The New Infinite Sequence Complexion Solutions of a Kind of Nonlinear Evolutionary Equations

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Abstract

The method combining the function transformation with the auxiliary equation is presented and the new infinite sequence complexion solutions of a class of nonlinear evolutionary equations are constructed. Step one, according to two function transformations, a class of nonlinear evolutionary equations is changed into two kinds of ordinary differential equations. Step two, using the first integral of the ordinary differential equations, two first order nonlinear ordinary differential equations are obtained. Step three, using function transformation, two first order nonlinear ordinary differential equations are changed to the ordinary differential equation that could be integrated. Step four, the new solutions, Bäcklund transformation and the nonlinear superposition formula of solutions of the ordinary differential equation that could be integrated are applied to construct the new infinite sequence complexion solutions of a class of nonlinear evolutionary equations. These solutions are consisting of two-soliton solutions, two-period solutions and solutions composed of soliton solutions and period solutions.

Keywords
Nonlinear Evolutionary Equation, Function Transformation, Nonlinear Superposition Formula of Solutions, Complexion New Solutions

1. Introduction

In Refs. [1]-[5], some methods were applied to research the following two nonlinear coupling systems, and a finite number of one soliton solutions were obtained.

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\[ \sigma_{xx} = -\sigma + \sigma^3 + l \rho \sigma^2, \quad (1) \]
\[ \rho_{xx} = (h-l) \rho + m \rho^3 + l \rho \sigma^2, \quad (2) \]

where \( h, l \) and \( m \) are constants; \( \rho \) and \( \sigma \) are scalars.

\[ \sigma_a - c^2 \sigma_{xx} = a \sigma - b \sigma^3 - h \sigma \rho^3, \quad (3) \]
\[ \rho_a - c^2 \rho_{xx} = (a - 4 e) \rho - b \rho^3 - b \rho \sigma^2, \quad (4) \]

where \( a, b, c^2 \) and \( e \) are constants; \( \rho \) and \( \sigma \) are scalars.

In Refs. [6]-[9], the new two-soliton solutions of the following a class of nonlinear evolutionary equations are constructed.

\[ u_{xt} = u + \frac{1}{6} (u^3 + 3uv^2)_{xx}, \quad v_{xt} = v + \frac{1}{6} (v^3 + 3u^2v)_{xx}. \quad (5) \]
\[ u_{xt} = u + \frac{1}{6} (u^3 - 3uv^2)_{xx}, \quad v_{xt} = v + \frac{1}{6} (v^3 - 3u^2v)_{xx}. \quad (6) \]
\[ u_{xt} = u + \frac{1}{6} (u^3)_{xx}, \quad v_{xt} = v + \frac{1}{2} (u^2v)_{xx}. \quad (7) \]
\[ u_{xt} = u + \frac{1}{6} (u^3 + uv^2)_{xx}, \quad v_{xt} = v + \frac{1}{6} (v^3 + u^2v)_{xx}. \quad (8) \]
\[ u_{xt} = u + \frac{1}{6} (u^3)_{xx}, \quad v_{xt} = v + \frac{1}{6} (u^2v)_{xx}. \quad (9) \]

In this paper, the method combining the function transformation with the auxiliary equation is applied to construct the new infinite sequence complexion solutions of a class of nonlinear evolutionary Equation (10) by the following steps.

Step one, in Part 2.1, according to function transformation (11), a class of nonlinear evolutionary Equation (10) is changed into two kinds of ordinary differential Equations (13), (14). And then with the help of the function transformation, ordinary differential Equations (13), (14) can be changed to ordinary differential Equations (19), (20). And the first integrals (21), (22) of the two ordinary differential equations are obtained. In Part 2.2, according to function transformation (12), a class of nonlinear evolutionary Equations (10) is changed into two kinds of ordinary differential Equations (25), (26). And then with the help of the function transformation, ordinary differential Equations (25), (26) can be changed to ordinary differential Equations (31), (32). And the first integrals (33), (34) of the two ordinary differential equations are obtained.

Step two, substituting the first integrals (21), (22) separately into the first equation of ordinary differential Equations (19), (20) yields two first order nonlinear ordinary differential Equations (23) and (24). Substituting the first integrals (33), (34) separately into the first equation of ordinary differential Equations (31), (32) yields two first order nonlinear ordinary differential Equations (35) and (36).

Step three, using function transformation, two first order nonlinear ordinary differential Equations (23) and (24) (or (23) and (24)) can be changed to the ordinary differential equation that could be integrated [10]-[13].

Step four, the new solutions, Bäcklund transformation and the nonlinear superposition formula of solutions of the ordinary differential equation that could be integrated [10]-[12] are applied to construct the new infinite sequence complexion solutions consisting of the Riemann \( \theta \) function, the Jacobi elliptic function, the hyperbolic function, the trigonometric function and the rational function of a class of nonlinear evolutionary equations (10). These solutions are consisting of two-soliton solutions, two-period solutions and solutions composed of soliton solutions and period solutions.

\[ \begin{aligned}
\delta_i u_{xx} + \alpha_i u_{xx} + \alpha_i u_{xx} &= \alpha_i u + \alpha_i u^3 + \alpha_i x u^4 + \alpha_i \left( \gamma_i u^3 + \gamma_i u v^2 \right)_{xx}, \\
\beta_i v_{xx} + \beta_i v_{xx} + \beta_i v_{xx} &= \beta_i v + \beta_i v^3 + \beta_i u^2 v + \beta_i \left( \gamma_i u^3 + \gamma_i u v^2 \right)_{xx}.
\end{aligned} \quad (10) \]

Here \( \alpha_i, \beta_i (i = j = 1, 2, \ldots, 7), \gamma_i, \gamma_j, \delta_i, \delta_j \) are constants.
2. The Method Combining the Function Transformation with the Auxiliary Equation

Assume the solutions of a class of nonlinear evolutionary Equation (10) as
\[
\begin{align*}
\left\{ 
\begin{array}{l}
u(x,t) = \frac{1}{2} \left[ P_1(\xi_1) - Q_1(\eta_1) \right], \\
u(x,t) = \frac{1}{2} \left[ P_1(\xi_1) + Q_1(\eta_1) \right].
\end{array} \right.
\end{align*}
\]
(11)

where \( \xi_1 = \lambda_1 x + \mu_1 t, \eta_1 = \nu_1 x + \omega_1 t, \lambda_j, \mu_j, \nu_j \) and \( \omega_j \) are constants to be determined, and \( \lambda_j \neq \nu_j, \mu_j \neq \omega_j \).

2.1. The First Kind of Function Transformation with the Two Kinds of Ordinary Differential Equations

Substituting the first kind of function transformation (11) into a class of nonlinear evolutionary Equation (10) yields the following conclusions.

Case 1. When the coefficients of a class of nonlinear evolutionary equations satisfy the following conditions, the complexion solutions exist.

When \( \alpha_i = \beta_i (i = 1, 2, \cdots, 7), \delta_i = \delta_2, \beta_8 = 3\alpha_4, \beta_9 = 3\gamma_2 \) a class of nonlinear evolutionary Equation (10) has the complexion solutions, and the solutions are determined by the following nonlinear ordinary differential equations.

\[
\begin{align*}
\frac{d^2 P_i(\xi)}{d\xi^2} &= -3\beta_i P_i'(\xi_1) - 36\beta_i \lambda^2 \gamma_1 \left[ P_i'(\xi_1) \right]^2 + 6\left[ \lambda^2 \gamma_1 \right] \left[ P_i'(\xi_1) \right]^2, \\
\frac{d^2 Q_i(\eta)}{d\eta^2} &= -3\beta_i Q_i'(\eta_1) - 36\beta_i \nu_1 \gamma_i \left[ Q_i'(\eta_1) \right]^2 + 6\left[ \nu_1 \gamma_i \right] \left[ Q_i'(\eta_1) \right]^2.
\end{align*}
\]
(13)

Case 2. The first integral of the second order nonlinear ordinary differential equations.

By the function transformation, the nonlinear ordinary differential Equations (13) and (14) are changed into the following two ordinary differential equations.

\[
\begin{align*}
\frac{dP_i}{d\xi_1} &= Y_i, \\
\frac{dY_i}{d\xi_1} &= -3\beta_i P_i - \alpha P_i + 6\left[ \lambda^2 \gamma_1 \right] Y_i^2, \\
\frac{dQ_i}{d\eta_1} &= X_i, \\
\frac{dX_i}{d\eta_1} &= -3\beta_i Q_i - \alpha Q_i + 6\left[ \nu_1 \gamma_i \right] X_i^2.
\end{align*}
\]
(15)

And by the following transformation (17) and (18), the two ordinary differential Equations (15) and (16) can be expressed by the form of (19) and (20).

\[
\begin{align*}
\frac{d\xi_1}{d\tau} &= 3 \left[ -3\beta_i \lambda^2 \gamma_1 + \left( \beta_i \mu^2 + \lambda^2 \delta + \lambda_i \mu_i \alpha_i \right) \right] d\tau,
\end{align*}
\]
(17)
\[ \frac{d\eta}{d\tau} = 3\left[-3\beta\gamma\chi v_1^2 + (v_1^2 + \nu \alpha \omega + \beta \omega) Q_1^2 \right] d\zeta, \]  
(18)

\[ \frac{dP}{d\tau} = 3Y_1 \left[-3\beta\gamma\chi\lambda^2_1 + (\beta^2 \mu_1 + \lambda^2 \delta_2 + \lambda \mu \alpha) P_1^2 \right], \]  
(19)

\[ \frac{dY_1}{d\tau} = \frac{1}{P_1}\left[ -3\beta P_1^4 - \alpha P_1^2 + \left[ 6\lambda^2 \delta_1 + \beta^2 \mu_1 + \lambda \mu \alpha \right] P_1^2 - 36\beta \lambda^2 \gamma_1 \right] Y_1^2. \]  
(20)

By calculating, the following first integral of the ordinary differential Equations (19) and (20) are obtained

\[ Y_1^2 = \frac{6c_1 P_1^5 + M_1 P_1^6 + M_2 P_1^5 - 2\alpha \beta \gamma \lambda^2 P_1^2}{6\left(-3\beta \gamma \chi \lambda^2 + \left[ \delta_2 \lambda^2 + \mu \left( \alpha \lambda + \beta \mu \right) \right] P_1^2 \right)^2}, \]  
(21)

\[ X_1^2 = \frac{6c_2 Q_1^5 + N_1 Q_1^6 + N_2 Q_1^5 - 2\alpha \beta \gamma \chi \lambda^2 Q_1^2}{6\left(-3\beta \gamma \chi \lambda^2 + \left[ \delta_2 \lambda^2 + \omega \left( \alpha \lambda + \beta \mu \right) \right] Q_1^2 \right)^2}. \]  
(22)

Substituting the first integral (21) and (22) severally into the first equation of the ordinary differential Equations (19) and (20) yields the following two ordinary differential equations

\[ \left( \frac{dP}{d\tau} \right)^2 = \frac{3}{2} \left( 6c_1 P_1^5 + M_1 P_1^6 + M_2 P_1^5 - 2\alpha \beta \gamma \lambda^2 P_1^2 \right), \]  
(23)

\[ \left( \frac{dQ}{d\tau} \right)^2 = \frac{3}{2} \left( 6c_2 Q_1^5 + N_1 Q_1^6 + N_2 Q_1^5 - 2\alpha \beta \gamma \lambda^2 Q_1^2 \right), \]  
(24)

where \( M_1 = 6\beta \delta \lambda^2 + 6\alpha \beta \lambda \mu_1 + 6\beta \beta \mu_1 \), \( M_2 = -9\beta \beta \chi \lambda^2 + \alpha \delta \lambda^2 + \alpha \gamma \lambda \mu_1 + \alpha \beta \mu_1 \), \( N_1 = 6\beta \beta \chi \lambda^2 + 6\alpha \beta \chi \omega + 6\beta \beta \omega \), \( N_2 = -9\beta \beta \chi \lambda^2 + \alpha \delta \lambda^2 + \alpha \gamma \chi \omega + \alpha \beta \omega \), \( c_1 \) and \( c_2 \) are arbitrary constants.

### 2.2. The Second Kind of Function Transformation with the Two Kinds of Ordinary Differential Equations

Substituting the second kind of function transformation (12) into a class of nonlinear evolutionary Equation (10) yields the following conclusions.

#### Case 1. When the coefficients of a class of nonlinear evolutionary equations satisfy the following conditions, the complexion solutions exist.

When \( \alpha_i = \beta_i (i = 1, 2, \ldots, 7) \), \( \delta_1 = \delta_2, \beta_3 = 3\alpha_4, \beta_4 = 3\gamma_2 \), a class of nonlinear evolutionary Equation (10) has the complexion solutions, and the solutions are determined by the following nonlinear ordinary differential equations.

\[ \frac{d^2 P_i}{d\xi^2} = \frac{3\beta_i + \left[ 27 \beta_\gamma \chi \lambda^2 P_i \left( \xi_1^2 \right) + \alpha_1 \right] \exp \left[ 2 \left[ P_i \left( \xi_1 \right) d\xi_1 \right] - \left( 3\beta \mu^2 + 3\lambda \mu \alpha \xi_1 + 3\lambda^2 \delta \xi_1 \right) P_i^2 \left( \xi_1 \right) \right]}{3 \left(-\lambda^2_1 \lambda \beta \mu \alpha \right) \exp \left[ 2 \left[ P_i \left( \xi_1 \right) d\xi_1 \right] - \lambda \mu \alpha \delta \beta \mu \right] P_i^2 \left( \xi_1 \right)}, \]  
(25)

\[ \frac{d^2 Q_i}{d\eta^2} = \frac{3\beta_i + \left[ 27 \beta_\gamma \chi \chi^2 Q_i \left( \eta_1 \right) + \alpha_1 \right] \exp \left[ 2 \left[ Q_i \left( \eta_1 \right) d\eta_1 \right] - \left( 3\beta \omega^2 + 3\nu \omega \alpha \eta_1 + 3\nu^2 \delta \eta_1 \right) Q_i^2 \left( \eta_1 \right) \right]}{3 \left(-\nu^2_1 \nu \beta \omega \right) \exp \left[ 2 \left[ Q_i \left( \eta_1 \right) d\eta_1 \right] - \alpha \omega \nu \beta \omega \right] Q_i^2 \left( \eta_1 \right)}, \]  
(26)

The ordinary differential Equations (25) and (26) can be expressed by the following forms.
\[ \frac{dY}{d\zeta} = 2PY, \]
\[ \frac{dP}{d\zeta} = -3\beta_1 + \alpha_1 Y_z - \left( 3\beta Y_z + 3\lambda\mu Y_z + 3\lambda^2\delta_z - 27\beta Y_z Y_z, \right) \frac{P_z}{2}. \]  
(27)

\[ \frac{dX}{d\eta} = 2Q, \]
\[ \frac{dQ}{d\eta} = -3\beta_1 + \alpha_1 X_z - \left( 3\beta Y_z + 3\lambda\mu Y_z + 3\lambda^2\delta_z - 27\beta Y_z Y_z, \right) \frac{Q_z}{2}. \]  
(28)

And by the following transformation (29) and (30), the two ordinary differential Equations (27), (28) can be expressed by the form of (31), (32).

\[ d\zeta = 3\left( \lambda^2\delta_z + 3\beta Y_z Y_z, \right) d\zeta, \]
\[ d\eta = 3\left( -\lambda^2 Y_z Y_z, \right) d\eta. \]  
(29)
\( (30) \)

\[ \frac{dY}{d\tau} = 6PY, \]
\[ \frac{dP}{d\tau} = \left[ 3\beta_1 + \alpha_1 Y_z - \left( 3\beta Y_z + 3\lambda\mu Y_z + 3\lambda^2\delta_z - 27\beta Y_z Y_z, \right) \frac{P_z}{2} \right] \frac{d\zeta}{d\tau}. \]  
(31)

\[ \frac{dX}{d\zeta} = 6Q, \]
\[ \frac{dQ}{d\zeta} = \left[ 3\beta_1 + \alpha_1 X_z - \left( 3\beta Y_z + 3\lambda\mu Y_z + 3\lambda^2\delta_z - 27\beta Y_z Y_z, \right) \frac{Q_z}{2} \right] \frac{d\zeta}{d\tau}. \]  
(32)

Case 2. The first integral of the second order nonlinear ordinary differential equations.
By calculating, the following first integral of the ordinary differential Equations (31), (32) are obtained.

\[ P_z^2 = \frac{6c_3 - 2\alpha_1 Y_z Y_z, + M_z Y_z^2 + M_4 X_z}{6 \left[ -3\beta Y_z Y_z, + \delta Y_z Y_z, + \mu_2 \left( \beta Y_z + \alpha_1 Y_z \right) \right] Y_z}. \]  
(33)

\[ Q_z^2 = \frac{6c_3 - 2\alpha_1 Y_z Y_z, + M_z Y_z^2 + M_4 X_z}{6 \left[ -3\beta Y_z Y_z, + \delta Y_z Y_z, + \mu_2 \left( \beta Y_z + \alpha_1 Y_z \right) \right] X_z}. \]  
(34)

Substituting the first integral (33) and (34) severally into the first equation of the ordinary differential Equations (31) and (32) yields the following two nonlinear ordinary differential equations

\[ \left( \frac{dY}{d\tau} \right)^2 = 6\left[ -2\alpha_1 Y_z Y_z, + M_z Y_z^2 + M_4 X_z^2 + 6c_5 Y_z \right], \]  
(35)

\[ \left( \frac{dX}{d\zeta} \right)^2 = 6\left[ -2\alpha_1 Y_z Y_z, + M_z Y_z^2 + M_4 X_z^2 + 6c_5 Y_z \right], \]  
(36)

where \( M_z = -9\beta_1 Y_z Y_z, + \alpha_2 \left( \delta Y_z Y_z, + \alpha_2 Y_z \mu + \beta_1 \mu \right), \) \( M_4 = 6\beta_4 \left[ \delta Y_z Y_z, + \mu_2 \left( \beta Y_z + \alpha_1 Y_z \right) \right] \), \( N_z = -9\beta_1 Y_z Y_z, + \alpha_2 \left( \delta Y_z Y_z, + \alpha_2 Y_z \mu + \beta_1 \mu \right), \) \( N_4 = 6\beta_4 \left[ \delta Y_z Y_z, + \mu_2 \left( \beta Y_z + \alpha_1 Y_z \right) \right], \) \( c_3 \) and \( c_4 \) are arbitrary constants.

3. The New Infinite Sequence Complexion Solutions of a Class of Nonlinear Evolutionary Equations
In some cases, according to the ordinary differential Equations (23) and (24) (or (35) and (36)), the new infinite
sequence complexion solutions of a class of nonlinear evolutionary Equation (10) are constructed.

3.1. The New Infinite Sequence Complexion Solutions Consisting of the Hyperbolic Function, the Trigonometric Function and the Rational Function

When \( c_1 = c_2 = 0, \alpha, \beta, \gamma \neq 0 \), by the following superposition formula, the new infinite sequence complexion solutions consisting of the hyperbolic function, the trigonometric function and the rational function of a class of nonlinear evolutionary Equation (10) are obtained.

\[
\begin{align*}
    u_m(x,t) &= \frac{1}{2} \left[ (P)_m(\xi) + (Q)_m(\eta) \right] \quad (m = 1, 2, \cdots), \\
    v_n(x,t) &= \frac{1}{2} \left[ (P)_n(\xi) - (Q)_n(\eta) \right] \quad (n = 1, 2, \cdots), \\
    d\xi &= 3 \left[ -3\beta_1\gamma_1 \lambda_i^2 + (\beta_1\mu_i^2 + \lambda_i^2 \delta_i^2 + \lambda_i \mu_i \alpha_i) P_i^2 \right] d\tau, \\
    d\eta &= 3 \left[ -3\beta_1\gamma_1 \eta_i^2 + (\eta_i^2 \delta_i^2 + \eta_i \alpha_i \omega_i + \beta_1 \alpha_i^2) Q_i^2 \right] d\zeta,
\end{align*}
\]

where \((P)_m(\xi)\) and \((Q)_n(\eta)\) are determined by the following superposition formula [10]-[13].

\[
\begin{align*}
    (P)_m(\tau_i) &= \pm \frac{\sqrt{-A_1 + (V)_m^2(\tau_i)}}{B_1 - 2 \sqrt{C_1 (V)_m^2(\tau_i)}} \quad (m = 1, 2, \cdots), \\
    (V)_m(\tau_i) &= \frac{-M_0 + \sqrt{M_0^2 - 4L_0N_0}}{2 \left[ L_0 + M_0 (V)_{m-1}^2(\tau_i) + N_0 (V)_{m-1}^2(\tau_i) \right]} \left( L_0 - N_0 R_1 \right) (V)_{m-1}(\tau_i) \quad (m = 1, 2, \cdots), \\
    (V)_0(\tau_i) &= -R_1 \tanh \left( \sqrt{R_1} \tau_i \right) \quad (-R_1 = A_1 = -3\alpha_1 \beta_1 \gamma_1 \lambda_i^2 < 0), \\
    B_1 &= \frac{3}{2} \left[ -9\beta_1 \beta_1 \gamma_1 \lambda_i^2 + \alpha_i \delta_i \lambda_i^2 + \alpha_2 \alpha_1 \lambda_i \mu_i + \alpha_2 \beta_i \mu_i^2 \right] \quad C_1 = \frac{3}{2} \left[ 6\beta_1 \delta_i \lambda_i^2 + 6\alpha_2 \beta_1 \lambda_i \mu_i + 6\beta_1 \beta_i \mu_i^2 \right].
\end{align*}
\]
\[
\begin{align*}
(Q_n)_n(\zeta_i) &= \pm \frac{\sqrt{-A_2 + (V_2)_n^2(\zeta_i)}}{\sqrt{B_2 - 2\sqrt{C_2}(V_2)_n(\zeta_i)}} \quad (n = 1, 2, \ldots), \\
(V_2)_n(\zeta_i) &= \left( -M_0 + \sqrt{M_0^2 - 4L_0N_0} \right) \frac{R_2 + 2(L_0 - N_0R_2)(V_2)_{n+1}(\zeta_i) + \left( M_0 + \sqrt{M_0^2 - 4L_0N_0} \right)(V_2)_{n-1}(\zeta_i)}{2 \left[ L_0 + M_0(V_2)_{n+1}(\zeta_i) + N_0(V_2)_{n-1}(\zeta_i) \right]}, \\
(V_2)_0(\zeta_i) &= -\sqrt{R_2} \tanh \left( \sqrt{R_2} \zeta_i \right) \quad (R_2 = A_2 = -3\alpha_2\beta_2\gamma_2\nu^2_i < 0), \\
C_2 &= \frac{3}{2} \left[ -9\beta_2\delta_2\gamma_2\nu^2_i + \alpha_2\delta_2\nu^2_i + \alpha_2\alpha_2\nu_i\omega_i + \alpha_2\beta_2\omega^5_i \right], \\
V_2_0(\zeta_i) &= \sqrt{R_2} \tan \left( \sqrt{R_2} \zeta_i \right) \quad (R_2 = A_2 = -3\alpha_2\beta_2\gamma_2\nu^2_i > 0), \\
B_2 &= \frac{3}{2} \left[ -9\beta_2\delta_2\gamma_2\nu^2_i + \alpha_2\delta_2\nu^2_i + \alpha_2\alpha_2\nu_i\omega_i + \alpha_2\beta_2\omega^5_i \right], \\
(V_2)_0(\zeta_i) &= \frac{2L_0(V_2)_{n+1}(\zeta_i) + \left( M_0 + \sqrt{M_0^2 - 4L_0N_0} \right)(V_2)_{n-1}(\zeta_i)}{2 \left[ L_0 + M_0(V_2)_{n+1}(\zeta_i) + N_0(V_2)_{n-1}(\zeta_i) \right]}, \\
(V_2)_0(\zeta_i) &= -\frac{1}{\zeta_i} \quad (R_2 = A_2 = -3\alpha_2\beta_2\gamma_2\nu^2_i = 0), \\
B_2 &= \frac{3}{2} \left[ -9\beta_2\delta_2\gamma_2\nu^2_i + \alpha_2\delta_2\nu^2_i + \alpha_2\alpha_2\nu_i\omega_i + \alpha_2\beta_2\omega^5_i \right].
\end{align*}
\]

In (38)-(43), \( L_0, M_0, \) and \( N_0 \) are arbitrary constants not all equal to zero.

Case 1. The hyperbolic function type new infinite sequence complexion two-soliton solutions of a class of nonlinear evolutionary Equation (10).

Substituting the solution determined by the superposition formula (38), (41) into (37) yields the hyperbolic function type new infinite sequence complexion two-soliton solutions.

Case 2. The new infinite sequence complexion solutions consisting of the hyperbolic function and the trigonometric function of a class of nonlinear evolutionary Equation (10).

Substituting the solution determined by the superposition formula (38), (42) into (37) yields the new infinite sequence complexion solutions composed of soliton solutions and the period solutions.

Case 3. The new infinite sequence complexion solutions consisting of the hyperbolic function and the rational function of a class of nonlinear evolutionary Equation (10).

Substituting the solution determined by the superposition formula (38) and (43) into (37) yields the new infinite sequence complexion solutions consisting of the hyperbolic function and the rational function.

Case 4. The trigonometric function type new infinite sequence two-period solutions of a class of nonlinear evolutionary Equation (10).

Substituting the solution determined by the superposition formula (39), (42) into (37) yields the trigonometric function type new infinite sequence two-period solutions.
Case 5. The new infinite sequence complexion solutions consisting of the trigonometric function and the rational function of a class of nonlinear evolutionary Equation (10).

Substituting the solution determined by the superposition formula (39), (43) into (37) yields the new infinite sequence complexion solutions consisting of the trigonometric function and the rational function.

Case 6. The rational function type new infinite sequence solutions of a class of nonlinear evolutionary Equation (10).

Substituting the solution determined by the superposition formula (40) and (43) into (37) yields the rational function type new infinite sequence solutions.

When \( c_3 = c_4 = 0 \), with the help of the ordinary differential Equations (35) and (36), the new infinite sequence complexion two-soliton and two-periods solutions consisting of the hyperbolic function, the trigonometric function and the rational function of a class of nonlinear evolutionary Equation (10) can also be obtained (not given here).

### 3.2. The New Infinite Sequence Complexion Solutions Consisting of the Riemann Function, the Jacobi Elliptic Function, the Hyperbolic Function, the Trigonometric Function and the Rational Function

With the help of the relative conclusions of the second kind of elliptic equation [11]-[13], by analyzing, the infinite sequence solutions of the ordinary differential Equations (35) and (36) are obtained. Then substituting these solutions into function transformation (44) yields the new infinite sequence complexion solutions consisting of the Riemann \( \theta \) function, the Jacobi elliptic function, the hyperbolic function, the trigonometric function and the rational function of a class of nonlinear evolutionary Equation (10) in some kinds of cases. These solutions are consisting of two-soliton solutions, two-period solutions and solutions composed of soliton solutions and period solutions (not given concretely here).

\[
\begin{align*}
\frac{u_{mn}(x,t)}{2} &= \frac{1}{2} \exp \left[ \int \left( P_m(x, \xi_2) \right) d\xi_2 \right] + \exp \left[ \int \left( Q_m(x, \eta_2) \right) d\eta_2 \right] \quad (m, n = 1, 2, \ldots),

\frac{v_{mn}(x,t)}{2} &= -\frac{1}{2} \exp \left[ \int \left( P_m(x, \xi_2) \right) d\xi_2 \right] - \exp \left[ \int \left( Q_m(x, \eta_2) \right) d\eta_2 \right] \quad (m, n = 1, 2, \ldots),

d\xi_2 &= 3 \left( -\lambda_2^2 \delta_2 + 3 \beta_0 \gamma_1 \lambda_2^2 Y_2 - \lambda_2 \mu \alpha_2 - \beta_1 \mu_1^2 \right) d\tau_2,

d\eta_2 &= 3 \left( -\nu_2^2 \delta_2 + 3 \beta_0 \gamma_1 \nu_2^2 X_2 - \alpha_2 \omega_1 \nu_2 - \beta_1 \omega_1^2 \right) d\zeta_2.
\end{align*}
\]

Case 1. When \( \gamma_1 = 0, \alpha_0 \beta_0 \neq 0, c_3 c_4 \neq 0 \), the new infinite sequence complexion solutions consisting of the Riemann \( \theta \) function, the Jacobi elliptic function, the hyperbolic function, the trigonometric function and the rational function of a class of nonlinear evolutionary Equation (10) are obtained.

Case 2. When \( \beta_0 = 0, \alpha_0 \gamma_1 \neq 0, c_3 c_4 \neq 0 \), the new infinite sequence complexion solutions consisting of the Riemann \( \theta \) function, the Jacobi elliptic function, the hyperbolic function, the trigonometric function and the rational function of a class of nonlinear evolutionary Equation (10) are obtained.

Case 3. When \( \alpha_0 = 0, \beta_0 \gamma_1 \neq 0, c_3 c_4 \neq 0 \), the new infinite sequence complexion solutions consisting of the Riemann \( \theta \) function, the Jacobi elliptic function, the hyperbolic function, the trigonometric function and the rational function of a class of nonlinear evolutionary Equation (10) are obtained.

### 4. Conclusions

In Refs. [1]-[5], a finite number of one function type one soliton new solutions consisting of the Jacobi elliptic function, the hyperbolic function, the trigonometric function and the rational function of the nonlinear evolutionary Equations (1)-(4) were constructed. In Refs. [6]-[9], a finite number of complexion function type two-soliton new solutions consisting of the Jacobi elliptic function, the hyperbolic function, the trigonometric function and the rational function of the nonlinear evolutionary Equation (5) were constructed.

In this paper, the method combining the function transformation with the auxiliary equation is presented, and the problem of solving the solutions of a class of nonlinear evolutionary Equation (10) concluding the nonlinear evolutionary Equations (1)-(9) is considered; the new infinite sequence complexion solutions consisting of the Riemann \( \theta \) function, the Jacobi elliptic function, the hyperbolic function, the trigonometric function and the
rational function are obtained. These solutions are consisting of two-soliton solutions, two-period solutions and solutions composed of soliton solutions and period solutions.

When the coefficients of nonlinear evolutionary Equations (1) are \( \alpha = \beta = h - l = 1, \quad \alpha = \beta = m = 1, \) then they satisfy the condition \( \alpha = \beta = 1, \) \( i = 1, 2, \ldots , 7, \delta = \delta = \beta_3 = 3\gamma_1, \beta_4 = 3\gamma_2 \) that the new infinite sequence complexion solutions exist. So, according to the method given in the Part 3, the new infinite sequence complexion two-soliton and two-periods and so on solutions of nonlinear evolutionary Equation (1) are constructed.

When the coefficients of a class of nonlinear evolutionary Equation (10) are

\[
\delta_1 = \delta_2 = \alpha_1 = \beta_1 = \alpha_4 = \beta_4 = \beta_3 = 0, \quad \alpha_2 = \beta_2 = \alpha_3 = \gamma_1 = \gamma_2 = 1, \alpha_6 = \beta_6 = \frac{1}{6}, \alpha_7 = \beta_7 = 3,
\]

it satisfies the condition that the new infinite sequence complexion solutions exist. Then according to the conclusions given in Part 3 in the paper, the new infinite sequence complexion two-soliton solutions, two-period solutions and solutions composed of soliton solutions and period solutions of nonlinear evolutionary Equation (5) are constructed.

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