R}^n \). According to [4], if \( o \in K \subseteq L \subseteq \mathbb{R}^n \), we call a pair \( (x, y) \in \mathbb{R}^n \times \mathbb{R}^n \) a contact pair for \( (K, L) \) if it satisfies:
1) \( x \in K \cap \partial L \),
2) \( y \in L \cap \partial K^* \),
3) \( \langle x, y \rangle = 1 \).

If \( x, y \in \mathbb{R}^n \), we denote by \( x \otimes y \) the rank one projection defined by \( x \otimes y(u) = \langle x, u \rangle y \) for all \( u \in \mathbb{R}^n \).

The geometric distance \( \delta_0(K, L) \) of the convex bodies \( K \) and \( L \) is defined by

\[
\delta_0(K, L) = \inf\{\alpha \beta : \alpha > 0, \beta > 0, (1 / \beta) L \subset K \subset \alpha L\}.
\]

3. Proof of Main Results

First, we prove that \( \tilde{I}_p() \) is a norm with respect to \( L_p \)-radial combination in \( S^*_n \). Apparently, \( \tilde{I}_p(K) \geq 0 \) and \( \tilde{I}_p(K) = 0 \) if and only if \( K = \{0\} \). At the same time, \( \tilde{I}_p(cK) = c \tilde{I}_p(K) \) if real constant \( c > 0 \). In addition, it follows that

\[
\tilde{I}_p(K) \leq \tilde{I}_p(K) + \tilde{I}_p(L).
\]

Indeed, we have

\[
\begin{aligned}
\tilde{I}_p(K) &= \left( \int_{\mathbb{R}^n} \rho_{K}^p(x) dx \right)^{\frac{1}{p}} \\
&\leq \left( \int_{\mathbb{R}^n} \rho_{K}^p(x) dx + \int_{\mathbb{R}^n} \rho_{L}^p(x) dx \right)^{\frac{1}{p}} \\
&= \tilde{I}_p(K) + \tilde{I}_p(L).
\end{aligned}
\]

Therefore, \( \tilde{I}_p() \) is a norm with respect to \( L_p \)-radial combination and \( S^*_n \) is normed space for \( \tilde{I}_p() \).

Now, we prove the optimization theorem of John [2] (see [10] also).

**Lemma 3.1.** Let \( F : \mathbb{R}^N \to \mathbb{R} \) be a \( C^1 \)-function. Let \( S \) be a compact metric space and \( \mathcal{G} : \mathbb{R}^N \times S \to \mathbb{R} \) be continuous. Suppose that for every \( s \in S \), \( \nabla_1 \mathcal{G}(z, s) \) exists and is continuous on \( \mathbb{R}^N \times S \).

Let \( A = \{ z \in \mathbb{R}^N : \mathcal{G}(z, s) \geq 0 \text{ for all } s \in S \} \) and \( z_0 \in A \) satisfy

\[
F(z_0) = \min_{z \in A} F(z).
\]

Then, either \( \nabla_1 \mathcal{G}(z_0) = 0 \) or, for some \( 1 \leq m \leq N \), there exist \( s_1, s_2, \ldots, s_m \in S \) and \( \lambda_1, \lambda_2, \ldots, \lambda_m \in \mathbb{R} \) such that \( \mathcal{G}(z_0, s_i) = 0, \lambda_i \geq 0 \) for \( 1 \leq i \leq m \), and

\[
\nabla_1 F(z_0) = \sum_{i=1}^{m} \lambda_i \nabla_1 \mathcal{G}(z_0, s_i).
\]

Using a similar argument as that in [1], we give the proof of Theorem 1.1.

**Proof of Theorem 1.1.** For \( N = n^2 \), we define \( \mathcal{F} : \mathbb{R}^N \to \mathbb{R} \) by

\[
\mathcal{F}(\phi) = \tilde{I}_p(\phi K) = \left( \int_{\mathbb{R}^n} \| \phi^i x \|_p^p d\gamma_n(x) \right)^{\frac{1}{p}},
\]

where \( \phi \in \mathbb{R}^N \) is the linear mapping from \( \mathbb{R}^n \) to \( \mathbb{R}^n \). Clearly \( \mathcal{F} \) is \( C^1 \). For \( S = K \times L^* \), define \( \mathcal{G} : \mathbb{R}^N \times S \to \mathbb{R} \) by
The set
\[ A = \{ z \in \mathbb{R}^N : G(z, s) \geq 0, s \in S \} \]
is just the set of elements \( \phi \in \mathbb{R}^N \) such that \( \phi K \subseteq L \). If \( K \) is in extremal position of \( \min \{ I_\phi(\phi K) : o \in \phi K \subseteq L, \phi \in \text{GL}(n) \} \), then \( \mathcal{F} \) attains its minimum on \( A \) at \( I_a \), namely,
\[ \mathcal{F}(I_a) = I_\phi(K) = \min \{ I_\phi(\phi K) : o \in \phi K \subseteq L, \phi \in \text{GL}(n) \}. \]

Now we prove \( \nabla_\phi \mathcal{F}(I_a) \). It follows from (3.1) that
\[
\mathcal{F}(\phi) = \left( \int_{\mathbb{R}^N} \| \phi^{-1}x \|_K^p \, d\gamma_n(x) \right)^{\frac{1}{p}}
\]
\[
= \left( (2\pi)^{-\frac{n}{2}} \int_{\mathbb{R}^N} \| \phi^{-1}x \|_K^p e^{-\frac{|x|^2}{2}} \, dx \right)^{\frac{1}{p}}
\]
\[
= \left( (2\pi)^{-\frac{n}{2}} (\det \phi) \int_{\mathbb{R}^N} \| x \|_K^p e^{-\frac{|x|^2}{2}} \, dx \right)^{\frac{1}{p}}
\]
It is easy to obtain that for non-degenerate \( \phi \), we have
\[
\nabla_\phi \mathcal{G}(\phi, (x, y)) = -\nabla_\phi \langle \phi x, y \rangle = \nabla_\phi \langle x \otimes y, \phi \rangle = -x \otimes y
\]
and
\[
\nabla_\phi \mathcal{F}(\phi) = \frac{1}{p} \left( (2\pi)^{-\frac{n}{2}} (\det \phi) \int_{\mathbb{R}^N} \| x \|_K^p e^{-\frac{|x|^2}{2}} \, dx \right)^{\frac{1}{q}}
\]
\[
\times \left[ (2\pi)^{-\frac{n}{2}} (\det \phi)(\phi^{-1})^* \int_{\mathbb{R}^N} \| x \|_K^p e^{-\frac{|x|^2}{2}} \, dx \right.
\]
\[
- (2\pi)^{-\frac{n}{2}} (\det \phi) \int_{\mathbb{R}^N} \| x \|_K^p e^{-\frac{|x|^2}{2}} x \otimes dx \right],
\]
where \( \frac{1}{p} + \frac{1}{q} = 1 \), \( (\phi^{-1})^* \) denotes conjugate of transposed transformation of \( \phi^{-1} \), and \( \phi^{-1} \) is inverse transform of \( \phi \in \text{GL}(n) \).

Since \( \mathcal{F} \) attains its minimum on \( A \) at \( z_a = I_a \), combining with Lemma 3.1, it follows that for some \( m \leq N \), there exist \( \lambda_i \geq 0, s_i \in S, s_i = (x_i, y_i) \), \( 1 \leq i \leq m \), such that
\[ \langle x_i, y_i \rangle = 1 - \mathcal{G}(I_a, (x_i, y_i)) = 1, \ 1 \leq i \leq m, \]
and
\[
\nabla_\phi \mathcal{F}(I_a) = \frac{1}{p} \left( I_\phi(K) \right)^{-\frac{n}{2}} \int_{\mathbb{R}^N} \| I_n - x \otimes x \|_K^p \, d\gamma_n(x)
\]
\[
= \sum_{i=1}^{m} \lambda_i \nabla_\phi \mathcal{G}(I_a, (x_i, y_i)) \tag{3.2}
\]
\[=-\sum_{i=1}^{m} \lambda_i x_i \otimes y_i. \]
From \( \{ x_i, y_i \} = 1, x_i \in K \subseteq L, y_i \in L' \subseteq K^* \), we yield \( x_i \in \partial L \) and \( y_i \in \partial K^* \).

Taking the trace in (3.2), we have

\[
\text{Tr} \left( \nabla_x \mathcal{F}(I_n) \right) = \text{Tr} \left( \frac{1}{p} \left( \mathcal{I}_p(K) \right)^{\frac{p}{2}} \int_{\mathbb{R}^n} (I_n - x \otimes x) \| x \|_K^p \, d\gamma_n(x) \right)
\]

\[
= \frac{1}{p} \left( \mathcal{I}_p(K) \right)^{\frac{p}{2}} \left[ n \int_{\mathbb{R}^n} \| x \|_K^p \, d\gamma_n(x) - \int_{\mathbb{R}^n} |x|^p \| x \|_K^p \, d\gamma_n(x) \right]
\]

\[
= \frac{1}{p} \left( \mathcal{I}_p(K) \right)^{\frac{p}{2}} \left[ n \int_{\mathbb{R}^n} r^{n-p-1} e^{-\frac{|x|^2}{2}} \, dr - \int_{\mathbb{R}^n} r^{n-p} e^{-\frac{|x|^2}{2}} \, dr \right] \int_{\partial \Omega} \theta \| x \|_K^p \, dS(\theta)
\]

\[
= \frac{1}{p} \left( \mathcal{I}_p(K) \right)^{\frac{p}{2}} \left( p \int_{\mathbb{R}^n} \| x \|_K^p \, d\gamma_n(x) \right) = \mathcal{I}_p(K).
\]

Suppose \( \lambda_i = c_i \mathcal{I}_p(K) \). Together with (3.2), we obtain

\[
\int_{\mathbb{R}^n} (x \otimes x - I_n) \| x \|_K^p \, d\gamma_n(x) = p(\mathcal{I}_p(K))^p (\sum_{i=1}^m c_i x_i \otimes y_i),
\]

where \( \sum_{i=1}^m c_i = 1 \). This completes the proof.

If \( L = B_2^n \) and \( G(\phi, x) = 1 - |x|^2 \), then using the same method in the proof of Theorem 1.1, we obtain

**Corollary 3.2.** Let \( K \) be a convex body such that \( o \in K \subseteq B_2^n \). If \( K \) is in extremal position of (1.7), then there exist contact points \( u_1, u_2, \ldots, u_m \in \partial K \cap S^{n-1} \) with \( m \leq n^2 \) and \( c_1, c_2, \ldots, c_m > 0 \), such that,

\[
I_n = \int_{\mathbb{R}^n} (x \otimes x) d \tilde{\mu}(x) - p \sum_{i=1}^m c_i u_i \otimes u_i, \quad \sum_{i=1}^m c_i = 1,
\]

where \( d \tilde{\mu}(x) \) is the probability measure on \( \mathbb{R}^n \) with normalized density

\[
d \tilde{\mu}(x) = \| x \|_K^p \, d\gamma_n(x) / (\mathcal{I}_p(K))^p.
\]

**Proof of Theorem 1.2.** Suppose that \( \phi \in L(\mathbb{R}^n, \mathbb{R}^n) \) and \( \epsilon > 0 \) is small enough. Then

\[
\phi_\epsilon \coloneqq (\min_{u \in S^{n-1}} \| u - \epsilon \phi u \|_K) (I_n - \epsilon \phi)^{-1}
\]

satisfies \( \phi_\epsilon K \subseteq B_2^n, \phi_\epsilon K \cap S^{n-1} \neq \emptyset \). Therefore

\[
\int_{\mathbb{R}^n} \| x - \epsilon \phi x \|_K^p \, d\gamma_n(x) \leq (\mathcal{I}_p(K))^p (\min_{u \in S^{n-1}} \| u - \epsilon \phi u \|_K)^{-p}.
\]

Let \( u_\epsilon \) be a point on \( S^{n-1} \) at which the minimum is attained. Observe that

\[
\| x - \epsilon \phi x \|_K^p = \| x \|_K^p + \epsilon p \| x \|_K^{p-1} \left( \nabla h_\epsilon(x), \phi x \right) + O(\epsilon^3)
\]

and

\[
|u_\epsilon - \epsilon \phi u_\epsilon|_K^p = 1 + \epsilon p \langle u_\epsilon, \phi u_\epsilon \rangle + O(\epsilon^2).
\]

Since \( u_\epsilon \in S^{n-1} \) and \( \| \cdot \|_K \), we have

\[

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\[
\int_{\mathbb{R}^n} p \|x\|_K^{p-1} \left\langle \nabla h_{K^p}(x), \phi x \right\rangle \, d\gamma_n(x) + O(\epsilon)
\]
\[
\leq \left( \bar{I}_p(K) \right)^p \left( \min_{\omega \in S^{n-1}} \|u - \varepsilon \phi \omega\|_K \right)^p - 1
\]
\[
\leq \left( \bar{I}_p(K) \right)^p \frac{|u_{x\varepsilon} - \varepsilon \phi u_{x\varepsilon}|^p - 1}{\varepsilon}
\]
\[
= \left( \bar{I}_p(K) \right)^p \left( p \{u_{x\varepsilon}, \phi u_{x\varepsilon}\} + O(\epsilon) \right).
\] (3.3)

If \( u \) is a contact point of \( K \) and \( B_2^n \), then
\[
1 + \varepsilon \|\phi\|_K \|u - \varepsilon \phi \omega\|_K \|u_{x\varepsilon} - \varepsilon \phi u_{x\varepsilon}\|_K \|\|
\]
\[
\leq \|u_{x\varepsilon}\|_K \leq 1 + 2\varepsilon \|\phi\|_K. \quad (3.4)
\]

It follows that
\[
1 \leq \|u_{x\varepsilon}\|_K \leq 1 + 2\varepsilon \|\phi\|_K. \quad (3.4)
\]

In order to obtain a sequence \( \varepsilon_k \to 0 \) and a point \( u \in S^{n-1} \) such that \( u_{x\varepsilon_k} \to u \). If \( k \to \infty \), it follows from (3.4) that \( \|u\|_K = \lim_{k \to \infty} \|u_{x\varepsilon_k}\|_K = 1 \). Namely, \( u \) is a contain point of \( K \) and \( B_2^n \). By (3.3), we obtain
\[
\int_{\mathbb{R}^n} \|x\|_K^{p-1} \left\langle \nabla h_{K^p}(x), \phi x \right\rangle \, d\gamma_n(x) \leq \left( \bar{I}_p(K) \right)^p \left\langle u, \phi u \right\rangle.
\]

Taking \( \phi \) for \( -\phi \), we can find another contact point \( u' \) of \( K \) and \( B_2^n \) such that
\[
\int_{\mathbb{R}^n} \|x\|_K^{p-1} \left\langle \nabla h_{K^p}(x), \phi x \right\rangle \, d\gamma_n(x) \geq \left( \bar{I}_p(K) \right)^p \left\langle u', \phi u' \right\rangle.
\]
Choosing \( \phi_x(x) = \langle x, \theta \rangle \theta \) with \( \theta \in S^{n-1} \), we get (1.6). \( \square \)

4. Estimate of the Distance

**Lemma 4.1.** (see [16]) Let \( x = (x_1, x_2, \ldots, x_n) \in \mathbb{R}^n \) and \( y = (y_1, y_2, \ldots, y_n) \in \mathbb{R}^n \). If
\[
0 < m_i \leq x_i \leq M_i, \quad 0 < m_2 \leq y_i \leq M_2, \quad k = 1, \ldots, n,
\]
then
\[
\left( \sum_{i=1}^n x_i^2 \right) \left( \sum_{i=1}^n y_i^2 \right) \leq \left( \frac{M_1 M_2}{m_1 m_2} + \sqrt{\frac{m_1 m_2}{M_1 M_2}} \right)^2 \left( \sum_{i=1}^n x_i y_i \right)^2.
\]

Lemma 4.1 implies that if \( x, y \in \mathbb{R}^n \), then there exist a constant \( c \in (0,1) \) such that
\[
| \langle x, y \rangle | \geq c |x||y|. \quad (4.1)
\]

Suppose that \( K \) is a centrally symmetric convex body in \( \mathbb{R}^n \) such that \( K \) is in the extremal position of (1.7). Now we estimate the geometric distance between \( K \) and \( B_2^n \).

**Theorem 4.1.** Let \( K \subseteq B_2^n \) be a centrally symmetric convex body in \( \mathbb{R}^n \). If \( K \) is in the extremal position of (1.7) and \( 1 \leq p < 3 \), then
\[
\tilde{c}_{n,p} B_2^n \subseteq K \subseteq B_{2^n}^n,
\]
where

\[ \tilde{c}_{n,p} = \frac{\tilde{I}_p(B^n_2)}{\sqrt{n}} \left( \frac{\sqrt{\pi} (cp + 1)}{2^{\frac{3}{2}} \Gamma \left( \frac{3-p}{2} \right)} \right)^{\frac{1}{p}}, \quad c \in (0,1). \]

**Proof.** It follows from Corollary 3.2 that \( K \) satisfies

\[ I_n = \int_{\mathbb{R}^n} (x \otimes x) d \tilde{\mu}(x) - p \sum_{i=1}^m c_i u_i \otimes u_i, \quad \sum_{i=1}^m c_i = 1, \]

where \( d \tilde{\mu}(x) \) is the probability measure on \( \mathbb{R}^n \) with normalized density

\[ d \tilde{\mu}(x) = x^n \| x \|^p d \gamma_n(x) / (\tilde{I}_p(K))^p. \]

For \( y \in K^\circ \) and \( u_i \in S^{n-1} \). By (4.1), there exists a constant \( c \in (0,1) \) such that \( \langle y, u_i \rangle \geq c | y | \). So we obtain

\[ \int_{\mathbb{R}^n} (\| x \|^2 - \| y \|^2) d \tilde{\mu}(x) \geq cp | y |^2 \sum_{i=1}^m c_i = cp | y |^2. \]

That is,

\[ (cp + 1) | y |^2 \leq \int_{\mathbb{R}^n} (\| x \|^2 - \| y \|^2) d \tilde{\mu}(x). \]

Since \( \| x \|_k \geq \| x, y \| \), we have

\[
\int_{\mathbb{R}^n} (\| x \|^2 - \| y \|^2) d \gamma_n(x) \leq \int_{\mathbb{R}^n} (\| y \|^2 - \| x \|^2) d \gamma_n(x) = (2\pi)^{\frac{n}{2}} \int_{S^{n-1}} (\| \theta \|^2 - \| y \|^2) dS(\theta) \int_0^\infty r^{n-p+1} e^{-\frac{r^2}{2}} dr = 2^{1-\frac{p}{2}} \pi^{\frac{n}{2}} \Gamma \left( \frac{3-p}{2} \right) | y |^{2-p}.
\]

From John’s theorem, for every centrally symmetric convex body \( K \) in \( \mathbb{R}^n \), there is a corresponding to the ball \( \lambda B^n_2 \) such that \( \lambda B^n_2 \subseteq K \subseteq \lambda B^n_2 \) (\( \lambda > 0 \)).

Take \( \lambda = 1 / \sqrt{n} \). We obtain

\[ \frac{1}{\sqrt{n}} B^n_2 \subseteq K \subseteq B^n_2. \]

Thus,

\[ \frac{1}{\sqrt{n}} \tilde{I}_p(B^n_2) \leq \tilde{I}_p(K) \leq \tilde{I}_p(B^n_2). \]

Therefore, we get

\[ \left| y \right| \leq \frac{\sqrt{n}}{\tilde{I}_p(B^n_2)} \left( \frac{2^{\frac{3}{2}} \Gamma \left( \frac{3-p}{2} \right)}{\sqrt{\pi} (cp + 1)} \right)^{\frac{1}{p}}, \]

and the result yields.

\( \Box \)

Giannopoulos et al. in [5] proved that if \( K \) is in a position of maximal volume in \( L \), then \( K \subseteq L \subseteq nK \), which is equivalent to \( \frac{1}{n} \| x \|_k \leq \| x \|_k \leq \| x \|_k \) for all \( x \in \mathbb{R}^n \). Hence it follows that
Furthermore, let $\phi \in \text{GL}(n)$. Since $\phi K \subseteq B^n_2$ is in the maximal volume position of $K$ contained in $B^n_2$, we have $\frac{1}{\sqrt{n}} B^n_2 \subseteq \phi K \subseteq B^n_2$. Thus

$$\frac{1}{\sqrt{n}} \leq \frac{\bar{I}_n(\phi K)}{\bar{I}_p(B^n_2)} \leq 1.$$ 

Finally, we propose the following concept of $l_0$-norm: Let $K$ be a convex body in $\mathbb{R}^n$, we define $l_0$-norm by

$$l_0(K) = \exp \left( \int_{\mathbb{R}^n} \log \| x \|_K \gamma_n(x) \right).$$

We propose an open question as follows: How should we solve the extreme problem

$$\min \{ l_0(\phi K) : \phi \in \text{GL}(n), \phi K \subseteq L \}?$$

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**Conflicts of Interest**

The author declares no conflicts of interest regarding the publication of this paper.

**References**


