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Quantum Local Causality in Non-Metric Space

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Abstract

The possibility that quantum mechanics is founded on non-metric space has been previously introduced as an alternative consequence of Bell inequalities violation. This work develops the concept further by an analysis of the iconic Heisenberg gedanken experiment. No lower bound is found in the gedanken uncertainly relation for a non-metric spatial background. This result has the fundamental consequence that the quantum particle trajectory is retained in non-metric space and time. Assignment of measurement number-values to unmeasured incompatible variables is found to be mathematically incorrect. The current disagreement between different formulations of the empirically verified error-disturbance relations can be explained as a consequence of the structure of space. Quantum contextuality can likewise be explained geometrically. An alternative analysis of the extended EPR perfect anti-correlation configuration is given. The consensus that local causality is the sole assumption is found to be incorrect. There is also the additional assumption of orientation independence. Inequalities violation does not therefore mandate rejection of local causality. Violation of the assumption of orientation independence implies rejection of metric, non-contextual variables algebraically representing physical quantities.

Keywords

Quantum Foundations, Heisenberg Gedanken Experiments, Error-Disturbance Relations, Quantum Locality, Non-Metric Space

1. Introduction

Bell's theory is widely considered to be the critical empirical testing ground on the foundation of quantum mechanics (QM) [1] [2]. Experimental violation of testable inequalities can be interpreted to require rejection of local causality as a fact of nature. Even the meaning of realism is being questioned [3] [4] [5] [6] [7].

There are nevertheless dissident views. Although now resolved, questions persisted on experimental design and possible loopholes. Also, additional assumptions underpinning derivation of Bell inequalities have been identified [8] [9] [10] [11] [12]. These focus not on the question of locality but on the mathematical characteristics of physical quantities. The investigation being developed here is just one such possibility. Quantum non-locality nevertheless remains the consensus option.

While experimental violation of Bell inequalities has been well established, only recent experiments are claimed to be loophole-free [13] [14] [15]. What have been termed the locality loophole and fair-sampling assumptions have been addressed in improved experimental design. Bell inequalities violation can now be said to be experimentally verified beyond reasonable doubt.

This alternative investigation is a consequence of the assumption in Bell inequalities that physical quantities are represented by metric variable-type [10]. This seemingly innocuous assumption has its origins in classical theories where physical quantities are mathematically represented by such variables. Metric variables have two fundamental properties: their algebraic structure is a field, and hence can be represented by number-values, and have measurability as a geometrical-mathematical property.

Algebraic representation of classical quantities by metric variables is a result of the underlying geometrical architecture. Hilbert showed that combinations of Euclidean geometrical points obey a field algebraic structure [16]. This algebra of points is isomorphic with the field of natural numbers. Isomorphism allows points in Euclidean geometry to be mathematically represented by number-values: as they obey the same rules. To the extent that physical points in space and time can be represented by points in Euclidean geometry then they likewise can be associated with number-values.

Basic geometrical principles also apply in defining measurability as a mathematical, as distinct from an operational property. Since distance between two geometrical points is invariant, measurability of distance is definable within the axiomatic structure of Euclidean geometry [17]. Variables in a mathematical theory with a Euclidean geometrical architecture can then be metric. That is, assignment of measurement number-values for classical quantities is a consequence of metric space.

This small discourse on foundations of geometry shows that there is a nexus between variable-type and underlying geometry. If inequalities violation is attributed to an incorrect algebraic representation of quantities alternative options become available—including the possibility that quantum phenomena are founded on a non-metric geometry. The geometrical modification being proposed requires space to be non-metric described by Projective geometry while time remains metric [11] [18] [19]. Under special case invariance conditions Euclidean geometry emerges from Projective.

For events on the space-time dual geometry the axiom of order cannot be consistently defined [11]. Consequently, the ensuring algebra of points is a ring *i.e.* the multiplication operation is non-commutative. Following Hilbert's isomorphic reasoning, points on the dual geometry can only be mathematically represented by algebraic entities which also form a ring. A direct connection with the mathematical formulation of QM is possible. Dirac q-numbers, which are subject to ring algebra and are the basis of Dirac's quantum formulation, are isomorphic with points on the dual geometry. Quantum mechanics can thereby be understood to be based on a spatial-temporal platform.

It is worthwhile to briefly outline the geometrical principles of quantum metrics in projective space [11]. While projective transformations are in general non-metric, isometric transformations, which are a subset, do preserve Euclidean invariance. Measurability can then be defined geometrically under special case conditions for subsets of points subject to isometric invariance [17] [18] [19]. Due to the symmetry properties of QM the Schrodinger equation, and hence the Schrodinger wave function in position space, is subject to isometric invariance. The wave function is thereby embedded in a Euclidean subspace. Quantum metrics become a characteristic of the geometrical properties of the wave function. Position and momentum space are not simultaneous representations of the wave function, and so, not simultaneously isometric invariant. Hence, measurability is not a simultaneous property of the two physical quantities.

Non-metric space has the consequence that physical observables are simultaneously in multiple contexts with differing geometrical properties, and so, differing algebraic representation. Quantum contextuality can thereby be understood geometrically.

Suggesting that the Schrodinger equation is sourced in a non-metric space may seem rather exotic. Nevertheless, as an example of a Sturm-Liouville equation it can be reconstructed by the Schwarzian from a specifically defined parameterized curve in projective geometry. If it is supposed that space is non-metric, there is at least a mathematical connection between such space and the time independent Schrodinger equation.

Obviously, the structure of classical space is being questioned. Einstein-Podolsky-Rosen (EPR)testing experiments, together with the various delayed-choice configurations, are at present the empirical frame of reference on quantum foundations [20] [21] [22]. If these experiments are to be taken seriously, and no simpler explanations are forthcoming, some kind of departure from classical understanding is the obvious option in seeking alternative explanations. Superluminal influences or an irreducible randomness are the current benchmark.

This paper is as follows. Analysis of the Heisenberg gedanken experiments is presented. The assumption of metric space is fundamental to the ionic constructions. Rejection of this basic assumption enables derivation of position-momentum indeterminacy relation in a non-metric space. No lower bound is found. This result has implications for the ensemble error-disturbance relations. Naive counter factuality, *i.e.* assignment of measurement number-values to unmeasured incompatible variables, is found to be invalid. An alternative analysis of Bell's extended EPR perfect anti-correlation configuration is presented. The claim that local causality is the sole assumption, which defines all other possible assumptions, is found to be incorrect.

2. Heisenberg Gedanken Experiments

The iconic Heisenberg microscope is considered to be the definitive gedanken experiment which encapsulates the fundamental physical basis of QM. Although its applicability has been questioned, the celebrated construct does provide a conceptual framework from which to explore quantum foundations. While Bohr and Heisenberg where in agreement that the uncertainty relation expressed a fundamental characteristic of physical reality, there was disagreement on the origins of indeterminacy. Scholars argue that contrary to textbook misrepresentations, the nuances disengagement between the Masters was never resolved. Heisenberg originally was of the view that intrinsic to the act of measurement was a discontinuous change in momentum. For Bohr however, indeterminacy was sourced in wave-particle complementarity. Heisenberg, at least initially, recalibrated to the Bohr position [23].

Only Heisenberg's interpretations of the gedanken experiments will be considered, and only as a point of reference. Heisenberg's reasoning is somewhat indirect. Following Bohr's earlier definition of complementarity, the empirically based particle picture is assumed to critique the limits of applicability of the wave picture. Interchanging pictures places limits on the applicability of both. The ensuing mathematical relation defines the now famous inverse proportionality between position and momentum indeterminacies, resulting from a positive lower bound of the order of Plank's constant. Where indeterminacies tend to zero the relation exhibits a singularity. From this simple mathematical relation very far reaching consequences are extrapolated.

Firstly, the particle path as "a sequence of points in space" is rejected. Secondly, two fundamental aspects of a physical theory—that natural phenomena be explained by exact laws, and by relations between objects existing in space and time—are mutually exclusive, and so cannot be both fulfilled by quantum theory [23]. Thirdly, ensemble uncertainty relations motivated by the gedanken experiments are obtained. These are the initial Kennard relation (preparation uncertainty), and the more recent error-disturbance relations (measurement uncertainty). Controversy exists about the appropriate formulation of the latter [24]. The ensemble relations are open to experimental verification.

The term "uncertainty principle" is commonly understood to refer to both the gedanken and also ensemble relations. These are however fundamentally different. As Ballentine emphatically pointed out about the Kennard relation, but also applicable to the other ensemble expressions, these relations use standard statistical definitions as measures of dispersion *i.e.* standard deviations, mean square error etc. to obtain relations between different distributions [25]. As statistical analysis they do not necessarily apply to the individual case. This is a principle of statistics it is not an interpretation. Imposing ensemble conclusions to the individual case introduces an irreducible randomness, which as Ballentine again points out, is the source of much of quantum perplexities.

The ensemble relations are obtained from the mathematical apparatus of standard QM which is not contradicted by the geometrical explanation being proposed.

Irrespective of the source of indeterminacy, whether Heisenberg's act-of-observation-disturbance or Bohr complementarity, the gedanken configurations all assume a metric background Euclidean space. It therefore becomes a possibility that the gedanken relation can also be interpreted as defining limits on this basic geometrical assumption.

Position indeterminacy is defined in the gedanken experiments as the distance between extremities. For example, with the microscope experiment it is the distance between endpoints of an image, whereas with the interference configurations it is the width of the slit. Momentum indeterminacy is likewise the difference between points, usually the initial and final values. Essentially then, Heisenberg defines indeterminacy as a length or range.

For non-metric space length is not an invariant. However, metric characteristics can be defined at least in generic form [19]. Briefly, the distance between two points P and Q denoted symbolically by (PQ), possesses the following properties:

$$(PP) = 0 \tag{1a}$$

$$(PQ) = -(QP) \tag{1b}$$

$$(PQ) + (QR) + (RP) = 0 \tag{1c}$$

where the distance between two coinciding points is zero, reversing the order changes the sign, and the sum of distances between three collinear points is given by relation (1c).

For M_1, M_2 arbitrarily chosen distinct points, the projective distance between P and Q is given by:

$$(PQ) = k \ln\{M_1, M_2, P, Q\} \tag{2}$$

This expression encompasses the three basic conditions, where $\{M_1, M_2, P, Q\}$ denotes the cross ratio and k is a scaling constant. Choosing the points of reference: A_0 the origin, A_1 the point at infinity and I the identity point,

$$\{A_0, A_1, Q, I\} = q \tag{3}$$

where q is the non-homogeneous coordinate of Q. The projective distance between Q_1 and Q_2 is then:

$$(Q_1, Q_2) = k[\ln q_2 - \ln q_1] \tag{4}$$

Returning to examining the gedanken experiments on a non-metric space, there are three possibilities: position space is metric and momentum space non-metric, the reverse, and both spaces are non-metric. The first two definitions would be the basis architecture of the Schrodinger equation and wave function. However, it is the third definition which would be the geometrical platform to describe the quantum path where position is a function of time.

For the Schrodinger configurations indeterminacies can be defined as:

$$\Delta x = \alpha\left(x_2 - x_1\right) \tag{5a}$$

$$\Delta p = \beta\left(\ln p_2 - \ln p_1\right) \tag{5b}$$

Alternatively,

$$\Delta q = \alpha\left(\ln q_2 - \ln q_1\right) \tag{6a}$$

$$\Delta p^m = \beta\left(p_2^m - p_1^m\right) \tag{6b}$$

where α, β are dimensionality constants for position and momentum. Relation (5a) refers to position indeterminacy in metric *i.e.* Euclidean defined space; while (5b) refers to momentum indeterminacy in non-metric *i.e.* projective defined space. Definitions are reversed in relations (6a) and (6b). Configuration (5a), (5b) would accommodate an error in the measurement of position with the resulting disturbance in momentum, using Heisenberg definitions. For this configuration the product of indeterminacies would be:

$$\Delta x \otimes \Delta p = \frac{1}{2}\left[\Delta x \Delta p + \Delta p \Delta x\right] \tag{7a}$$

$$\Delta x \otimes \Delta p = \frac{1}{2}\alpha\beta\left\{\left(x_2 - x_1\right)\left(\ln p_2 - \ln p_1\right) + \left(\ln p_2 - \ln p_1\right)\left(x_2 - x_1\right)\right\} \tag{7b}$$

where, the standard QM definition of half the sums of non-commutating products is used.

These expressions should not be understood within the mindset of standard QM. The variables do not represent operators in Hilbert space, at least not directly. Considering for example relations (5a) and (5b), x_i, p_i represent geometrical points. To the extent that such abstract constructions represent position in space, and instantaneous momentum, these geometrical points represent physical quantities in space and time in a manner analogous to Classical Mechanics. In relation to quantum theory the quantities will then refer to "*hidden variables*", using Bell-EPR terminology. As such they are not to be confused with, for example, the momentum differential operator as generator of translation in Euclidean position space, defined from the isometric symmetry of the Schrodinger equation and wave function. In relation (7), x as a c-number and p as a Dirac q-number commute *i.e.* $\left[x, p\right] = 0$; which differs from x, p defined in Euclidean position space where the variables do not commute.

Following some straightforward algebra gives:

$$\Delta x \otimes \Delta p = \alpha\beta\left\{\left(\ln p_2 - \ln p_1\right)\left(x_2 - x_1\right)\right\} \tag{8}$$

Significantly, the RHS of relation (8) does not have a lower bound, tending instead to zero as indeterminacies tend to zero. Unlike the gedanken relation there is no singularity. Further, while momentum is non-metric, a measurement of the position error can be zero. The alternative definition, *i.e.* (6a), (6b), leads to a similar conclusion but while position is non-metric, measurement of momentum disturbance can be zero.

For the third geometrical configuration the indeterminacies are:

$$\Delta q = \alpha\left(\ln q_2 - \ln q_1\right) \tag{9a}$$

$$\Delta p = \beta\left(\ln p_2 - \ln p_1\right) \tag{9b}$$

Here the coordinates can more generally refer to the particle at position *Q*, at time *t* with momentum *P*; Δq defines position indeterminacy around *Q*, and correspondingly for momentum indeterminacy. As Dirac q-numbers *p*, *q* of relation (9) are non-commuting, the appropriate non-commutation relation is [26]:

$$\left[f(q), g(p)\right] = (-1)\sum_{k=1}^{\infty}(-c)^k f^{(k)}(q)\, g(p)^{(k)} \big/ k! \tag{10a}$$

$$\ln q \ln p - \ln p \ln q = (-1)\sum_{n=1}^{\infty}\frac{(-1)^n (n-1)!}{nq^n p^n} \quad \text{where } c = \left(\frac{i\hbar}{\alpha\beta}\right) \tag{10b}$$

Identifying the natural logarithms function in the general commutation relation for functions of operators (10a) gives relation (10b).

The product of indeterminacies is:

$$\begin{aligned}&\Delta q \otimes \Delta p\\ &= \frac{1}{2}\alpha\beta\left\{\left(\ln q_2 - \ln q_1\right)\left(\ln p_2 - \ln p_1\right) + \left(\ln p_2 - \ln p_1\right)\left(\left(\ln q_2 - \ln q_1\right)\right)\right\}\end{aligned} \tag{11}$$

Following the same algebra as the Schrodinger configurations leads to the same conclusion of a non-singularity. Note that the non-commutation relation, which is said to be the primal mathematical source of the uncertainty relations in standard QM, has been explicitly used in obtaining these expressions.

Assuming that the Heisenberg gedanken experiments are applicable (rejected by Ballentine), there are two possibilities for the product of indeterminacies: where both are metric *i.e.* share a common Euclidean space, the product is bounded by the order of Plank's constant; where one is metric and the other non-metric or where both are non-metric, the product is zero.

The validity of Heisenberg's conclusion is not in question. Rather, the domain of its applicability is restricted to Euclidean space, where a causal description of motion is not possible. This does not however, mean an irreducible randomness in the single particle position. It refers instead to an incomplete description of motion in metric space. Ontologically, there is no required departure from classical theory in the movement of a particle in non-metric space and time. The difference is in its geometrical-mathematical description.

Retaining the reality of a physical particle trajectory has been previously considered in relation to de Broglie-Bohm mechanics where both locality and causality are preserved [11].

While relations (5) and (6) are specifically formulated for indeterminacies, the underlying geometrics describe any simultaneous factual – counterfactual measurement of position and momentum. Since counter factuality features in EPR testing experiments, it is worthwhile exploring the concept in the context of these relations.

Factual position measurement is described by the metric relation (5a) where position can be assigned measurement number-values. Alternatively, a factual momentum measurement is in like manner described by the metric relation (6b). EPR motivated experiments involve simultaneity of the factual measurement of one of the physical quantities and a counterfactual measurement (not actually performed) of the incompatible partner. A possibility is the factual momentum measurement (6b) combined with the counterfactual measurement of position. By operational definition the outcome of the latter is its factual outcome if that measurement was actually performed.

Counterfactual position measurement then imposes the metric properties of relation (5a) onto the properties of relation (6a). However, position in the non-metric relation (6a) does not have such properties. The two relations are subject to different geometries and hence have different mathematical properties which then require different variable-type. Measurement number-values cannot be assigned to position in the configuration where it is the momentum which is measured. That incompatible variables require different variable-types is significant in the analysis of EPR perfect anti-correlations configuration, which follows.

Quantum orthodoxy operationally defines measurement as an interaction between a classical apparatus and quantum system. Orthodoxy also introduces the notion of an intrusive observation as a feature of measurement. Quantum and classical measurability would then be radically different. For this proposal however, only the geometrical architecture of the classical/quantum interaction is relevant. A classical apparatus performs a measurement in the context of metric space. Operationally, it will always register a metric outcome for the measurement. This is consistent with classical theory whose mathematical basis is founded on a metric geometry. That is, the geometrical context of measurement, *defined operationally*, is the same as the geometrical context of the theory, *defined axiomatically.* Furthermore, the geometrical context of measurement and the classical system being measured are the same. This symmetry must exist if the theory is to predict the measured true value.

The geometrics of the gedanken relations are more complex since the contexts of measurement and quantum system can differ. However, the symmetry of measurement, theory and physical system remains a necessary condition for the predicted value to be the true value.

Considering for example position, the same reasoning applies also to momentum, relation (5a) and (6a) can refer to the same physical points, which can simultaneously be in different geometrical contexts. With relation (5a), the points are contextually in a metric subspace of the position space wave function where measurability is axiomatically definable. For relation (6a), the same points are in a different context of a more general non-metric space where only the generics of measurement are axiomatically defined. Essentially, simultaneous multiple contexts are possible due to the special-case relation between Euclidean and Projective geometry.

Supposing an in principle measurement of quantum position:

1) The apparatus will register a metric value which may or may not be the true value.

2) The system is simultaneously in metric and non-metric contexts.

3) Quantum theory has two basis—position and momentum space. For position space, the hidden variables relation (5a) predicts the registered value as the true value. Whereas, for momentum space the corresponding hidden variable relation (6a) does not predict the registered measurement value which cannot then be the true value. However, a measurement of momentum with this configuration will register the momentum true value.

The true value can be defined to be the metric value in the metric context of the system. However, this is only definitional. Both metric and non-metric contexts of the system are equally fundamental, differing only in their invariance properties. Measurement number-values, as a consequence of the particular invariance characteristics of metric contextuality, cannot (at least from geometrical reasoning) be considered more fundamental than any other algebraic representation.

Operationally, position and momentum measurement are interchanged by a switch in the apparatus from the position to momentum “meter”. However, if an intrusive act-of-observation is not introduced, the reality of the particle position and momentum must pre-exist measurement. Mathematically, the same reality has two descriptions. Since only one of which assigns numeracy, pre-existing (to measurement) number-values cannot be fundamental. Furthermore, since measurability is a characteristic of only one description, measurability cannot be assumed to be the necessary criteria for realism.

Obviously, in this discussion measurement number-values are not fundamental, being instead defined contextually as a consequence of geometrical invariance properties. This view is contrary to instrumentalism where measurability is imposed *philosophically* as an a priori fact of nature. Although it may seem otherwise, Classical theory however does not require instrumentalism, since measurement number-values are a consequence of its foundational geometry.

Multiple contexts do not infer an alternative quantum reality. Rather, classical theory describes a restricted domain where the classical system is in a single metric context.

Ensemble error-disturbance relations are motivated by Heisenberg’s intuition that a measurement of position disturbs the momentum by an amount bounded by the inverse proportionality relation [24]. Two verifiable formulations have been developed. These, along with other significant ensemble relations, are:

$$\text{Kennard}: \sigma(q)\sigma(p) \ge \hbar/2 \tag{12a}$$

$$\text{Heisenberg}: \varepsilon(q)\eta(p) \ge \hbar/2 \tag{12b}$$

$$\text{Ozawa}: \varepsilon(q)\eta(p) + \varepsilon(q)\sigma(p) + \sigma(q)\eta(p) \ge \hbar/2 \tag{12c}$$

$$\text{BLW}: (\Delta Q)(\Delta P) \ge \hbar/2 \tag{12d}$$

The critical difference is that for the Ozawa relation the product of position error and momentum disturbance does not have a lower bound, contrary to the "Heisenberg" error-disturbance relation. This is clearly not the case with the Busch, Lahti and Werner (BLW) relation which reproduces the usual mathematical form. Currently, there is passionate debate on the appropriateness of these relations. Discussion focuses on the definitions of error and disturbance [24]. Nevertheless, both forms are rigorously obtained from standard QM and both are experimentally verified.

There is some controversy whether relation (12b) should be correctly referred to as "Heisenberg". BLW asserts that Heisenberg's motivation was to give an intuitive heuristic understanding of why atomic orbits are unobserved. Nevertheless, Ozawa gives Heisenberg's reasoning in obtaining the relation from the Kennard relation. Appleby, and then also Ozawa, have shown this relation is not in general valid.

This perspective is not unsympathetic to BLW's view. Heisenberg's deepest concern was to understand and explain the departure of quantum from classical mechanics at the foundations, for which the γ-ray microscope is only a heuristic tool to explore fundamental assumptions. Heisenberg's conclusion, as is well known, was to reject that an object moves from one position to another by a sequence of continuous points in space and time.

With the ensemble error-disturbance relations there are conceptual difficulties in defining true or precise values as distinct from approximate or measured values. There are also statistical questions on appropriate definitions of measures of dispersions. Definitional complexities notwithstanding, a critical difference between the two verified relations is also that the different formulations have different underlying geometrical architecture.

For BLW the experimental realization is in three stages. Firstly, the precise position and precise momentum distributions are obtained by separate measurements *i.e.* not by joint measurement. A joint probability distribution of the approximate position and approximate momentum is then obtained by a joint commuting measurement. The marginal approximate position and marginal approximate momentum distributions are obtained from their joint distribution. Using the Wassenstein-2 measure of distance between distributions, the distance between the precision and approximate distributions is then obtained leading to the BLW relation.

All three experimentally determined distributions are simultaneously in a background Euclidean space, as would be both marginal distributions. In which case, the BLW relation is defined on a background Euclidean (metric) space, as is assumed in the Heisenberg gedanken experiment. The Kennard, and "Heisenberg" error-disturbance relation from which it is obtained, are likewise based on the same background metrics.

Realization of the Ozawa relation is geometrically different (see Lund and Wiseman). Position error and momentum disturbance are measured jointly on the single particle. In principle, position error measurement is governed by relation (5a), permitting zero error. Correspondingly, the momentum disturbance measurement is governed by relation (6b), again permitting zero disturbance. Critically, the position/momentum measurements are made in metric/non-metric contexts. Since by relation (5a) and (6b) position and momentum differences can be individually zero that their expectation values in the Ozawa relation can likewise be zero is not unexpected. Appleby has also shown that for error-error relations the "Heisenberg" relation is specifically violated for joint measurements.

This alternative perspective raises the possibility that at issue with the error-disturbance relations is also the nature of space, separate from differences in error definitions.

3. Bell-EPR Perfect Anti-Correlation Configuration

Recent papers have again raised the persistent question of what assumptions underpin Bell inequalities [27]. Stephen Boughn attributes violation to a broadly defined classicality rather than locality. In reply, Laudisa reaffirms the consensus that locality, as a fact of nature, is the single primitive assumption [28].

Zukowski and Brukner also question locality being the sole assumption, in particular identifying counterfactual definiteness as an addition assumption. Their work however remains within the framework of quantum orthodoxy concluding that "*individual events may have spontaneous, acasual nature*". Obviously, this is contrary to Bell's motivation for developing the inequalities.

The following discussion follows closely the work of Norsen and more recent clarification of Tumulka [29] [30] [31] [32]. Firstly, it is important to stress that philosophically this presentation is aligned with rejecting Bohr-Heisenberg orthodoxy. Bell's elegant formulation of local causality is not questioned, and unlike the consensus, its rejection is not advocated. Furthermore, the view that Bell inequalities are reality-neutral in their assumptions, beyond Norsen's metaphysical realism, is accepted [31] [32]. At issue is neither locality not realism but rather the mathematical treatment of physical quantities, most particularly that of incompatible variables. As shown for position-momentum, incompatible variables are individually different in the algebraic representation of incompatible observables.

Bell's reasoning, which is also used by Norsen, is said to be a two-stage argument. The fundamental principle of the first-stage analysis is an operational conjunction of locality as *an assumption* together with EPR perfect anti-correlations as *empirically verified* outcomes. That locality is the sole assumption is said to be established by an extended EPR analysis which is followed by derivation of the inequality [29] [30]. A source of confusion may be that the latter does rely on additional assumptions which may be characterized as classicality [30]. Before re-examining Bell's first-stage argument it is necessary to clarify some mathematical preliminaries placing the question of underlying assumptions in a broader context.

The mathematical description of the standard particle-pair experimental configuration is extended to include an additional variable representing a second simultaneous outcome/apparatus-setting at each location. The motivation for this extension is to focus on the presence of incompatible variables. Doing so emphasizes the complication that not all variables represent actual measurement outcomes. Incompatible variables are crucial if Bell inequalities are to describe quantum experiments. Bell inequalities cannot be formulated exclusively on observable phenomena [29].

Accordingly, an extended joint probability is defined,

$$P\left(A,A',B,B' \mid a,a',b,b',\lambda\right)$$

where λ again refers to a complete description of the particle-pair state, with upper and lower case letters representing outcomes and apparatus settings respectively. Basic rules of probability give:

$$\begin{aligned}&P\left(A,A',B,B' \mid a,a',b,b',\lambda\right)\\&=P\left(A,A' \mid B,B',a,a',b,b',\lambda\right)P\left(B,B' \mid a,a',b,b',\lambda\right)\end{aligned} \tag{13a}$$

$$=P\left(A,A' \mid a,a',\lambda\right)P\left(B,B' \mid b,b',\lambda\right)\ \text{(by locality)} \tag{13b}$$

$$=P\left(A \mid A',a,a',\lambda\right)P\left(B \mid B',b,b',\lambda\right)P\left(A' \mid a,a',\lambda\right)P\left(B' \mid b,b',\lambda\right) \tag{13c}$$

Introducing the assumption of orientation independence *i.e.* the outcome/apparatus-setting at either location is independent of the simultaneous outcome/apparatus-setting at the *same* location gives:

$$\begin{aligned}&P\left(A,A',B,B' \mid a,a',b,b',\lambda\right)\\&=P\left(A \mid a,\lambda\right)P\left(B \mid b,\lambda\right)P\left(A' \mid a',\lambda\right)P\left(B' \mid b',\lambda\right)\end{aligned} \tag{14}$$

The joint probability of a single measurement at each location gives four combinations such as:

$$P\left(A,B \mid a,b,\lambda\right)=P\left(A \mid a,\lambda\right)P\left(B \mid b,\lambda\right) \tag{15}$$

These combinations are just the standard definitions for Bell locality used in derivations of the inequalities.

While the generality of Bell's probability definitions is under serious challenge elsewhere, this analysis remains consistent with Bell [8] [33]. For these definitions locality and orientation independence are pre-existing, independent assumptions. Their violations have very different physical implications. Violation of locality requires superluminal influences between two different particles at space-separated locations, implying some kind of action-at-a-distance, inferring non-locality as a fact of nature. For orientation independence, violation implies influences between different orientations for the same particle at the same location. This assumption has implications for the algebraic representation of observables.

An objection can be raised that the extended joint probability is classical; referring to the simultaneous measurements, at different orientation, for the single particle. However, the probability expression can be modified to $P\left(A, A'_{a'}, B, B'_{b'} \mid a, a', b, b', \lambda\right)$ where the variable pair $\left(A, A'_{a'}\right)$ refers to incompatible variables, one actual and the other latent, only one of which is actually measured in keeping with quantum mechanics. Repeating the above steps leads to the same conclusions. The joint probability combinations, like relation (15), are now mixtures of measured and unmeasured *i.e.* actual and latent variables.

There is a further potential objection that the latent variables are mathematical constructs with no physical reality. While this reasoning has validity within orthodox quantum mechanics it is not the case within Bell's program. A resolution to this issue will be given shortly.

The basic functional relation of Bell inequalities has two forms. There is the "generic" form *i.e.* that which specifies all potential variables involved in the inequality:

$$f = f\left(\left\{\left[A_a, A'_{a'}\right] \text{or} \left[A'_a, A_{a'}\right]\right\}, \left\{\left[B_b, B'_{b'}\right] \text{or} \left[B'_b, B_{b'}\right]\right\}\right) \tag{16}$$

The prime and un-prime upper case letters refer to the latent and actual variables for incompatible pair-wise variables. This form is however non-computational by the restricted mathematical apparatus of Bell inequalities. The focal issue here is the mathematical property of each of the pair-wise variables *i.e.* $\left[A_a, A'_{a'}\right]$. While the actual variable *i.e.* A_a can be metric, or at least have the mathematical property of measurability while its numeracy is to be determined, the mathematical property of its latent partner *i.e.* $A'_{a'}$ is non-metric. Similar reasoning applies for the other location.

The computational functional relation where all variables are simultaneously mathematically defined is:

$$f = f\left(A_a, A_{a'}, B_b, B_{b'}\right) \tag{18}$$

Inequalities do vary in the number of outcomes and apparatus settings which may differ from those specified here. However, this reasoning applies generally. Transformation from the generic to computational form is achieved by the introduction of various conditions and assumptions. Subject to which the inferred mathematical properties of the latent variable are equivalent to the actual variable, such that:

$$\left[A_a, A'_{a'}\right] \equiv \left[A'_a, A_{a'}\right] \equiv \left[A_a, A_{a'}\right] \tag{19}$$

While all variables are algebraically metric, the computational form is nevertheless a mixture of measured and unmeasured variables.

At issue is the nature of the introduced conditions/assumptions. The question is said to be resolved by Bell's first-stage analysis. Experimental outcomes of EPR perfect anti-correlation configuration are said to define number-values, and most critically, all necessary additional assumptions are purported to be solely a consequence of local causality (or Bell Locality). However, this is not the case.

Following Norsen, the usual joint probability for the same orientation *i.e.* $\boldsymbol{n}_1$ at two locations, subject to Bell Locality, is [29]:

$$P(A,B \mid \boldsymbol{n}_1,\boldsymbol{n}_1,\lambda) = P(A \mid \boldsymbol{n}_1,\lambda)P(B \mid \boldsymbol{n}_1,\lambda) \tag{20}$$

The variables A, B refer to actual measurement outcomes taking values ±1. This outcome condition then defines the numeracy of Bell non-incompatible variables, which, together with the property of measurability, means such variables are metric. Quantum complications of incompatibility do not apply along the same orientation. Applying perfect anti-correlation outcome conditions leads to shorthand notations for conditional probability functions [29]:

$$A(\boldsymbol{n}_1,\lambda) = \pm 1 \text{ and } B(\boldsymbol{n}_1,\lambda) = \mp 1 \tag{21}$$

Outcomes are empirically observed facts with Bell Locality and the encoded properties of λ the only determining factors. Clearly there are no additional assumptions: not realism (beyond Norsen's metaphysical), variable type, non-contextuality etc.

Norsen extends the analysis to three orientations, *i.e.* $\boldsymbol{n}_1,\boldsymbol{n}_2,\boldsymbol{n}_3$ as required by some inequalities. For the purpose of this discussion two orientations are sufficient. For a specific outcome, a possible combination is:

$$\begin{gathered}\{A(\boldsymbol{n}_1,\lambda) = +1 \text{ and } B(\boldsymbol{n}_1,\lambda) = -1\} \\ \text{and / or } \{A(\boldsymbol{n}_2,\lambda) = +1 \text{ and } B(\boldsymbol{n}_2,\lambda) = -1\}\end{gathered} \tag{22}$$

Distinguishing between the "and/or" options is critical. The "or" option refers to orientation-by-orientation (called parallel by Bell) while the "and" refers to simultaneous orientations, which are not experimentally realizable. Quantum incompatibility features in the latter, bringing with it the complications of defining simultaneous mathematical properties to two variables only one of which is actually measured. Bell inequalities involve non-parallel configurations.

However, Bell's first-stage operational analysis does not extend to such orientations. The operational analysis does not therefore define a relation between different orientations simultaneously. Additional justifications are required to determine the mathematical properties of the variables involved. That is, additional assumptions are required to verify relation (22) for the non-parallel case.

The corresponding extended functions for simultaneous orientations for a particular outcome at one location are:

$$A_1(\boldsymbol{n}_1, A_2', \lambda) = +1 \text{ and } A_2'(\boldsymbol{n}_1, A_1, \lambda) = q \quad \text{for orientation } \boldsymbol{n}_1 \tag{23a}$$

$$A_2(\boldsymbol{n}_2, A_1', \lambda) = +1 \text{ and } A_1'(\boldsymbol{n}_2, A_2, \lambda) = q \quad \text{for orientation } \boldsymbol{n}_2 \tag{23b}$$

Their mathematical characteristics follow from the indeterminacies relations (5) and (6) discussed above. The un-primed upper case letters refer to the actual variable along a given measured orientation together with the primed upper case referring to the corresponding incompatible latent variable along the non-measured orientation. Similar expressions follow for the second location and for different outcome combinations. While the actual variable along either orientation is metric, the corresponding incompatible latent variable is non-metric, denoted by q. Conservation of spin is still maintained for the latent variables *i.e.* $q_A + q_B = 0$. Dirac q-numbers still obey basic rules of addition and subtraction. The obvious critical question is then how numeracy can assigned to the non-measured variables [29] [32].

Applying the operational analysis of extended EPR for functions (23a) and (23b) along orientation $\boldsymbol{n}_1$: suppose a measurement $A_1(\boldsymbol{n}_1, A_2', \lambda) = +1$ with corresponding measurement $B_1(\boldsymbol{n}_1, B_2', \lambda) = -1$ and corresponding incompatible latent variable $B_2'(\boldsymbol{n}_1, B_1, \lambda) = q$. A switch in apparatus-setting at B along $\boldsymbol{n}_2$ will give $B_2(\boldsymbol{n}_2, B_1', \lambda) = \pm 1$ with $B_1'(\boldsymbol{n}_2, B_2, \lambda) = q$. A settings switch interchanges actual/latent variables. Notice that the non-measured latent variable B_1' when the apparatus-setting is along $\boldsymbol{n}_2$ has different algebraic representation (subject to different algebraic rules) to the actual measurement variable B_1 when the apparatus-setting is along $\boldsymbol{n}_1$. Since the non-metric variable is numerically undefined it is not possible to obtain Bell inequalities for the extended functions.

Returning to the question of the realism of the latent variable, if the actual is reality-based so must the latent. For non-orthodox quantum mechanics a change in apparatus -settings does not change Norsen's metaphysical reality.

For the Bell-Norsen functions (22), following the same EPR operational analysis of a measurement $A(\boldsymbol{n}_1, \lambda) = +1$ and $B(\boldsymbol{n}_1, \lambda) = -1$ along orientation $\boldsymbol{n}_1$. For a switch in apparatus-setting at B along $\boldsymbol{n}_2$, on an orientation-by-orientation basis it is reasoned that [32]:

$$B_1'(\text{latent in } \boldsymbol{n}_2 \text{ orientation}) \equiv B_1(\text{actual in } \boldsymbol{n}_1 \text{ orientation}) = -1 \tag{24}$$

However, this would only be true under the assumption (additional to locality causality) of orientation independence *i.e.* measurement along one orientation does not affect other orientations at the same location. An important consequence is that the observables involved can be algebraically represented by metric variables simultaneously which is also an assumption of non-contextuality. The issue however, is not the veracity of this algebraic representation, but rather that it is not a consequence of local causality.

Alternatively, the mathematical equivalence of actual and latent variables can be established by imposing the realism criteria of pre-existing measurement number values. However, this would forfeit the realism neutrality of assumptions criteria by imposing a naive realism which Norsen argues is already redundant [31].

Norsen following Bell, introduces a more sophisticated reasoning for the locality alone argument [29]. According to this reasoning at issue is not the quantum prohibited simultaneous assignment of number values. Rather, the question becomes whether "*a Bell Local theory predicting perfect anti-correlations posit this detail structure in the state description*? *The answer is unambiguously yes.*"

Norsen clarifies further that "*all theories respecting a certain locality condition* (*Bell Locality*) *must*, *in order to successfully reproduce a certain class of empirically well-confirmed correlations*, *posit that the outcomes of all possible spin-component measurements to be made on the particles are encoded in the pre-measurement state of the particles*, *such that the outcome in one wing of the experiment is determined once the state of the particle pair and the orientation of the nearby apparatus are specified. In rough terms*, *the particles must carry* '*instructions sets*' *which predetermine the outcome of spin measurements. Since such* '*instruction sets*' *go beyond what is attributed to the states by orthodox quantum theory*, *the kind of theory we have argued for may be termed a hidden-variable*." There is no fundamental disagreement with this statement.

In essence, the candidate hidden-variable theory must be subject to Bell Locality and reproduce EPR perfect anti-correlations outcomes. Norsen further reasons that since a hidden-variable theory of this type cannot account for the empirically tested correlations of Bell-type experiments the theory is not empirically viable. Norsen thereby concludes that "*Bell Locality cannot be maintained*". However, this is not the case.

The candidate hidden-variable theory must also meet the third criteria of being a mathematical theory. Bringing the issue (central to this presentation) of identifying the appropriate algebraic representation of the physical quantities involved.

There are now two candidate theories. That of relations (23a) and (23b) is Bell Local, mathematically represents physical quantities by different hidden variable-types *i.e.* metric and non-metric, and reproduces EPR perfect anti-correlations outcomes. However, it does not define Bell inequalities. Since the inequalities are not empirically verified this is not an issue. The Bell-Norsen hidden-variable theory, that of relation (22), differs in that it represents all physical quantities by metric hidden variables, and does define the empirically invalid Bell inequalities.

The critical issue is that there are two candidate Bell Local hidden-variable theories. Simultaneous metric variables (characteristic of only one of the candidate theories) is therefore not a consequence of locality but must be introduced as an additional assumption.

The question arises whether Bell's candidate theory actually reproduces EPR perfect anti-correlation experiments, as claimed. While by relation (22) simultaneous metric variables are clearly correct in predicting outcomes of measureable observables, they also predict values for observables which cannot be measured. Classically, this is not a problem. However, for the candidate hidden-variable theory to correctly predict quantum EPR experiments it should also predict the conditions under which observables are non-measurable. On first impression this may seem unnecessary. However, a hidden-variable theory explaining standard QM must also explain the non-classical measurability of quantum theory. Since Bell's candidate hidden-variable theory fails to do so (measurability is always predicted) it fails to predict quantum experiments; both Bell-type and EPR.

Norsen gives a detailed analysis of the Clauser, Horne, Shimony and Holt (CHSH) inequality to examine the claim that the inequality is based solely on the assumption of locality [30]. A provisional conclusion is given that violation can be attributed to either locality or non-contextuality. However, Norsen reintroduces the Bell argument that the critical question is not directly about measurement outcomes but rather about theories and predictions. In which case, it is reasoned that it becomes meaningful to "*simultaneously talk of* $A(\boldsymbol{n}_1,\lambda)$ *and* $A(\boldsymbol{n}_2,\lambda)$" as simply the values the theory yields for measurement outcomes. Hence, it would then emerge the CHSH inequality follows from locality alone.

However, the inequality is defined from the four combinations of conditional probability functions of relation (15). As shown, locality and orientation independence are pre-existing, independent assumptions. The functions $A(\boldsymbol{n}_1,\lambda)$ and $A(\boldsymbol{n}_2,\lambda)$ are simultaneous metric variables only with Bell's candidate local theory of relations (22). As with Bell's first-stage analysis, violation can also be attributed to the wrong Bell Local theory.

It can then be concluded that violation of Bell inequalities prohibits *locally causalmetric* hidden variable theories.

Since quantum observables are algebraically represented by non-metric variables, it is not unexpected that possible hidden variables would likewise be non-metric. This conclusion may thereby seem unexceptionable. However, the basic purpose of the inequalities is to answer EPR's original foundational question on the completeness of quantum theory. Violation answers EPR by affirming that a *locally causal* hidden variables theory is possible.

Recently Cabello introduced a new line of investigation where local contextuality is identified as a priori to quantum non-locality in explaining inequalities violation. An experimental configuration has been defined and subsequently realised [34] [35]. Empirical results are in agreement with QM while violating non-contextual local hidden variables inequalities. Concerns about experimental design have nevertheless again been expressed [36].

Cabello inequalities associate violation with the properties of the physical quantities involved *i.e.* contextuality, rather than, primarily, the introduction of non-local influences. Indeed, in the mathematical expression of the inequalities it is the assumption of non-contextuality which leads to contradiction not that of locality. The term containing any potential action-at-a-distance influence is explained by local hidden variables.

The assumption of orientation independence in Bell inequalities is also an assumption of non-contextuality. Cabello inequalities violation does not contradict the alternative possibility being explored here.

Algebraic representation of physical quantities is also an issue with EPR-Steering and Hardy non-locality.

In summary, inequalities violation can be attributed to Bell Locality. Alternatively, it can be attributed to classicality, where observables are incorrectly represented by classical, non-contextual metric variables. A third possibility is that no significant inferences can be drawn because the inequalities are simply too restricted. Bell's theorem would then be a most rigorous theoretical-experimental and conceptual proof that quantum mechanics is not classical. On the other hand, there may be no need to go beyond orthodoxy accepting instead the irreducible random nature of events.

A prudent conclusion may be that it is important to keep an open mind to what is an open question. *Bell inequalities only establish possibilities.* The celebrated constructs are not sufficiently rigorous to be more definitive.

4. Conclusion

This proposal is an alternative to the mainstream on the question of quantum foundations. Neither quantum orthodoxy nor the consensus on inequalities violation, are advocated. No departure from traditional concepts of space, time and geometry is found to be necessary in explaining the foundations of QM. Further, the proposal is consistent with EPR-testing experiments, the conceptual Heisenberg gedanken experiments and gives an alternative explanation to account for the difference in ensemble error-disturbance relations. There is no ontological contradiction with other physical theories; a plague of orthodoxy. Non-classical quantum measurability is explained geometrically without the need to introduce an intrusive observation. Since locality can be assumed there is no contradiction with relativity.

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