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Delay Harmonic Oscillator

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Abstract

The motion of two point objects at the end of a spring is analyzed. The objects interact by an elastic wave propagating through the spring. A new comprehensive method, Reaction Mechanics, for the analysis of this motion is used. This analysis is valid when the propagation of the interaction through the spring wire takes less time than the period of the oscillating frequency. The propagation delay couples the oscillating and center of mass motions. If the masses are equal, the center of mass velocity is a constant, and the objects oscillate with a frequency which is a modification of the oscillation frequency with no delay. If the masses are not equal, the center of mass also oscillates. In the case of zero delay, the motion of the objects reverts to the motion of a Simple Harmonic Oscillator.

Keywords

Lagrangian, Equation of Motion, Hamiltonian, Delay, Oscillation

1. Introduction

The motion of two objects connected by a spring is analyzed. The interaction propagates between the objects as a mechanical wave through the spring as shown in Figure 1. The mechanical wave propagates through the spring with a velocity [1] that depends on the stiffness of the spring and the mass of the spring per unit length. Therefore, there is a time delay τ required for the deformation applied to one end of the spring to reach the other end. The interaction is not instantaneous. This is seldom taken into consideration when systems of springs and masses are calculated.

In this paper, a new comprehensive method called Reaction Mechanics [2] is used for the analysis of the motion of a system that takes the propagation delay of the interaction into account.

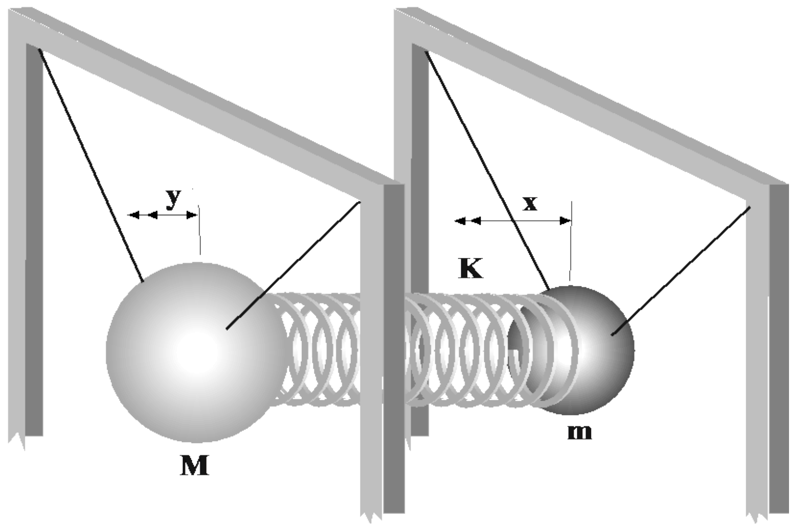

Figure 1. Two objects with unequal masses m and M connected by a spring with a spring constant K. The interaction travels between the objects as a mechanical wave through the spring.

The interaction travels between the two objects, through the spring, with a velocity v_S equal to the square root of the ratio of the spring stiffness constant in Newtons, divided by the mass density of the spring in kg per m. The spring stiffness is equal to an elastic constant times a cross sectional area. The analysis is valid when the propagation of the interaction through the spring wire takes less time than the period of the oscillating frequency. The propagation delay through the spring couples the Oscillatory and Center of Mass motions of the objects.

It is demonstrated that the delayed interaction [2] is completely causal. A force F_R that was radiated by object TWO in the past is received currently by object ONE. Object ONE also radiates a force F_A currently that object TWO might or might not receive in the future.

By Newton's law of action and reaction object ONE experiences a recoil force F_{Recoil} equal in magnitude and opposed in direction to the force F_A it radiated. The recoil force is independent whether or not object TWO receives the radiated force in the future. The forces are continually radiated and received.

Two forces act on object ONE. The force F_R radiated by object TWO in the past and the recoil force F_{Recoil}. These two forces are equal to the mass of object ONE times its acceleration. Thus, the formulation is causal [2].

The calculations show that the two masses and the spring oscillate. But, if the two masses are not equal, the Center of Mass of the objects will also oscillate.

The results of this paper will be useful for the analysis of machines with coil springs and masses. The most obvious application is the vehicle suspension [3]. The vehicle suspension has two masses connected by coil springs and vibration dampers. However, the vibration dampers are not part of this analysis. The upper mass is the body, chassis and engine, and the lower mass is the axels and wheels. The upper mass is much heavier than the lower mass. Therefore, the center of mass will vibrate. If the vibration dampers were not there, it would cause a large vibration of the upper mass.

If one uses the standard calculation, neglecting the propagation delay through the spring, one would then obtain that the two masses vibrate about the stationary center of mass.

This type of motion also occurs when an electric motor is mounted on a platform that is suspended on elastic supports. Even if the rotation of the motor is well balanced, the platform might still vibrate because there are different masses on both sides of the elastic support.

The intake and exhaust valves with their springs of an internal combustion engine can cause oscillations in the engine block.

2. Equation of Motion

In this analysis, the masses are treated as point objects and the model of the spring includes the propagation delay. The propagation delay is modeled by considering that the forces, implicit in Equation (1), with components $Kx_{\mu k}$ and $Ky_{\mu k-2}$ at the two ends of the spring, do not occur simultaneously. It is assumed that the time difference $t_k - t_{k-2}$ between the forces occurring is equal to the propagation delay through the spring. This eliminates the necessity of describing the spring by a Continuum Mechanics model. It is assumed that the mechanical signal propagates through the spring, from one object to the other, in a shorter time than the period of oscillation of the masses and the spring. In this case, the causal Lagrangian [4] for the Delay Harmonic Oscillator (DHO) can be approximated as follows:

$$L_k = \frac{1}{2} m \dot{x}_{\mu k} \dot{x}_{\mu k} + \frac{1}{2} M \dot{y}_{\mu k} \dot{y}_{\mu k} - \frac{1}{4} K \left(x_{\mu k} - y_{\mu k-2} \right) \left(x_{\mu k} - y_{\mu k-2} \right) - \frac{1}{4} K \left(y_{\mu k} - x_{\mu k-1} \right) \left(y_{\mu k} - x_{\mu k-1} \right) \tag{1}$$

where summation over repeated Greek indices is implied. The subscripts $\mu = 1, 2, 3$, range over the tree dimensions of the coordinates. Single superior dots $\dot{x}_{\mu k}$ denote time derivatives and double superior dots $\ddot{x}_{\mu k}$ denote second time derivatives. Here $x_{\mu k}$ at time step t_k is a coordinate component, and m is the mass of Object ONE, and $y_{\mu k}$ at time step t_k is a coordinate component, and M is the mass of Object TWO. K is a spring constant in Newtons per m.

Forming a sum S of Lagrangians.

$$S = \sum_{k=-\infty}^{k=\infty} L_k \tag{2}$$

Only the Lagrangians L_k and L_{k+1} contain the coordinate components $x_{\mu k}$ and $y_{\mu k}$ occurring at time step t_k. The Lagrangian L_k is advanced to the next time step L_{k+1} by advancing the coordinates to their next nodal points, $x_{\mu k}$ advances to $x_{\mu k+1}$, $x_{\mu k-1}$ advances to $x_{\mu k}$, $y_{\mu k}$ advances to $y_{\mu k+2}$ and $y_{\mu k-2}$ advances to $y_{\mu k}$.

As shown in Equation (1) and Equation (3), t_{k-1} is a time step of object ONE in the past and t_{k-2} is a time step of object TWO in the past. The time t_{k-2} in the past, at which object TWO radiated an interaction that object ONE is receiving currently at time t_k, is not the same as the time t_{k-1} in the past at which Object ONE was radiating an interaction that object TWO is receiving currently. A similar condition exists for the future coordinate times. The future forces radiated by the objects do not act on the objects. Recoil forces due to the objects radiating interactions act on the objects. The Lagrangian at time step t_{k+1} is:

$$\begin{aligned} L_{k+1} &= \frac{1}{2} m \dot{x}_{\mu k+1} \dot{x}_{\mu k+1} + \frac{1}{2} M \dot{y}_{\mu k+2} \dot{y}_{\mu k+2} - \frac{1}{4} K \left(x_{\mu k+1} - y_{\mu k} \right) \left(x_{\mu k+1} - y_{\mu k} \right) \\ &\quad - \frac{1}{4} K \left(y_{\mu k+2} - x_{\mu k} \right) \left(y_{\mu k+2} - x_{\mu k} \right) \end{aligned} \tag{3}$$

Making a transformation of variables [4].

$$\text{a)}\ r_{\mu k} = x_{\mu k} - y_{\mu k} \qquad \text{b)}\ R_{\mu k} = \frac{m}{m + M} x_{\mu k} + \frac{M}{m + M} y_{\mu k} \tag{4}$$

where $r_{\mu k}$ and $R_{\mu k}$ are also in rectangular coordinates. Here $r_{\mu k}$ is a oscillation coordinate component and $R_{\mu k}$ is a center of mass coordinate component. Inverting Equation (4).

$$\text{a)}\ x_{\mu k} = \frac{M}{m + M} r_{\mu k} + R_{\mu k} \qquad \text{b)}\ y_{\mu k} = -\frac{m}{m + M} r_{\mu k} + R_{\mu k} \tag{5}$$

Substituting the transformation of variables, Equation (5) into the Lagrangian of Equation (1).

$$\begin{aligned} L_k &= \frac{1}{2} m_R \dot{r}_{\mu k} \dot{r}_{\mu k} + \frac{1}{2} (m + M) \dot{R}_{\mu k} \dot{R}_{\mu k} \\ &\quad - \frac{1}{4} K \left(\frac{M}{m+M} r_{\mu k} + R_{\mu k} + \frac{m}{m+M} r_{\mu k-1} - R_{\mu k-1} \right) \\ &\quad \times \left(\frac{M}{m+M} r_{\mu k} + R_{\mu k} + \frac{m}{m+M} r_{\mu k-1} - R_{\mu k-1} \right) \\ &\quad - \frac{1}{4} K \left(-\frac{m}{m+M} r_{\mu k} + R_{\mu k} - \frac{M}{m+M} r_{\mu k-1} - R_{\mu k-1} \right) \\ &\quad \times \left(-\frac{m}{m+M} r_{\mu k} + R_{\mu k} - \frac{M}{m+M} r_{\mu k-1} - R_{\mu k-1} \right) \end{aligned} \tag{6}$$

where m_R is the reduced mass.

$$m_R \equiv \frac{mM}{m + M} \tag{7}$$

As described previously, the coordinate components $x_{\mu k-1}$ and $y_{\mu k-2}$ do not occur simultaneously. However, a simplification is made in Equation (6) and Equation (8) that assumes that those coordinate components do occur simultaneously. Substituting the transformation of variables, Equation (5) into the Lagrangian of Equation (3)

$$
\begin{aligned}
L_{k+1} = {} & \frac{1}{2} m_R \dot{r}_{\mu k+1} \dot{r}_{\mu k+1} + \frac{1}{2}(m+M) \dot{R}_{\mu k+1} \dot{R}_{\mu k+1} \\
& - \frac{1}{4} K \left(\frac{M}{m+M} r_{\mu k+1} + R_{\mu k+1} + \frac{m}{m+M} r_{\mu k} - R_{\mu k} \right) \\
& \times \left(\frac{M}{m+M} r_{\mu k+1} + R_{\mu k+1} + \frac{m}{m+M} r_{\mu k} - R_{\mu k} \right) \\
& - \frac{1}{4} K \left(-\frac{m}{m+M} r_{\mu k+1} + R_{\mu k+1} - \frac{M}{m+M} r_{\mu k} - R_{\mu k} \right) \\
& \times \left(-\frac{m}{m+M} r_{\mu k+1} + R_{\mu k+1} - \frac{M}{m+M} r_{\mu k} - R_{\mu k} \right)
\end{aligned} \tag{8}
$$

The Euler Lagrange equations of motion of Reaction Mechanics [2] for the oscillating and center of mass motions are:

$$
\text{a)}\ \frac{\mathrm{d}}{\mathrm{d}t_k} \frac{\partial S}{\partial \dot{r}_{\mu k}} - \frac{\partial S}{\partial r_{\mu k}} = 0 \qquad \text{b)}\ \frac{\mathrm{d}}{\mathrm{d}t_k} \frac{\partial S}{\partial \dot{R}_{\mu k}} - \frac{\partial S}{\partial R_{\mu k}} = 0 \tag{9}
$$

Substituting the Lagangians of Equation (6) and Equation (8) into Equation (2) and the resulting sum of Lagrangians into the Euler Lagrange equation of motion, Equation (9a) to obtain the oscillation equation of motion.

$$
\begin{aligned}
& m_R \ddot{r}_{\mu k} + \frac{M}{m+M} \frac{K}{2} \left(\frac{M}{m+M} r_{\mu k} + R_{\mu k} + \frac{m}{m+M} r_{\mu k-1} - R_{\mu k-1} \right) \\
& + \frac{m}{m+M} \frac{K}{2} \left(\frac{m}{m+M} r_{\mu k} - R_{\mu k} + \frac{M}{m+M} r_{\mu k-1} + R_{\mu k-1} \right) \\
& + \frac{m}{m+M} \frac{K}{2} \left(\frac{M}{m+M} r_{\mu k+1} + R_{\mu k+1} + \frac{m}{m+M} r_{\mu k} - R_{\mu k} \right) \\
& + \frac{M}{m+M} \frac{K}{2} \left(\frac{m}{m+M} r_{\mu k+1} - R_{\mu k+1} + \frac{M}{m+M} r_{\mu k} + R_{\mu k} \right) = 0
\end{aligned} \tag{10}
$$

Performing a first step of collecting terms:

$$
\begin{aligned}
& m_R \ddot{r}_{\mu k} + K r_{\mu k} - \frac{MmK}{(m+M)^2} \left(2 r_{\mu k} + r_{\mu k-1} + r_{\mu k+1} \right) \\
& + \frac{M-m}{M+m} \frac{K}{2} \left(2 R_{\mu k} - R_{\mu k-1} - R_{\mu k+1} \right) = 0
\end{aligned} \tag{11}
$$

Performing a second step of collecting terms:

$$
\begin{aligned}
& m_R \ddot{r}_{\mu k} + K r_{\mu k} - \frac{m_R K}{m+M} \left(\frac{r_{\mu k} r_{\mu k}}{\dot{r}_{\mu k} \dot{r}_{\mu k}} \right) \frac{2 r_{\mu k} - r_{\mu k-1} - r_{\mu k+1}}{\tau^2} \\
& + \frac{M-m}{2(m+M)} K \left(\frac{r_{\mu k} r_{\mu k}}{\dot{r}_{\mu k} \dot{r}_{\mu k}} \right) \frac{2 R_{\mu k} - R_{\mu k-1} - R_{\mu k+1}}{\tau^2} = 0
\end{aligned} \tag{12}
$$

where the delay time τ is approximately:

$$
\tau^2 = \frac{r_{\mu k} r_{\mu k}}{\dot{r}_{\mu k} \dot{r}_{\mu k}} \tag{13}
$$

Substituting the Lagrangians of Equation (6) and Equation (8) into Equation (2) and the resulting sum of Lagrangians into the Euler Lagrange equation of motion, Equation (9b) to obtain the center of mass equation of motion.

$$
\begin{aligned}
&(m+M)\ddot{R}_{\mu k}+\frac{1}{2}K\left(\frac{M}{m+M}r_{\mu k}+R_{\mu k}+\frac{m}{m+M}r_{\mu k-1}-R_{\mu k-1}\right)\\
&\quad+\frac{1}{2}K\left(-\frac{m}{m+M}r_{\mu k}+R_{\mu k}-\frac{M}{m+M}r_{\mu k-1}-R_{\mu k-1}\right)\\
&\quad+\frac{1}{2}K\left(-\frac{M}{m+M}r_{\mu k+1}-R_{\mu k+1}-\frac{m}{m+M}r_{\mu k}+R_{\mu k}\right)\\
&\quad+\frac{1}{2}K\left(\frac{m}{m+M}r_{\mu k+1}-R_{\mu k+1}+\frac{M}{m+M}r_{\mu k}+R_{\mu k}\right)=0
\end{aligned}
\tag{14}
$$

Collecting terms:

$$
(m+M)\ddot{R}_{\mu k}+K\tau^2\frac{2R_{\mu k}-R_{\mu k-1}-R_{\mu k+1}}{\tau^2}+\frac{K\tau^2}{2}\frac{M-m}{m+M}\frac{2r_{\mu k}-r_{\mu k-1}-r_{\mu k+1}}{\tau^2}=0 \tag{15}
$$

where the second and third terms have been multiplied and divided by the square of the delay time τ^2. Approximating the second differences divided by τ^2 such as

$$
\frac{2r_{\mu k}-r_{\mu k-1}-r_{\mu k+1}}{\tau^2}
$$

by the second derivatives times the square τ^2 of the propagation delay time τ.

$$
\text{a)}\ \frac{r_{\mu k+1}-2r_{\mu k}+r_{\mu k-1}}{\tau^2}\approx\ddot{r}_{\mu k}\qquad \text{b)}\ \frac{R_{\mu k+1}-2R_{\mu k}+R_{\mu k\text{-}1}}{\tau^2}\approx\ddot{R}_{\mu k} \tag{16}
$$

Substituting Equation (16) into Equation (12) and dividing by the reduced mass m_R.

$$
\left(1-\frac{K\tau^2}{m+M}\right)\ddot{r}_{\mu}+\frac{K}{m_R}r_{\mu}+\frac{M-m}{2(m+M)}\frac{K\tau^2}{m_R}\ddot{R}_{\mu}=0 \tag{17}
$$

Substituting Equation (16) into Equation (15).

$$
\left(1-\frac{K\tau^2}{m+M}\right)\ddot{R}_{\mu}-\frac{K\tau^2}{2}\frac{M-m}{(m+M)^2}\ddot{r}_{\mu}=0 \tag{18}
$$

Solving Equation (18) for $\ddot{R}_{\mu}$ and substituting the resulting equation into Equation (17) to obtain the oscillation equation of motion of the Delay Harmonic Oscillator.

$$
\boxed{\left[\left(1-\frac{K\tau^2}{m+M}\right)^2+\frac{K^2\tau^4}{4mM}\frac{(M-m)^2}{(m+M)^2}\right]\ddot{r}_{\mu}+\frac{K}{m_R}\left(1-\frac{K\tau^2}{m+M}\right)r_{\mu}=0} \tag{19}
$$

Solving Equation (17) for $\ddot{r}_{\mu}$ and substituting the resulting equation into Equation (18) to obtain the center of mass equation of motion of the Delay Harmonic Oscillator.

$$
\boxed{\left[\left(1-\frac{K\tau^2}{m+M}\right)^2+\frac{K^2\tau^4}{4mM}\frac{(M-m)^2}{(m+M)^2}\right]\ddot{R}_{\mu}+\frac{K^2\tau^4}{2mM}\frac{M-m}{m+M}r_{\mu}=0} \tag{20}
$$

The final results are Equation (19) and Equation (20), the Equations of Motion of the Delay Harmonic Oscillator.

In the limit when both masses are equal, $m = M$, the center of mass acceleration components $\ddot{R}_\mu$ are equal to zero, and the center of mass, at most, moves at a constant velocity and the objects oscillate with an approximate modified angular frequency.

$$\omega_{m=M}^2 \approx \frac{K}{m_R\left(1-\frac{K\tau^2}{m+M}\right)} \tag{21}$$

Note from Equation (13) that the delay time τ is not constant.

In the limit when the delay time τ goes to zero, the equations of motions 19 and 20 revert to the equations of motion of a Simple Harmonic Oscillator [4].

$$\text{a)}\ m_R\ddot{r}_\mu + Kr_\mu = 0 \qquad \text{b)}\ \ddot{R}_\mu = 0 \tag{22}$$

3. Conservation of Energy

In this conservative system, it is assumed that the energy is equal to the Hamiltonian H of the system. In order to calculate the Hamiltonian, we must first calculate the momentum component p_μ of the system. The total momentum component p_μ is:

$$p_\mu = \frac{\partial S}{\partial \dot{r}_\mu} + \frac{\partial S}{\partial \dot{R}_\mu} \tag{23}$$

Substituting Equation (6) and Equation (8) into Equation (2) and the resulting equation into Equation (23),

$$p_{\mu k} = m_R\dot{r}_{\mu k} + (m+M)\dot{R}_{\mu k} \tag{24}$$

The Hamiltonian sum is equal to the Legender Transform of Sum of Lagrangians with respect to the velocities $\dot{r}_{\mu k}$ and $\dot{R}_{\mu k}$ [2]:

$$H = m_R\dot{r}_{\mu k}\dot{r}_{\mu k} + (m+M)\dot{R}_{\mu k}\dot{R}_{\mu k} - L_k - L_{k+1} \tag{25}$$

where the Lagrangian L_k at time step t_k is given by equation 6 and the Lagrangian L_{k+1} at time step t_{k+1} is given by Equation (8). Taking the derivative with respect to the time t_k of the Hamiltonian of Equation (25) using Equation (6) and Equation (8) one obtains:

$$\begin{aligned}
\frac{\mathrm{d}H}{\mathrm{d}t_k} &= m_R\ddot{r}_{\mu k}\dot{r}_{\mu k} + (m+M)\ddot{R}_{\mu k}\dot{R}_{\mu k} \\
&+\frac{1}{2}K\left(\frac{M}{m+M}r_{\mu k} + R_{\mu k} + \frac{m}{m+M}r_{\mu k-1} - R_{\mu k-1}\right)\left(\frac{M}{m+M}\dot{r}_{\mu k} + \dot{R}_{\mu k}\right) \\
&+\frac{1}{2}K\left(-\frac{m}{m+M}r_{\mu k} + R_{\mu k} - \frac{M}{m+M}r_{\mu k-1} - R_{\mu k-1}\right)\left(-\frac{m}{m+M}\dot{r}_{\mu k} + \dot{R}_{\mu k}\right) \\
&+\frac{1}{2}K\left(\frac{M}{m+M}r_{\mu k+1} + R_{\mu k+1} + \frac{m}{m+M}r_{\mu k} - R_{\mu k}\right)\left(\frac{m}{m+M}\dot{r}_{\mu k} - \dot{R}_{\mu k}\right) \\
&+\frac{1}{2}K\left(-\frac{m}{m+M}r_{\mu k+1} + R_{\mu k+1} - \frac{M}{m+M}r_{\mu k} - R_{\mu k}\right)\left(\frac{M}{m+M}\dot{r}_{\mu k} - \dot{R}_{\mu k}\right)
\end{aligned} \tag{26}$$

By multiplying Equation (10) by $\dot{r}_{\mu k}$ and Equation (14) by and $\dot{R}_{\mu k}$ and adding the result one obtains an equation identical to Equation (26). Since Equation (10) and Equation (14) are equal to zero, the change of the Hamiltonian H with time is equal to zero, and therefore the Hamiltonian and thus the energy are constant.

$$\text{a)}\ \frac{\mathrm{d}H}{\mathrm{d}t_k}=0 \qquad \text{b)}\ H=\text{constant} \qquad c)\ \text{Energy}=\text{constant} \tag{27}$$

4. Conclusions

In this paper, the propagation delay of a deformation traveling through the spring is considered, and therefore, is called the Delay Harmonic Oscillator. The Equations of Motion of two masses at the end of a spring through which a deformation propagates with a finite velocity are derived. The effect of the delay is modeled by assuming that the forces at both ends of the spring are not simultaneous. The difference in time of the occurrence of these forces is equal to the propagation delay of the deformation signal traveling through the spring. This effect couples the oscillatory and center of mass motions. But this derivation is valid when the propagation of the interaction through the spring wire takes less time than the period of the oscillating frequency. It demonstrates that if the masses are unequal, the center of mass of this system oscillates. Also, the effect of the finite propagation delay causes the oscillation to be non sinusoidal. In the case of zero delay, the motion of the masses reverts to the motion of a Simple Harmonic Oscillator. The effect of the propagation delay through the spring is rarely discussed in literature.

The Equations of Motion derived here are useful for the design of mechanical systems with coil springs and masses. The vehicle suspention [3] is the most obvious application. It has two masses connected by coil springs and vibration dampers. However, the vibration dampers are not part of this analysis. The upper mass consists of the body, chassis and engine and the lower mass consists of the axels, differential and wheels. The upper mass is much heavier than the lower mass. Therefore, the center of mass will vibrate. If the vibration dampers were not there, this would cause a large vibration of the upper mass.

If one uses the standard calculation neglecting the propagation delay through the spring, one would obtain that the two masses vibrate about the stationary center of mass.

Another example of this type of motion occurs when an electric motor is mounted on a platform that is suspended on elastic supports. Even if the rotation of the motor is well balanced, the platform may still vibrate because there are different masses on both sides of the elastic support.

The Equations of Motion of the Delay Harmonic Oscillator are derived in this paper.

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References

[1] Achenbach, J.D. (1984) Wave Propagation in Elastic Solids. North Holland. Elsevier Science, Amsterdam.

[2] Kornreich, P. (2017) Reaction Mechanics of Point Objects. *Journal of Applied Mathematics and Physics*, **5**, 137-152. https://doi.org/10.4236/jamp.2017.51015

[3] Lavanayal, N., Rao Sampath, P. and Redd Promod, M. (2014) Design and Analysis of a Suspension Coil Spring for Automotive Vehicle. *Journal of Engineering Research and Applications*, **4**, 151-157.

[4] Goldstein, H. (1981) Classical Mechanics. 2nd Edition, Addison-Wesley Publishing Company, Boston.