New Exact Solutions of the (2 + 1)-Dimensional AKNS Equation

Yepeng Sun
School of Mathematics and Quantitative Economics, Shandong University of Finance and Economics, Jinan, China
Email: yepsun@163.com

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Abstract

N-soliton solutions and the bilinear form of the (2 + 1)-dimensional AKNS equation are obtained by using the Hirota method. Moreover, the double Wronskian solution and generalized double Wronskian solution are constructed through the Wronskian technique. Furthermore, rational solutions, Matveev solutions and complexitons of the (2 + 1)-dimensional AKNS equation are given through a matrix method for constructing double Wronskian entries. The three solutions are new.

Keywords

(2 + 1)-Dimensional AKNS Equation, Rational Solutions, Matveev Solutions, Complexitons

1. Introduction

It is one of the most important topics to search for exact solutions of nonlinear evolution equations in soliton theory. Moreover, various methods have been developed, such as the inverse scattering transformation [1], the Darboux transformation [2], the Hirota method [3], the Wronskian technique [4] [5], source generation procedure [6] [7] and so on. In 1971, Hirota first proposed the formal perturbation technique to obtain N-soliton solution of the KdV equation. Satsuma gave the Wronsian representation of the N-soliton solution to the KdV equation [8]. Then the Wronsian technique was developed by Freeman and Nimmo [4] [5]. In 1992, Matveev introduced the generalized Wronsian to obtain another kind of exact solutions called Positons for the KdV equation [9]. Recently, Ma first introduced a new kind of exact solution called complexitons [10]. By using these methods, exact solutions of many nonlinear soliton equations are obtained [11]-[16].

The AKNS (Ablowitz-Kaup-Newell-Segur) equation is one of the most important physical models [17]-[19]. In 1997, Lou and Hu have obtained the (2 + 1)-dimensional AKNS equation from the inner parameter dependent symmetry constraints of the KP equation [20]. Moreover, Lou et al. have studied Painlevé integrability of the
(2 + 1)-dimensional AKNS equation [21]. In this paper, we will apply the Hirota method and the Wronskian technique to obtain new exact solutions of the (2 + 1)-dimensional AKNS equation.

This paper is organized as follows. In Section 2, the bilinear form of the (2 + 1)-dimensional AKNS equation and its N-soliton solutions are obtained through the Hirota method. In Section 3, the double Wronskian solution and generalized double Wronskian solution are constructed by using the Wronskian technique. In Sections 4 and 5, rational solutions and Matveev solutions are given. In Section 6, complexitons of the (2 + 1)-dimensional AKNS equation are provided. Finally, we give some conclusions.

2. N-Soliton Solutions of the (2 + 1)-Dimensional AKNS Equation

We consider the following (2 + 1)-dimensional AKNS equation [21]

$$p_t + p_{xx} + pu_x = 0, \quad q_t - q_{xx} - qu_x = 0, \quad u_t + 2pq = 0. \quad (2.1)$$

Through the dependent variable transformation

$$p = \frac{g}{f}, \quad q = \frac{h}{f}, \quad u = 2 \frac{f_x}{f}, \quad (2.2)$$

Equation (2.1) is transformed into the following bilinear form

$$(D_t + D_x^2)g \cdot f = 0, \quad (2.3a)$$

$$(D_t - D_x^2)h \cdot f = 0, \quad (2.3b)$$

$$D_tD_x^2 f \cdot g = (\partial_x - \hat{\partial}_x)^n (\partial_x - \hat{\partial}_x)^n f(t,x)g(t,x)|_{\xi,\eta=\alpha}. \quad (2.3c)$$

where $D$ is the well-known Hirota bilinear operator defined by

$$D^n f = \left(\frac{\partial}{\partial t} - \frac{\partial}{\partial x}\right)^n f(t,x)\big|_{\xi,\eta=\alpha}.$$ 

Expanding $f, g$ and $h$ as the series

$$f(t,x,y) = 1 + f^{(2)}e^2 + f^{(4)}e^4 + \cdots + f^{(2j)}e^{2j} + \cdots, \quad (2.4a)$$
$$g(t,x,y) = g^{(1)}e + g^{(3)}e^3 + \cdots + g^{(2j+1)}e^{2j+1} + \cdots, \quad (2.4b)$$
$$h(t,x,y) = h^{(1)}e + h^{(3)}e^3 + \cdots + h^{(2j+1)}e^{2j+1} + \cdots. \quad (2.4c)$$

Substituting Equation (2.4) into (2.3) and comparing the coefficients of the same power of $e$ yields

$$g_t^{(1)} + g_{xx}^{(1)} = 0, \quad g_t^{(3)} + g_{xx}^{(3)} = 0, \quad (2.5a)$$

$$g_t^{(5)} + g_{xx}^{(5)} = 0, \quad h_t^{(1)} - h_{xx}^{(1)} = 0, \quad h_t^{(3)} - h_{xx}^{(3)} = 0, \quad (2.5b)$$

$$f_t^{(2)} = -g^{(1)}h^{(1)}, \quad 2f_t^{(4)} = -D_tD_x f^{(2)} \cdot f^{(2)} - 2\left(g^{(1)}h^{(3)} + g^{(3)}h^{(1)}\right), \quad (2.5c)$$

Taking

$$g^{(1)} = e^\zeta, \quad \zeta_t = \alpha t + k_x x + l_y y + \zeta_t^{(0)}, \quad (2.5a)$$
$$h^{(1)} = e^\eta, \quad \eta_t = \alpha t + \beta x + \gamma y + \eta_t^{(0)}. \quad (2.5b)$$

we can obtain
\[ f^{(2)} = -\frac{1}{(k_1 + \beta_j)(l_1 + \gamma_1)} e^{\xi_j + \eta_j}, \quad f^{(m)} = 0, \quad m = 4, 6, \cdots, \]
\[ g^{(n)} = 0, \quad h^{(n)} = 0, \quad n = 3, 5, \cdots, \quad \alpha_1 = -k_1^2, \quad \alpha_i = \beta_i^2. \]

Letting \( \varepsilon = 1 \), then \( g_1 = g^{(1)} \), \( h_1 = h^{(1)} \), \( f_1 = f^{(2)} \). Thus, the one-soliton solution is given as follows.

\[ p = \frac{e^{\xi_1}}{1 + e^{\xi_1 + \eta_1}}, \quad q = \frac{e^{\eta_1}}{1 + e^{\xi_1 + \eta_1}}, \quad u = \frac{2(k_1 + \beta_1)e^{\xi_1 + \eta_1}}{1 + e^{\xi_1 + \eta_1}}, \quad (2.6) \]

where \( e^{A_3} = -\frac{1}{(k_1 + \beta_1)(l_1 + \gamma_1)} \).

In the same way, we can obtain the following N-soliton solutions of Equation (2.3).

\[ f_\mu = \sum_{\mu = 0, 1} A_\mu(\mu) \exp \left[ \sum_{j=1}^{2n} \mu_j \xi_j + \sum_{j\leq j'}^{2n} \mu_j \mu_{j'} \theta_{j j'} \right], \quad (2.7a) \]
\[ g_\mu = \sum_{\mu = 0, 1} A_\mu(\mu) \exp \left[ \sum_{j=1}^{2n} \mu_j \xi_j + \sum_{j\leq j'}^{2n} \mu_j \mu_{j'} \theta_{j j'} \right], \quad (2.7b) \]
\[ h_\mu = \sum_{\mu = 0, 1} A_\mu(\mu) \exp \left[ \sum_{j=1}^{2n} \mu_j \xi_j + \sum_{j\leq j'}^{2n} \mu_j \mu_{j'} \theta_{j j'} \right], \quad (2.7c) \]

where

\[ \omega_j = -k_j^2, \quad \alpha_j = \beta_j^2, \quad \xi_{\omega_j} = \eta_j, \quad (j = 1, \cdots, n), \quad (2.8a) \]
\[ e^{\eta_j} = 2(k_j - k_{j'}) (l_j - l_{j'}), \quad (j < \rho = 2, 3, \cdots, n), \quad (2.8b) \]
\[ e^{\theta_{(j \leq j')}_{\omega_j}} = 2(\beta_j - \beta_{j'}) (\gamma_j - \gamma_{j'}), \quad (j < \rho = 2, 3, \cdots, n), \quad (2.8c) \]
\[ e^{\theta_{(j \leq j')}_{\omega_j}} = -\frac{1}{(k_j + \beta_j)(l_j + \gamma_j)}, \quad (j, \rho = 1, 2, \cdots, n), \quad (2.8d) \]

\( A_\mu(\mu), A_\mu(\mu) \) and \( A_\mu(\mu) \) take over all possible combinations of \( \mu_j = 0, 1 (j = 1, 2, \cdots, 2n) \) and satisfy the following condition

\[ \sum_{j=1}^{2n} \mu_j = \sum_{j=1}^{2n} \mu_{\omega_j}, \quad \sum_{j=1}^{2n} \mu_j = 1 + \sum_{j=1}^{2n} \mu_{\omega_j}, \quad 1 + \sum_{j=1}^{2n} \mu_j = \sum_{j=1}^{2n} \mu_{\omega_j}. \]

### 3. The Double Wronskian Solution and Generalized Double Wronskian Solution

Let us first specify some properties of the Wronskian determinant. As is well known, the double Wronskian determinant is

\[ W_N^M(\varphi; \psi) = \det \left( \varphi, \varphi, \varphi, \cdots, \varphi, \psi, \psi, \cdots, \psi \right) \]

where \( \varphi = (\varphi_1(x), \varphi_2(x), \cdots, \varphi_{N^2}(x))^T \) and \( \psi = (\psi_1(x), \psi_2(x), \cdots, \psi_{N^2}(x))^T \). The following two determinant identities were often used [4] [5]. The one is

\[ |D, a, b||D, c, d|-|D, a, c||D, b, d|+|D, a, d||D, b, c|=0, \quad (3.1) \]

where \( D \) is a \( N \times (N - 2) \) matrix and \( a, b, c \) and \( d \) represent \( N \) column vectors. The other is

\[ \sum_{j=1}^{N} |\alpha_1, \cdots, \alpha_j, \cdots, \alpha_N| = \left( \sum_{j=1}^{N} b_j \right) |\alpha_1, \cdots, \alpha_N|, \quad (3.2) \]
where \( \alpha_j \) are \( N \) column vectors and \( bA_j \) denotes \( (b_1, a_1, b_2, a_2, \ldots, b_N, a_N) \).

Employing the Wronskian technique, we have the following result.

**Theorem 1.** The \((2 + 1)\)-dimensional AKNS Equation (2.3) has the double Wronskian solution

\[
g = 2W^{N+2,M}\varphi, \quad f = W^{N+1,M+1}\psi, \quad h = -2W^{N,M+2}\varphi, \]

where \( \varphi \) and \( \psi \) satisfy the following conditions

\[
\varphi_{j,x} = -\varphi_{j,t}, \quad \psi_{j,x} = 2\psi_{j,t}, \quad (j = 1, 2, \ldots, N + M + 2),
\]

\[
\varphi_{j,y} = \varphi_{j,x}, \quad \psi_{l,y} = 2\psi_{l,x}, \quad (l = 1, 2, \ldots, N + M + 2).
\]

**Proof.** In the following, we use the abbreviated notation of Freeman and Nimmo for the Wronskian and its derivatives [4] [5], then Equation (3.3) becomes

\[
g = 2\left[\hat{N} + 1; \hat{M} - 1\right], \quad f = \left[\hat{N}; \hat{M}\right], \quad h = -2\left[\hat{N} - 1; \hat{M} + 1\right]. \tag{3.5}
\]

First, we calculate various derivatives of \( g \) and \( f \) with respect to \( x \) and \( t \).

\[
f_x = 2\left[\hat{N} - 1, N + 1; \hat{M}\right] + \left[\hat{N}; \hat{M} - 1, M + 1\right],
\]

\[
f_{xx} = 2\left[\hat{N} - 2, N, N + 1; \hat{M}\right] + \left[\hat{N} - 1, N + 2, \hat{M}\right] + 2\left[\hat{N} - 1, N + 1; \hat{M} - 1, M + 1\right]
+ \left[\hat{N} - N - 2, M, M + 1; \hat{M}\right] + \left[\hat{N} - N - 1, M - 1, M + 1\right],
\]

\[
g_x = 2\left[\hat{N} - 1, N + 2, \hat{M} - 1\right] + 2\left[\hat{N} - 1, \hat{M} - 2, \hat{M}\right],
\]

\[
g_{xx} = 2\left[\hat{N} - 1, N + 1, N + 2; \hat{M} - 1\right] + 2\left[\hat{N} - N - 1, N + 3; \hat{M} - 1\right] + 4\left[\hat{N} - N + 2, \hat{M} - 2, \hat{M}\right]
+ 2\left[\hat{N} + 1, \hat{M} - 3, M - 1, M + 1\right] + 2\left[\hat{N} + 1, \hat{M} - 2, M + 1\right],
\]

\[
f_t = -2\left[\hat{N} - 2, N + 1, \hat{M}\right] + \left[\hat{N} - 1, N + 2, \hat{M}\right] + 2\left[\hat{N} - 2, M + 1, M\right] + \left[\hat{N} - M + 2\right],
\]

\[
g_t = -4\left[\hat{N} - 1, N + 2, N + 1; \hat{M} - 1\right] + \left[\hat{N} - N + 3; \hat{M} - 1\right]
+ 4\left[\hat{N} + 1, \hat{M} - 3, M - 1\right] + \left[\hat{N} + 1, \hat{M} - 2, M + 1\right].
\]

Then a direct calculation gives

\[
g, f, f + g_{xx} f - 2g f_s + 6g_{fs} = 6\left[\hat{N}; \hat{M}\right]\left[\hat{N} - 1, N + 1, N + 2; \hat{M} - 1\right]
- 2\left[\hat{N}; \hat{M}\right]\left[\hat{N} - N + 3, \hat{M} - 1\right] - 2\left[\hat{N}, \hat{M} - 2, M + 1\right] - 2\left[\hat{N} - 2, N + 1; \hat{M}\right]
+ 6\left[\hat{N}; \hat{M}\right]\left[\hat{N} + 1, \hat{M} - 2, M + 1\right] - 2\left[\hat{N}, \hat{M} - 2, M + 1\right] - 2\left[\hat{N} - 1, N + 2, \hat{M}\right]
+ 2\left[\hat{N} + 1, \hat{M} - 1\right]\left[\hat{N} - 1, N + 2, \hat{M}\right] + 2\left[\hat{N} - 1, N + 1; \hat{M} - 1, M + 1\right]
+ 2\left[\hat{N} - 1, \hat{M} - 2, M + 1\right] - \left[\hat{N} - 1, \hat{M} - 1, M + 1\right]
- 4\left[\hat{N}; \hat{M}\right]\left[\hat{N} - 1, N + 1; \hat{M}\right] + \left[\hat{N}; \hat{M} - 1, M + 1\right]
- 4\left[\hat{N} + 1, \hat{M} - 2, M\right] + \left[\hat{N} - 1, N + 1, \hat{M}\right] + \left[\hat{N}; \hat{M} - 1, M + 1\right]. \tag{3.6}
\]
Utilizing Equation (3.2) and Equation (3.4), we get

\[ \left( \sum \frac{1}{2} \lambda_j \right) \left[ \tilde{N}, N + 2; M - 1 \right] = - \left[ \tilde{N}, N + 1, N + 2; M - 1 \right] - \left[ \tilde{N}, N + 2; M - 1 \right], \]  
\[ (3.7a) \]

\[ \left( \sum \frac{1}{2} \lambda_j \right) \left[ \tilde{N} + 1, M - 2, M \right] = - \left[ \tilde{N}, N + 2; M - 2, M \right] + \left[ \tilde{N} + 1; M - 1, M \right] + \left[ \tilde{N} + 1; M - 2, M + 1 \right], \]  
\[ (3.7b) \]

\[ \left( \sum \frac{1}{2} \lambda_j \right) \left[ \tilde{N} + 1, N + 1, \tilde{M} \right] = - \left[ \tilde{N} - 1, N + 1, \tilde{M} \right] - \left[ \tilde{N} + 1, N + 2, \tilde{M} \right] + \left[ \tilde{N} - 1, N + 1, \tilde{M} - 1, M + 1 \right], \]  
\[ (3.7c) \]

\[ \left( \sum \frac{1}{2} \lambda_j \right) \left[ \tilde{N}, \tilde{M} - 1, M + 1 \right] = - \left[ \tilde{N} - 1, N + 1, \tilde{M} - 1, M + 1 \right] + \left[ \tilde{N}; \tilde{M} - 2, M, M + 1 \right] + \left[ \tilde{N}; \tilde{M} - 1, M + 2 \right]. \]  
\[ (3.7d) \]

Noting

\[ \left( \sum \frac{1}{2} \lambda_j \right) \left[ \tilde{N}, N + 2; M - 1 \right] = \left[ \tilde{N}, N + 2; M - 1 \right] \left( \sum \frac{1}{2} \lambda_j \right) \left[ \tilde{N}; \tilde{M} \right], \]  
\[ (3.8a) \]

\[ \left( \sum \frac{1}{2} \lambda_j \right) \left[ \tilde{N} + 1, M - 2, M \right] = \left[ \tilde{N} + 1; M - 2, M \right] \left( \sum \frac{1}{2} \lambda_j \right) \left[ \tilde{N}; \tilde{M} \right], \]  
\[ (3.8b) \]

\[ \left( \sum \frac{1}{2} \lambda_j \right) \left[ \tilde{N} - 1, N + 1, \tilde{M} \right] = \left[ \tilde{N} - 1, N + 1, \tilde{M} \right] \left( \sum \frac{1}{2} \lambda_j \right) \left[ \tilde{N}; \tilde{M} \right], \]  
\[ (3.8c) \]

\[ \left( \sum \frac{1}{2} \lambda_j \right) \left[ \tilde{N}; \tilde{M} - 1, M + 1 \right] = \left[ \tilde{N}; \tilde{M} - 1, M + 1 \right] \left( \sum \frac{1}{2} \lambda_j \right) \left[ \tilde{N}; \tilde{M} \right]. \]  
\[ (3.8d) \]

Using Equation (3.7) and Equation (3.8), then Equation (3.6) becomes

\[ g, f = f + g + g, f = -2 g, f + g, f, \]
\[ = 8 \left[ \tilde{N} - 1, N + 1, N + 2; M - 1 \right] \left[ \tilde{N}; \tilde{M} \right] + 8 \left[ \tilde{N}; \tilde{M} - 2, M, M + 1 \right] \left[ \tilde{N} + 1; M - 1 \right] \]
\[ -8 \left[ \tilde{N}, N + 2; M - 1 \right] \left[ \tilde{N} - 1, N + 1; \tilde{M} \right] + 8 \left[ \tilde{N} + 1; M - 2, M + 1 \right] \left[ \tilde{N}; \tilde{M} \right], \]  
\[ (3.9) \]

According to (3.1), it is easy to see that Equation (3.9) is equal to zero. So, the proof of Equation (2.3a) is completed. Similarly Equations (2.3 b) and (2.3 c) can also be proved.

In the following, we give some exact solutions. From Equation (3.4), we deduce that

\[ \phi_j = e^{-z_j} c_j, \psi_j = e^{z_j} d_j, \xi_j = \frac{\lambda_j}{2} x + \frac{\lambda_j^2}{2}, \]  
\[ (3.10) \]

where \( c_j \) and \( d_j \) \( (j = 1, 2, \cdots, N + M + 2) \) are arbitrary real constants.

Taking \( c_j = d_j = 1 \), the double Wronskian solution of Equation (2.3) is obtained as follows:

\[ f = e^{-z_j}, \partial_x e^{-z_j}, \cdots, \partial_x^N e^{-z_j}, e^{z_j}, \partial_x e^{z_j}, \cdots, \partial_x^N e^{z_j}, \]  

\[ g = 2 e^{-z_j}, \partial_x e^{-z_j}, \cdots, \partial_x^{N+1} e^{-z_j}, e^{z_j}, \partial_x e^{z_j}, \cdots, \partial_x^{N+1} e^{z_j}, \]  

\[ h = -2 e^{-z_j}, \partial_x e^{-z_j}, \cdots, \partial_x^{N+1} e^{-z_j}, e^{z_j}, \partial_x e^{z_j}, \cdots, \partial_x^{N+1} e^{z_j}. \]

Letting \( N = 0 \) and \( M = 0 \) gives

\[ f = e^{-z_j} - e^{z_j}, g = (\lambda_1 - \lambda_2) e^{-z_j}, h = (\lambda_1 - \lambda_2) e^{z_j}, \]

then one-soliton solution of Equation (2.1) is
In the following, we will prove that Equation (2.3) has the generalized double Wronskian solution. First, we give the following lemma [19].

**Lemma 1.** Assume that $P = (p_{ij})$ is an $l \times l$ operator matrix and its entries $p_{ij}$ are differential operators. $B = (b_{ij})$ is an $l \times l$ function matrix with column vector set $b_i$ and row vector set $b'_i$ ($i = 1, 2, \cdots, l; j = 1, 2, \cdots, l$), then

$$\sum_{i=1}^{l} b_i \cdots p_i b_i \cdots b_i = \sum_{j=1}^{l} p'_j b'_j,$$

(3.11)
where \( p, q = (p_1 b_1, p_2 b_2, \cdots, p_N b_N)^T, \quad p', q' = (p'_1 b'_1, p'_2 b'_2, \cdots, p'_N b'_N)^T \).

Using the Lemma 1 and the Wronskian technique, we construct the following result.

**Theorem 2.** The \((2 + 1)\)-dimensional AKNS Equation (2.3) has the generalized double Wronskian solution
\[
g = 2W^{N+2,M} (\phi; \psi), \quad f = W^{N+1,M+1} (\phi; \psi), \quad h = -2W^{N,M+2} (\phi; \psi),
\]
where \( \phi_j \) and \( \psi_j \) satisfy the following conditions
\[
\begin{align*}
\phi_{j,i} &= -A\phi_j, \quad \phi_{j,2j} = -2\phi_{j,2i}, \quad \phi_{j,i} = \phi_{j,2i}, \quad (j = 1, 2, \cdots, N + M + 2), \\
\psi_{j,i} &= A\psi_j, \quad \psi_{j,2j} = 2\psi_{j,2i}, \quad \psi_{j,i} = \psi_{j,2i}, \quad (l = 1, 2, \cdots, N + M + 2),
\end{align*}
\]
\( A = (a_{ij}) \) is an \((N + M + 2) \times (N + M + 2)\) arbitrary real matrix independent of \( x \) and \( t \).

In fact, similar to the proof of Theorem 1, we only need to verify that identities (3.7) hold.

(1) If \( \text{tr}A \neq 0 \), setting
\[
p_{ij} = \begin{cases} -\frac{\partial}{\partial x} & 1 \leq i \leq N + M + 2; 1 \leq j \leq N + 1; \\
\frac{\partial}{\partial x} & 1 \leq i \leq N + M + 2; N + 2 \leq j \leq N + M + 2,
\end{cases}
\]
from Lemma 1, we can get
\[
\sum_{j=1}^{N+M+2} \phi_{i} \cdots \phi_{i}^N \psi_{i} \cdots \psi_{i}^N \phi_{i} \cdots \phi_{i}^N \psi_{i} \cdots \psi_{i}^N = \left[ N; M - 1, M + 1 \right] - \left[ N - 1, N + 1; \tilde{M} \right].
\]

Using Equation (3.13), the left-hand side of (3.14) is equal to
\[
\sum_{j=1}^{N+M+2} \sum_{i=1}^{N+M+2} a_{ij} \phi_{i} \cdots \phi_{i}^N \psi_{i} \cdots \psi_{i}^N \phi_{i} \cdots \phi_{i}^N \psi_{i} \cdots \psi_{i}^N = \sum_{j=1}^{N+M+2} a_{ij} \left[ N; M - 1, M + 1 \right] - \left[ N - 1, N + 1; \tilde{M} \right].
\]
Therefore,
\[
\text{tr}A \left[ N; M - 1, M + 1 \right] = \left[ N - 1, N + 1; \tilde{M} \right].
\]
From (3.15), we derive further
\[
\text{tr}A \left[ N, N + 2; M - 1 \right] = -\left[ N, N + 2; M - 1 \right] - \left[ N, N + 3; M - 1 \right] + \left[ N, N + 2; M - 2, M \right],
\]
\[
\text{tr}A \left[ N + 1, M - 2, M + 1 \right] = -\left[ N + 1, N + 2; M - 2, M \right] + \left[ N + 1, M - 3, M - 1, M \right] + \left[ N + 1, M - 2, M + 1 \right],
\]
\[
\text{tr}A \left[ N - 1, N + 1; \tilde{M} \right] = -\left[ N - 2, N, N + 1; \tilde{M} \right] - \left[ N - 1, N + 1; \tilde{M} \right] + \left[ N - 1, N + 1, \tilde{M} - 1, M + 1 \right],
\]
\[
\text{tr}A \left[ N; M - 1, M + 1 \right] = -\left[ N - 1, N + 1, M - 1, M + 1 \right] + \left[ N; M - 2, M, M + 1 \right] + \left[ N; M - 1, M + 2 \right],
\]
\[
(\text{tr}A)^2 \left[ N; \tilde{M} \right] = \left[ N - 2, N, N + 1; \tilde{M} \right] + \left[ N - 1, N + 1; \tilde{M} - 2, M \right] + \left[ N - 1, N + 2; \tilde{M} - 2, M \right] + \left[ N - 1, N + 1; \tilde{M} - 1, M + 1 \right]
+ \left[ N; M - 2, M, M + 1 \right] + \left[ N; M - 1, M + 2 \right].
\]
It is obvious that (3.7) hold.

(2) If \( trA = 0 \), we can consider this as a limit case where \( trA \) tends to zero. Then (3.15)-(3.17) become

\[
\begin{align*}
\hat{N}; M \rightarrow 1, M + 1 &= \hat{N} - 1, N + 1; M \\
\hat{N}; M + 2; M \rightarrow 2, M &= \hat{N} - 1, N + 1; M \\
\hat{N} + 1; M \rightarrow 2, M + 1 &= \hat{N} - 1, N + 1; M \\
\hat{N} - 1, N + 1; M \rightarrow 1, M + 1 &= \hat{N} - 1, N + 1; M \\
\hat{N}; M \rightarrow 1, M + 2 &= \hat{N} - 1, N + 1; M \\
N - 2, N + 1; M &= \hat{N} - 1, N + 1; M \\
\hat{N}; M \rightarrow 2, M + 1 &= \hat{N} - 1, N + 1; M \\
\end{align*}
\]

Using (3.18), Equation (3.12) still satisfies Equation (2.3).

From Equation (3.13), we can get the general solution

\[
\varphi = e^{-2x^2 - 4x^2 - 6y} C, \quad \psi = e^{-2x^2 - 4x^2 - 6y} D,
\]

where \( C = (c_1, c_2, \cdots, c_{N+M+2})^T \) and \( D = (d_1, d_2, \cdots, d_{N+M+2})^T \) are real constant vectors. Thus, we have the following result.

**Theorem 3.** \( A = (a_{ij}) \) is an \( (N + M + 2) \times (N + M + 2) \) arbitrary real matrix independent of \( x \) and \( t \). Equation (2.3) has double Wronskian solution (3.12), where \( \varphi \) and \( \psi \) are constructed by (3.19). The corresponding solution of Equation (2.1) can be expressed as

\[
p = 2 \frac{W^{N+2}}{W^{N+2} M + 1} (\varphi; \psi), \quad q = -2 \frac{W^{N+2} M + 1}{W^{N+2} M + 1} (\varphi; \psi), \quad u = 2 \frac{\ln W^{N+2} M + 1 (\varphi; \psi)}{2}.
\]

4. Rational Solutions

In the section, we will give rational solutions of the \((2 + 1)\)-dimensional AKNS Equation (2.1).

Expanding (3.19) leads to

\[
\begin{align*}
\varphi &= e^{-2x^2} e^{-4x^2 - 6y} C \sum_{i=0}^{1} \left[ \frac{1}{2} \sum_{i=0}^{N} \frac{(-1)^{i+j} 2^j}{i!(s-2i)!} (x+y)^{i-2j} \right] A^i C, \\
\psi &= e^{-2x^2} e^{-4x^2 - 6y} D \sum_{i=0}^{1} \left[ \frac{1}{2} \sum_{i=0}^{N} \frac{2^j}{i!(s-2i)!} (x+y)^{i-2j} \right] A^i D.
\end{align*}
\]

If

\[
A = \begin{pmatrix}
k_1 & 0 & \cdots & 0 \\
0 & k_2 & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
0 & \cdots & 0 & k_{N+M+2}
\end{pmatrix}, \quad k_i \neq k_j (i \neq j),
\]

we can obtain solution solutions of Equation (2.3), where

\[
\varphi_j = c_j e^{-2x^2 - k_{j+1} y - k_{j+1}^2}, \quad \psi_j = d_j e^{-2x^2 - k_{j+1} y - k_{j+1}^2} (j = 1, 2, \cdots, N + M + 2).
\]
If

\[
A = \begin{pmatrix}
0 & 0 & 0 \\
1 & 0 & \cdots \\
0 & 1 & 0
\end{pmatrix},
\]

it is obvious to know that \( A^{N+M+2} = 0 \). Thus (4.1) can be truncated as

\[
\varphi = \sum_{s=0}^{N+M+1} \left[ \sum_{j=0}^{s} \frac{(-1)^{s-j} 2^j}{j!} (x+y)^{s-2j} \right] A^s C, \tag{4.5a}
\]

\[
\psi = \sum_{s=0}^{N+M+1} \left[ \sum_{j=0}^{s} \frac{2^j}{j!} (x+y)^{s-2j} \right] A^s D. \tag{4.5b}
\]

The components of \( \varphi \) and \( \psi \) are

\[
\varphi_j = c_j - c_{j-1} (x+y) + c_{j-2} \left( -2t + \frac{(x+y)^2}{2} \right) + \cdots + c_{j-1} \sum_{j=0}^{j-1} \left[ \frac{(-1)^{j-1} 2^j}{j!} \right] (x+y)^{j-2j}, \tag{4.6a}
\]

\[
\psi_j = d_j + d_{j-1} (x+y) + d_{j-2} \left( 2t + \frac{(x+y)^2}{2} \right) + \cdots + d_{j-1} \sum_{j=0}^{j-1} \left[ \frac{2^j}{j!} \right] (x+y)^{j-2j}, \tag{4.6b}
\]

(\( j = 1, 2, \cdots, N+M+2 \)).

In (4.6), taking \( c_i = d_i = 1 \), \( c_k = d_k = 0 \) (\( k = 2, 3, \cdots, N+M+2 \)), then (4.6) becomes

\[
\varphi_j = \left[ \sum_{t=0}^{j-1} \frac{(-1)^{j-1-t} 2^t}{t!} \right] (x+y)^{j-2j}, \quad \psi_j = \left[ \sum_{t=0}^{j-1} \frac{2^t}{t!} \right] (x+y)^{j-2j}. \tag{4.7}
\]

Thus, we can calculate some rational solutions of Equation (2.1).

\[
p = -\frac{1}{x+y}, \quad q = -\frac{1}{x+y}, \quad u = \frac{2}{x+y}, \tag{4.8}
\]

\[
p = \frac{1}{(x+y)^2 + 2t}, \quad q = \frac{(x+y)^2 - 2t}{(x+y)^2 + 2t}, \quad u = \frac{x+y}{(x+y)^2 + 2t}, \tag{4.9}
\]

\[
p = \frac{2t + (x+y)^2}{2t - (x+y)^2}, \quad q = \frac{1}{2t - (x+y)^2}, \quad u = \frac{x+y}{2t - (x+y)^2}. \tag{4.10}
\]

5. Matveev Solutions

In the following, we will discuss Matveev solutions of the \((2 + 1)\)-dimensional AKNS equation.

Let \( A \) be a Jordan matrix

\[
A = \begin{pmatrix}
J(k_1) & 0 & \cdots \\
& J(k_2) & \cdots \\
& & \ddots
\end{pmatrix}_{(N+M+2) \times (N+M+2)}. \tag{5.1}
\]
Without loss of generality, we observe the following Jordan block (dropping the subscript of $k$)

$$J(k) = \begin{pmatrix} k & 1 & 0 \\ 1 & k & 0 \\ 0 & 1 & k \end{pmatrix}_{l \times l}, \quad E_i = \begin{pmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{pmatrix}_{l \times l},$$

(5.2)

where $I_l$ is an $l \times l$ unite matrix. We have

$$J^*(k) = (kI_l + E_i) \begin{pmatrix} 1 \\ \frac{\partial}{\partial k} \\ \frac{1}{2!} \frac{\partial^2}{\partial k^2} \\ \frac{1}{3!} \frac{\partial^3}{\partial k^3} \end{pmatrix} + \cdots + \frac{1}{s!} \frac{\partial^s}{\partial k^s},$$

(5.3a)

i.e.,

$$J^*(k) = T_k k^s, \quad T_k = \begin{pmatrix} 1 \\ \frac{\partial}{\partial k} \\ \frac{1}{2!} \frac{\partial^2}{\partial k^2} \\ \frac{1}{3!} \frac{\partial^3}{\partial k^3} \end{pmatrix} + \cdots + \frac{1}{(l-1)!} \frac{\partial^{l-1}}{\partial k^{l-1}},$$

(5.3b)

Substituting (5.2) into (4.1), we get

$$\varphi(k) = T_k e^{-2k^2 t - kx y} C, \quad \psi(k) = T_k e^{-2k^2 t + kx y} D.$$ (5.4)

The components of $\varphi(k)$ and $\psi(k)$ are

$$\varphi_j(k) = \frac{1}{(j-1)!} \frac{\partial^{j-1}}{\partial k^{j-1}} e^{-2k^2 t - kx y}, \quad (j = 1, 2, \ldots, l),$$

(5.5a)

$$\psi_j(k) = \frac{1}{(j-1)!} \frac{\partial^{j-1}}{\partial k^{j-1}} e^{2k^2 t + kx y}, \quad (j = 2, 3, \ldots, l).$$

(5.5b)

Specially, taking $c_i = d_i = 1, \quad c_j = d_j = 0 (j = 2, 3, \ldots, l)$, then (5.5) becomes

$$\varphi_j(k) = \frac{1}{(j-1)!} \frac{\partial^{j-1}}{\partial k^{j-1}} e^{-2k^2 t - kx y}, \quad \psi_j(k) = \frac{1}{(j-1)!} \frac{\partial^{j-1}}{\partial k^{j-1}} e^{2k^2 t + kx y}. $$

(5.6)

Thus, Matveev solutions of Equation (2.1) can be obtained, where

$$\varphi = (\varphi_1(k), \ldots, \varphi_{l_1}(k); \varphi_2(k), \ldots, \varphi_{l_2}(k); \ldots; \varphi_{l_M}(k), \ldots, \varphi_{l_M}(k))^T,$$

(5.7a)

$$\psi = (\psi_1(k), \ldots, \psi_{l_1}(k); \psi_2(k), \ldots, \psi_{l_2}(k); \ldots; \psi_{l_M}(k), \ldots, \psi_{l_M}(k))^T,$$

(5.7b)

$$(l_1 + l_2 + \cdots + l_M = N + M + 2).$$

In (5.7), taking

$$\varphi = (\varphi_1(k), \varphi_2(k))^T, \quad \psi = (\psi_1(k), \psi_2(k))^T,$$

(5.8)

where $\varphi_j(k)$ and $\psi_j(k)$ are generated from (5.6), we can obtain the Matveev solution of Equation (2.1).

$$p = -\frac{1}{4kt + x + y} e^{4k^2 t - 2kx - 2ky}, \quad q = -\frac{1}{4kt + x + y} e^{4k^2 t + 2kx + 2ky}, \quad u = \frac{2}{4kt + x + y} e^{4k^2 t + 2kx + 2ky}.$$ (5.9)
Similarly, choosing
\[ \phi = (\varphi_1(k), \varphi_2(k), \varphi_3(k))^T, \quad \psi = (\psi_1(k), \psi_2(k), \psi_3(k))^T, \] (5.10)
and \((N,M)=(1,0)\), we get
\[ p = \frac{1}{2t + (4kt + x + y)^2} e^{-2kt - 2x - 2y}, \] (5.11a)
\[ q = \frac{-2t + (4kt + x + y)^2}{2t + (4kt + x + y)^2} e^{2kt + 2x + 2y}, \] (5.11b)
\[ u = \frac{4(3kt + x + y)}{2t + (4kt + x + y)^2} - \frac{2k(4kt + x + y)^2}{2t + (4kt + x + y)^2}. \] (5.11c)

When \((N,M)=(0,1)\), we have
\[ p = \frac{2t + (4kt + x + y)^2}{2t - (4kt + x + y)^2} e^{2kt - 2x - 2y}, \] (5.12a)
\[ q = \frac{1}{2t - (4kt + x + y)^2} e^{2kt + 2x + 2y}, \] (5.12b)
\[ u = \frac{4(3kt + x + y)}{2t - (4kt + x + y)^2} + \frac{2k(4kt + x + y)^2}{2t - (4kt + x + y)^2}. \] (5.12c)

Assume that
\[ \phi = (\varphi_1(k_1), \varphi_2(k_1), \varphi_3(k_2))^T, \quad \psi = (\psi_1(k_1), \psi_2(k_1), \psi_3(k_2))^T, \] (5.13)
letting \((N,M)=(1,0)\) gives
\[ p = \frac{-(k_1 - k_2)^2}{1 + 2(k_2 - k_1)(4kt + x + y)} e^{2k_1} - e^{2k_2}, \] (5.14a)
\[ q = \frac{-2}{1 + 2(k_2 - k_1)(4kt + x + y)} \left( e^{2k_2} - e^{-2k_1} \right), \] (5.14b)
\[ u = 2 \frac{k_2 - 2k_1 - 2k_2(k_2 - k_1)(4kt + x + y)}{1 + 2(k_2 - k_1)(4kt + x + y)} \left( e^{2k_1} + (2k_1 - k_2) e^{2k_2} \right). \] (5.14c)

Similarly, taking \((N,M)=(0,1)\) yields
\[ p = 2 \frac{1 + 2(k_1 - k_2)(4kt + x + y)}{1 + 2(k_1 - k_2)(4kt + x + y)} \left( e^{2k_1} - e^{2k_2} \right), \] (5.15a)
\[ q = -2 \frac{(k_1 - k_2)^2}{1 + 2(k_1 - k_2)(4kt + x + y)} \left( e^{2k_1} - e^{-2k_2} \right), \] (5.15b)
\[ u = 2 \frac{k_2 - 2k_1 - 2k_1((k_1 - k_2)(4kt + x + y))}{1 + 2(k_1 - k_2)(4kt + x + y)} \left( e^{2k_1} + (2k_1 - k_2) e^{2k_2} \right). \] (5.15c)

6. Complexions of the (2 + 1)-Dimensional AKNS Equation

In the following, we would like to consider that \(A\) is a real Jordan matrix.
\[ A = \begin{pmatrix} J_1 & 0 \\ J_2 & \ddots \\ 0 & \ddots & J_h \end{pmatrix}, \quad (6.1) \]

where
\[
J_i = \begin{pmatrix} A_i & 0 \\ I_2 & A_i \\ & \ddots & \ddots \\ 0 & I_2 & A_i \end{pmatrix}, \quad A_i = \begin{pmatrix} \alpha_i & -\beta_i \\ \beta_i & \alpha_i \end{pmatrix},
\]

and \( \alpha_i, \beta_i \quad (i = 1, 2, \cdots, h) \) are real constants. Then, from (4.1), complexitons can be obtained. In order to prove that, we first observe the simplest case when
\[
A = \begin{pmatrix} \alpha & -\beta \\ \beta & \alpha \end{pmatrix} = \alpha I_2 + \beta \sigma_2, \quad \sigma_2 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}. \quad (6.2)
\]

Substituting (6.2) into (4.1a) yields
\[
\varphi = e^{-\frac{1}{2} \left(4 \alpha^2 \beta^2 \sigma_2 + \sigma_2 \right)} e^{\frac{1}{2} \left(4 \alpha \beta + \beta \sigma_2 \right)} \varphi. \quad (6.3)
\]

Expanding the above \( \varphi \) and taking advantage of \( \sigma_2^2 = -I_2 \), we have
\[
\varphi = e^{-\frac{1}{2} \left(4 \alpha^2 \beta^2 \sigma_2 + \sigma_2 \right)} \left[ \cos \left(4 \alpha \beta t + \beta (x + y)\right) I_2 - \sin \left(4 \alpha \beta t + \beta (x + y)\right) \sigma_2 \right] \varphi. \quad (6.4a)
\]

Similarly,
\[
\psi = e^{-\frac{1}{2} \left(4 \alpha^2 \beta^2 \sigma_2 + \sigma_2 \right)} \left[ \cos \left(4 \alpha \beta t + \beta (x + y)\right) I_2 + \sin \left(4 \alpha \beta t + \beta (x + y)\right) \sigma_2 \right] \psi. \quad (6.4b)
\]

Further, we consider the matrix \( A \) as a Jordan block \( J_i \)
\[
A = J_i = A' + E', \quad (6.5)
\]

\[
A' = I_i \otimes A = \begin{pmatrix} A_i & 0 \\ I_2 & A_i \\ & \ddots & \ddots \\ 0 & I_2 & A_i \end{pmatrix}, \quad E' = E_i \otimes I_i = \begin{pmatrix} 0 & 0 \\ I_2 & 0 \\ & \ddots & \ddots \\ 0 & I_2 & 0 \end{pmatrix}, \quad (6.5b)
\]

where the symbol \( \otimes \) denotes tensor product of matrices. Noting that \( A' E' = E' A' \), we get
\[
A' = (A' + E')^p = \left( I_{2k} + E' \partial_{a_{1i}} + \cdots + \frac{1}{j!} E' \partial_{a_{1i}}^j + \cdots + \frac{1}{s!} E' \partial_{a_{1i}}^s \right) A'. \quad (6.6)
\]

Employing the following formula
\[
\alpha_i A_i^p = \alpha_i (\alpha_i I_2 + \beta \sigma_2)^p = p (\alpha_i I_2 + \beta \sigma_2)^p, \quad (p = 1, 2, 3, \cdots), \quad (6.7)
\]

then (6.6) can be written as
\[
A' = \begin{pmatrix} I_2 & \cdots & I_2 \\ I_2 \partial_{a_{1i}} & \cdots & I_2 \partial_{a_{1i}} \\ & \ddots & \ddots \\ & & \ddots \end{pmatrix}, \quad A'' = T \left( \partial_{a_{1i}} \right) A'. \quad (6.8)
\]
Substituting (6.8) into (4.1) yields

$$
\varphi_j(\alpha_i) = T(\partial_{\alpha_i}) e^{-2\Delta_t^\alpha \xi(\alpha+y)} C = T(\partial_{\alpha_i}) (I_{\alpha_i} \otimes e^{-2\Delta_t^\alpha \xi(\alpha+y)}) C,
$$

(6.9a)

$$
\psi_j(\alpha_i) = T(\partial_{\alpha_i}) e^{2\Delta_t^\alpha \xi(\alpha+y)} D = T(\partial_{\alpha_i}) (I_{\alpha_i} \otimes e^{2\Delta_t^\alpha \xi(\alpha+y)}) D,
$$

(6.9b)

or

$$
\varphi_j(\alpha_i) = \frac{1}{(j-1)!} \partial_{\alpha_i}^{-j+1} e^{-2\Delta_t^\alpha \xi(\alpha+y)} c_1 + \cdots + \partial_{\alpha_i}^{-j+1} e^{-2\Delta_t^\alpha \xi(\alpha+y)} c_j + e^{-2\Delta_t^\alpha \xi(\alpha+y)} c_j,
$$

(6.10a)

$$
\psi_j(\alpha_i) = \frac{1}{(j-1)!} \partial_{\alpha_i}^{-j+1} e^{2\Delta_t^\alpha \xi(\alpha+y)} d_1 + \cdots + \partial_{\alpha_i}^{-j+1} e^{2\Delta_t^\alpha \xi(\alpha+y)} d_j + e^{2\Delta_t^\alpha \xi(\alpha+y)} d_j,
$$

(6.10b)

where

$$
\varphi_j(\alpha_i) = (\varphi_{j_1}(\alpha_i), \varphi_{j_2}(\alpha_i))^T, \quad \varphi(\alpha_i) = (\varphi_1(\alpha_i)^T, \varphi_2(\alpha_i)^T, \cdots, \varphi_h(\alpha_i)^T)^T,
$$

$$
\psi_j(\alpha_i) = (\psi_{j_1}(\alpha_i), \psi_{j_2}(\alpha_i))^T, \quad \psi(\alpha_i) = (\psi_1(\alpha_i)^T, \psi_2(\alpha_i)^T, \cdots, \psi_h(\alpha_i)^T)^T,
$$

$$
c_j = (c_{j_1}, c_{j_2})^T, C = (c_1^T, c_2^T, \cdots, c_j^T)^T, d_j = (d_{j_1}, d_{j_2})^T, D = (d_1^T, d_2^T, \cdots, d_j^T)^T.
$$

According to (6.4), Equation (6.10) can be expressed as the following explicit form:

$$
\varphi_j(\alpha_i) = \frac{1}{(j-1)!} \partial_{\alpha_i}^{-j+1} \left[ e^{-(a_i^2-b^2)\xi(\alpha+y)} C \right],
$$

(6.11a)

$$
\psi_j(\alpha_i) = \frac{1}{(j-1)!} \partial_{\alpha_i}^{-j+1} \left[ e^{(a_i^2-b^2)\xi(\alpha+y)} D \right],
$$

(6.11b)

Thus, the double Wronskian (3.12) is the complextion of Equation (2.3), where

$$
\varphi = \left( \varphi_1(\alpha_i)^T, \cdots, \varphi_h(\alpha_i)^T \right)^T, \quad \psi = \left( \psi_1(\alpha_i)^T, \cdots, \psi_h(\alpha_i)^T \right)^T,
$$

$$
(l_i + l_2 + \cdots + l_h = N + M + 2).
$$

On the other hand, for \( \partial_{\alpha_i} A^\alpha = -\sigma_{\alpha} \partial_{\beta_i} A^\alpha \), the partial derivative with respect to \( \alpha_i \) can be replaced by the partial derivative with respect to \( \beta_i \) in (6.10) and (6.11).

For example, taking \( N = M = 0 \), \( \xi = 2(a^2 - b^2) + \alpha (x + y) \), \( \eta = 4a\beta \beta + \beta (x + y) \) (dropping the subscript) and \( \varphi = (e^i \cos \eta, -e^{-i} \sin \eta)^T \), \( \psi = (e^i \cos \eta, e^i \sin \eta)^T \), we have

$$
p = -2\beta \frac{e^{-(a^2-b^2)\xi-2a\eta+y)}}{\sin 2(4a\beta \beta + \beta (x + y))},
$$

(6.12a)
\[
q = -2\beta \frac{e^{\frac{1}{2}(u^2 - \beta^2) + 2\alpha(x + y)}}{\sin 2\left(4\alpha\beta t + \beta(x + y)\right)},
\]
(6.12b)

\[
u = 4\beta \cot 2\left(4\alpha\beta t + \beta(x + y)\right).
\]
(6.12c)

7. Conclusion

In this paper, we have obtained N-solution solutions and the generalized double Wronskian solution of the (2 + 1)-dimensional AKNS equation through the Hirota method and the Wronskian technique, respectively. Moreover, we have given rational solutions, Matveev solutions and complexitons of the (2 + 1)-dimensional AKNS equation. According to our knowledge, the three solutions are novel.

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