Solutions of the Exponential Equation $y^y = x$

or $\frac{\ln x}{x} = \frac{\ln y}{y}$ and Fine Structure Constant

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Abstract

In this paper, we study the equation of the form of $y^y = x$ which can also be written as $\frac{\ln x}{x} = \frac{\ln y}{y}$. Apart from the trivial solution $x = y$, a non-trivial solution can be expressed in terms of Lambert $W$ function as

$$y = \frac{W \left( \frac{-\ln(x)}{x} \right)}{-\ln(x)}.$$

For $y > e$, the solutions of $x$ are in-between 1 and $e$. For integer $y$ values between 4 and 12, the solutions of $x$ written in base $y$ are in-between 1.333 and 1.389. The non-trivial solutions of the equations $y^{1/y^2} = x/y$ and $y^{1/y^3} = x/y^2$ written in base $y$ are exactly one and two orders higher respectively than the solutions of the equation $y^y = x$. If $y = 10$, the rounded nontrivial solutions for the three equations are 1.3713, 13.713 and 137.13, i.e. $10^{1.3713} \approx 137.13$. Further, $\ln(137.13)/137.13 \approx 0.2302$ and $W(-0.2302) \approx -2.302$. The value 137.13 is very close to the fine structure constant value of 137.036 within 0.1%.

Keywords

Exponential Equation, Lambert W Function, Fine Structure Constant
1. Introduction

Lambert \( W \) function is a transcendental function \([1]\) \([2]\) which has applications in many areas of science which include QCD renormalisation, Planck’s spectral distribution law, water movement in soil and population growth \([3]\)-\([8]\).

Considering the equation

\[
x^y = x
\]

\((1)\)

The Equation \((1)\) can be written as

\[
\log_y x = \frac{x}{y}
\]

\((2)\)

Converting the Equation \((2)\) in terms of natural log gives

\[
\frac{\ln x}{x} = \frac{\ln y}{y}
\]

\((3)\)

Equations \((1)-(3)\) have a trivial solution \(x = y\), but they also have a non-trivial solution.

**Figure 1** shows the plot of the function \(\frac{\ln x}{x}\). The plot indicates that, for any value of the function \(\frac{\ln x}{x}\) in the range of 1 to infinity, it has two different solutions of \(x\). *i.e.* for any value of \(y\) between 1 and infinity, a non-trivial solution of \(x\) can be found. The plot also indicates that, at \(y = e\), there is only one solution \(x = e\) and \(\frac{\ln x}{x} = 1/e = 0.3679\) (rounded). For any value of \(y\) between \(e\) and infinity, a solution for \(x\) can be found in-between 1 and \(e\).

![Figure 1](image-url)
The solution of Equations (1)-(3) can be written in terms of Lambert $W$ function [9],

$$y = \frac{W\left[ -\frac{\ln(x)}{x} \right]}{x}$$  \hspace{1cm} (4)

If $x = e$, $y = \frac{W\left[ -\frac{1}{e} \right]}{e}$ and according to Dence [2], $W\left[ -\frac{1}{e} \right] = -1$, hence $y = e$, which is the result obtained graphically and numerically.

Some variations of Equation (1) are:

$$y^{x/y^2} = x/y$$  \hspace{1cm} (5)

$$y^{x/y^3} = x/y^2$$  \hspace{1cm} (6)

Equation (5) can be written as

$$\frac{\ln x}{\ln y} = \frac{x}{y^2} + 1$$  \hspace{1cm} (7)

Equation (6) can be written as

$$\frac{\ln x}{\ln y} = \frac{x}{y^3} + 2$$  \hspace{1cm} (8)

Equation (5) and Equation (6) have trivial solutions of $x = y^2$ and $x = y^3$ respectively.

### 2. Non-Trivial Solutions

If $y = 10$ then Equation (1) becomes $10^{x/10} = x$ and $x = 1.3713$ (rounded) is the nontrivial solution, i.e. $10^{0.13713} = 1.3713$ and

$$\frac{\ln x}{x} = \frac{\ln y}{y} = 0.2302$$

If $y = 10$ then Equation (5) and Equation (6) become $10^{x/100} = x/10$ and $10^{x/1000} = x/100$ respectively and their solutions are 13.713 (rounded) and 137.13 (rounded) respectively. These solutions are exactly one and two orders larger than the solution of Equation (1).

Also if $x = 1.3713$ and $y = 10$, Equation (4) gives

$$W\left[ -\frac{\ln(1.3713)}{1.3713} \right] = 10\left[ -\frac{\ln(1.3713)}{1.3713} \right]$$

Hence $W(-0.2302) = -2.302$

For the range of integer $y$ values of 4 to 12, the non-trivial solutions for $x$ of Equations (1), (5) and (6) were obtained using iterative method. The solutions of $x$ are written in base 10 and in base $y$ (Table 1). Plots of $y$ vs $x$ with $x$ in base 10 and in base $y$ are shown in Figures 2-4 respectively.
3. Conclusions

The non-trivial solutions of Equations (1), (5) and (6) written in base $y$, differ exactly by one order. For $y$ values in the range of 4 to 12, the solutions of Equation (6) written in base $y$ are in the range of 133.33 to 138.99.

When $y = 10$, the rounded nontrivial solutions for Equation (1), Equation (5) and Equation (6) are 1.3713, 13.713 and 137.13, i.e. $10^{\log_{10} 1.3713} = 1.3713$, $\ln(1.3713)/1.3713 = 0.2302$ and $W(-0.2302) = -2.302$, i.e. for the argument values of 1.3713 and $-0.2302$, the function values are exactly one order higher. To our knowledge, these results were not reported before.

Table 1. Rounded non-trivial solutions for $x$ of Equations (1), (5) and (6) for $y$ values from 4 to 12 are written in base 10 and base $y$.

<table>
<thead>
<tr>
<th>$y$</th>
<th>Solutions of Equation (1)</th>
<th>Solutions of Equation (5)</th>
<th>Solutions of Equation (6)</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>In base 10</td>
<td>In base $y$</td>
<td>In base 10</td>
</tr>
<tr>
<td>12</td>
<td>1.3122</td>
<td>1.389</td>
<td>15.75</td>
</tr>
<tr>
<td>11</td>
<td>1.3389</td>
<td>1.380</td>
<td>14.73</td>
</tr>
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<td>10</td>
<td>1.3713</td>
<td>1.371</td>
<td>13.71</td>
</tr>
<tr>
<td>9</td>
<td>1.4114</td>
<td>1.363</td>
<td>12.70</td>
</tr>
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<td>8</td>
<td>1.4625</td>
<td>1.355</td>
<td>11.70</td>
</tr>
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<td>1.5301</td>
<td>1.350</td>
<td>10.71</td>
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<tr>
<td>4</td>
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<td>1.333..</td>
<td>8.00</td>
</tr>
</tbody>
</table>

Figure 2. Solutions of $x$ in base 10 and in base $y$ for Equation (1) for $y$ values of 1 to 13.
Figure 3. Solutions of $x$ in base 10 and in base $y$ for Equation (5) for $y$ values of 1 to 13.

Figure 4. Solutions of $x$ in base 10 and in base $y$ for Equation (6) for $y$ values of 1 to 13.

The trivial solutions of Equations ((1), (5) and (6)) can be written as 10, 100 and 1000 in base $y$ for any $y$ value.

The non-trivial solution for $x$ of Equation (6), 137.128857 is within 0.1% of the reciprocal value of the atomic fine structure constant $\alpha^{-1}$, 137.0359991.
4. Possible Connection to Fine Structure Constant

Allen suggested that \( m_e / M_p \sim 10\alpha^2 \) [10] however for the current values of \( m_e / M_p \) and \( \alpha \), the relationship is \( m_e / M_p = 10.227\alpha^2 \). Edward Teller suggested \( \ln T_0^{3/2} = \alpha^{-1} \), where \( T_0 \) is the age of the universe [11]. There could be a connection between Equations ((1) to (8)) and \( \alpha^{-1} \).

References


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