The Existence and Multiplicity of Solutions for Singular Boundary Value Systems with $p$-Laplacian

Zengxia Cai
School of Science, Linyi University, Linyi, China
Email: caizengxia64@163.com

Received January 2015

Abstract

This paper presents sufficient conditions for the existence of positive solutions for the fourth-order boundary value problem system with $p$-Laplacian operator. The existence of single or multiple positive solutions for the system is showed through the fixed point index theory in cones under some assumptions.

Keywords

Coupled Singular Boundary Value Problem, Positive Solution, Fixed Point Index Theorem

1. Introduction

In this paper, we are concerned with the existence and multiplicity of positive solutions for the system (BVP):

$$
\begin{align*}
\left( \phi_p \left( u^*(t) \right) \right)^{\prime} - a_1(t) f_1(u(t),v(t)) &= 0, \quad 0 < t < 1, \\
\left( \phi_p \left( v^*(t) \right) \right)^{\prime} - a_2(t) f_2(u(t),v(t)) &= 0, \quad 0 < t < 1, \\
u(0) &= u(1) = u^*(0) = u^*(1) = 0, \\
v(0) &= v(1) = v^*(0) = v^*(1) = 0,
\end{align*}
$$

where $\phi_p(s) = \left| s \right|^{p-2}s$, $s > 1$, $a_i(t) \in C\left( (0,1),(0,\infty) \right)$ and $f_i(u,v) \in C\left( [0,\infty) \times [0,\infty) \right)$, $a_i(t)$ is allowed to have singularity at $t = 0,1$, $i = 1,2$.

Several papers ([1]-[4]) have studied the solution of fourth-order boundary value problems. But results about fourth-order differential equations with $p$-Laplacian have rarely seen. Recently, several papers ([6]-[8]) have been devoted to the study of the coupled boundary value problem.

Motivated by the results mentioned above, here we establish some sufficient conditions for the existence of to (BVP) (1.1) under certain suitable weak conditions. The main results in this paper improve and generalize the results by others.

The following fixed-point index theorem in cones is fundamental.

**Theorem A** [9] Assume that $X$ is a Banach space, $K \subseteq X$ is a cone in $X$, and $0 < r < +\infty$, $\Omega_r = \{ x \in k : \|x\| \leq r \}$, if $T : \Omega_r \rightarrow X$ is a completely operator and $Tx \neq x$, $\forall x \in \partial \Omega_r$.

1) If for $\forall u \in \partial \Omega_r$, $\|u\| \leq \|Tx\|$, then $i(T, \Omega_r, K) = 0$.
2) If for $\forall u \in \partial \Omega_r$, $\|u\| \geq \|Tx\|$, then $i(T, \Omega_r, K) = 1$.

### 2. Preliminaries and Lemmas

In this paper, let $E = C([0,1])$ and $E^+ = \{ u \in E; u(t) \geq 0 \}$ is a concave function, $E^+ \times E^+$ is a Banach space with the norm $\|(u,v)\|_0 = \|u\| + \|v\|$, $\forall (u,v) \in E^+ \times E^+$, where $\|u\|_0 = \max \{ |u(t)| \}$, $\|v\|_0 = \max \{ |v(t)| \}$, then $X := E^+ \times E^+$ is a cone of $E \times E$. In this paper, $(u_1, v_1) \geq (u_2, v_2)$ i.e. $u_1 \geq u_2$, $v_1 \geq v_2$.

Suppose $G(t,s)$ is the Green function of the following boundary problem: $z = 0$, $0 < t < 1$, $z(0) = z(1) = 0$, then $G(t,s) = \begin{cases} s(1-t), & 0 \leq s \leq t \leq 1, \\ t(1-s), & 0 \leq t \leq s \leq 1, \end{cases}$

Obviously, $(1-t)s(1-s) \leq G(t,s) = G(s,t) \leq t(1-t)$, $0 \leq t$, $s \leq 1$.

Define a cone $K \subseteq X$ as follows $K = \{(u,v) \in X; [u,v] \geq (0,0), u(t) + v(t) \geq \|u,v\|, (1-t), t \in [0,1]\}$ and define an integral operator $A : K \rightarrow K$ by $A(u(t),v(t)) = (A_{u,v}(t), A_{u,v}(t))$, where

$$ A_i(u,v)(t) = \int_0^t G(s,t) \phi_t \left( \int_0^s G(s,\tau) a_i(\tau) f_i(u(\tau),v(\tau)) d\tau \right) ds, \quad i = 1, 2 $$

Let us list the following assumptions for convenience.

**Lemma 2.1** $(u,v)$ is a solution of BVP (1.1) if and only if $(u,v) \in K$, $A(u,v) = (u,v)$ has fixed points. It is easy to see that $(u,v) \in K$, $A(u,v) = (u,v)$ if $(u,v)$ is a solution of BVP (1.1).

**Lemma 2.2** Suppose that $(H)$ hold, then $AK \subseteq K$.

**Lemma 2.3** Suppose that $H$ hold. Then $A : K \rightarrow K$ is completely continuous.

**Proof** Firstly, assume $D \subseteq K$ is a bounded set, we have

$$ A_i(u,v)(t) \leq \int_0^t G(s,t) \phi_t \left( \int_0^s G(s,\tau) a_i(\tau) f_i(u(\tau),v(\tau)) d\tau \right) ds \leq \sup f_i(u,v) \epsilon = \infty, \quad \epsilon = 0 $$

Then $A_i(D) = \{ i = 1, 2 \}$ is bounded, therefore $A(D)$ is bounded.

Secondly, suppose $(u_n, v_n), (u_0, v_0) \in D$, $(u_n, v_n) \rightarrow (u_0, v_0)$ then $(u_n, v_n)$ is bounded, we get

$$ A(u_n, v_n)(t) - A(u_0, v_0)(t) $$

$$ \leq \int_0^t G(s,t) \phi_t \left( \int_0^s G(s,\tau) a_1(\tau) f_1(u_n(\tau),v_n(\tau)) d\tau \right) ds $$

$$ + \int_0^t G(s,t) \phi_t \left( \int_0^s G(s,\tau) a_2(\tau) f_2(u_n(\tau),v_n(\tau)) d\tau \right) ds $$

$$ \leq \frac{1}{4} \max_{0 \leq s \leq t} f_1^{\epsilon-1}(u_n(t),v_n(t)) - f_2^{\epsilon-1}(u_n(t),v_n(t)) \epsilon $$

Due to the continuity of $f_1, f_2$, by $H$ and above formula together with Lebesgue Dominated Convergence Theorem, then $A(u_n, v_n)(t) - A(u_0, v_0)(t) \rightarrow 0$ when $n \rightarrow \infty$. Therefore $A$ is continuous.

Lastly, since $G(t,s)$ is continuous in $[0,1] \times [0,1]$, so it is uniformly continuous. For all $\epsilon > 0$, $\exists \delta > 0$ for all $s \in [0,1]$, when $|t_1 - t_2| < \delta$, we get
\[ |G(t_i, s) - G(t_j, s)| < \varepsilon \left\lfloor \frac{\varepsilon}{2} \left( \phi_i \left( \int_0^T G(\tau, \tau) a_i(\tau) d\tau \right) + \phi_j \left( \sup_{(u, v) \in D} f_i(u, v) \right) \right) \right\rfloor, \quad i = 1, 2 \]

Then for all \((u, v) \in D\), we have
\[
\begin{align*}
|A(u, v)(t_i) - A(u, v)(t_j)| &\leq \int_0^T \left| G(t_i, s) - G(t_j, s) \right| \phi_i \left( \int_0^T G(\tau, \tau) a_i(\tau) d\tau \right) d\tau \cdot \phi_j \left( \sup_{(u, v) \in D} f_i(u, v) \right) ds \\
&\quad + \int_0^T \left| G(t_i, s) - G(t_j, s) \right| \phi_j \left( \int_0^T G(\tau, \tau) a_j(\tau) d\tau \right) d\tau \cdot \phi_i \left( \sup_{(u, v) \in D} f_j(u, v) \right) ds \\
&< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon,
\end{align*}
\]

So \( A \) is equiconvergent, by Arzela-Ascoli theorem we know \( AD \) is relatively compact. Therefore, \( A : K_{r, r} \to K \) is completely continuous.

For convenience we denote
\[
\begin{align*}
&f_{i0}(u, v) := \lim_{u+v \to 0} \frac{f_i(u, v)}{(u+v)^{p-1}}, \quad f_{i\infty}(u, v) := \lim_{u+v \to \infty} \frac{f_i(u, v)}{(u+v)^{p-1}}, \quad i = 1, 2; \\
&f_i^{\infty}(u, v) := \lim_{u+v \to 0} \frac{f_i(u, v)}{(u+v)^{p-1}}, \quad f_i^\infty(u, v) := \lim_{u+v \to \infty} \frac{f_i(u, v)}{(u+v)^{p-1}}, \quad i = 1, 2; \\
&\mu_i := \phi_i \left( \int_0^1 G(\tau, \tau) a_i(\tau) d\tau \right), \quad i = 1, 2; \\
&\nu_i := \min_{\tau \in [0, 1]} \int_0^1 G(s, \tau) \phi_i \left( \int_0^1 G(\tau, \tau) a_i(\tau) d\tau \right) ds, \quad i = 1, 2.
\end{align*}
\]

3. Main Results

**Theorem 3.1** Suppose that \( H \) holds. If the following conditions are satisfied:

\[
(H_1) \quad f_i^0(u, v) = 0, \quad i = 1, 2; \quad (H_2) \quad f_i(\infty, \infty) = \infty \quad \text{or} \quad f_{i+\infty}(u, v) = \infty
\]

Then the system (1.1) has at least one positive solution \((u(t), v(t))\), \( t \in (0, 1) \)

**Proof** By Lemma 2.3, we know \( A \) is completely continuous. By \( (H_1) \), there exists \( R > 0 \), when

\[
0 \leq u(t) + v(t) \leq r, \quad t \in [0, 1], \quad \text{we have} \quad f_i(u, v) \leq \left( \alpha_i(u+v) \right)^{p-1}, \quad \text{where} \quad \alpha_i > 0 (i = 1, 2)
\]

satisfies

\[
\max \left\{ \alpha_i, \mu_i, \sigma_i \right\} \leq 3.
\]

Let \( \Omega_1 = \left\{ (u, v) \in K \left| \|u,v\| < r \right\} \right. \), when \( (u, v) \in \partial \Omega_1 \cap K \), we get

\[
A_i(u(t), v(t)) \leq \frac{\alpha_i}{6} \|u,v\| + \mu_i \left( \int_0^1 G(\tau, \tau) a_i(\tau) d\tau \right) \leq \frac{\alpha_i}{6} \|u,v\| + \mu_i \|u,v\| = \frac{\alpha_i}{6} \|u,v\| + \frac{\alpha_i}{6} \|u,v\| = \frac{\alpha_i}{3} \|u,v\|
\]

Hence, \( \|A(u, v)\| \leq \frac{\|u, v\|}{2} \). Similarly, we have \( \|A_i(u(t), v(t))\| \leq \frac{\|u, v\|}{2} \), then \( \|A_i(u, v)\| \leq \frac{\|u, v\|}{2} \), therefore

\[
\left\{ \begin{array}{l}
\|A(u, v)\| = \|A_1(u, v)\| + \|A_0(u, v)\| = \|A_1(u, v)\| \leq \frac{\|u, v\|}{2}, \quad \forall (u, v) \in \partial \Omega_1 \cap K.
\end{array} \right.
\]

On the other hand, from \( (H_2) \), if \( f_i(\infty, \infty) = \infty \), there exists \( R_0 > R > 0 \), for \( \beta_i > 0 \) satisfying \( \beta_i \nu_i \geq 8 \), we get \( f_i(u, v) \geq (\beta_i(u+v))^{p-1} \) when \( u(t) + v(t) \geq R_0 \). Set \( R_1 > R_0 \) such that \( R_0 \leq \|u\| + \|v\| \leq R_1 \), let

\[
\Omega_2 = \left\{ (u, v) \in K \left| \|u,v\| < R_0 \right\} \right. \), when \( (u, v) \in \partial \Omega_2 \cap K \), \( t \in \left[ \frac{1}{2}, 1 \right] \), we get

\[
\frac{1}{8} \|u,v\| \geq \frac{1}{2} \|u,v\|, \quad \text{so}
\]

\[
A_i(u(t), v(t)) \geq \frac{\beta_i}{8} \|u, v\| \cdot \frac{1}{2} \|G(t, s) \phi_i \left( \int_0^1 G(\tau, \tau) a_i(\tau) d\tau \right) ds \geq \frac{\beta_i}{8} \|u, v\| \nu_i \geq \|u, v\|
\]
Hence, \( \|A_i(u,v)\| \geq \|u,v\| \), then \( \|A(u,v)\|_0 = \|A_1(u,v)\| + \|A_2(u,v)\| \geq \|u,v\| \), \( \forall (u,v) \in \partial \Omega_{R_1} \cap K \).

If \( f_{2_0}(u,v) = \infty \), with the similar proofs of the condition \( f_{1_0}(u,v) = \infty \), we get \( \|A_i(u,v)\| \geq \|u,v\| \). Then \( \|A(u,v)\|_0 = \|A_1(u,v)\| + \|A_2(u,v)\| \geq \|u,v\| \), \( \forall (u,v) \in \partial \Omega_{R_1} \cap K \). In either case, we always may set \( \|A(u,v)\|_0 \geq \|u,v\| \), \( \forall (u,v) \in \partial \Omega_{R_1} \cap K \). By Theorem A, \( i(A,\Omega_{R_1} \cap K, K) = 0 \). Through the additivity of the fixed point index we know that

\[ i(A,\Omega_{R_1} \cap K), (\Omega_{R_1} \cap K) = 0 \]

Therefore it follows from the fixed-point theorem that \( A \) has a fixed point \( (u,v) \in (\Omega_{R_1} \cap K), (\Omega_{R_1} \cap K) \), and thus \( (u(t), v(t)) \), \( t \in (0,1) \) is a positive solution of BVP (1.1).

**Theorem 3.2** Suppose that \( H \) holds. If the following conditions are satisfied:

\[ (H_3) \ f^{\infty}_i(u,v) = 0, \quad i = 1, 2; \quad (H_4) \ f_{10}(u,v) = \infty \quad \text{or} \quad f_{20}(u,v) = \infty, \]

Then the system (1.1) has at least one positive solution \( (u(t), v(t)) \), \( t \in (0,1) \).

**Proof** By lemma 2.3, we know \( A \) is completely continuous. From \( (H_3) \), if \( f_{10}(u,v) = \infty \), for \( \xi > 0 \) satisfying \( \xi \nu_v \geq 8 \), there exists \( r' > 0 \), such that \( 0 \leq u(t) + v(t) \leq r', \quad t \in [0,1] \), we have \( f_i(u,v) \geq \left( \xi (u+v) \right)^{p-1} \). Let \( \Omega_{r'} = \{(u,v) \in K; \|u,v\| < r' \} \), when \( (u,v) \in \partial \Omega_{r'} \cap K \), \( t \in \left[ 1, 1 - \frac{1}{4} \right] \), we get

\[ u + v \geq t(1-t)\|u,v\| \geq \frac{1}{8}\|u,v\| \]

\[ A_i(u(t), v(t)) \geq \frac{\xi}{8}\|u,v\| \int_0^1 G(t,s) \phi_i \left( \int_0^s G(s,\tau) a_i(\tau) d\tau \right) ds \geq \frac{\xi}{8}\|u,v\| \]

Hence, \( \|A_i(u,v)\| \geq \|u,v\| \), then \( \|A(u,v)\|_0 = \|A_1(u,v)\| + \|A_2(u,v)\| \geq \|u,v\| \), \( \forall (u,v) \in \partial \Omega_{R_1} \cap K \).

If \( f_{20}(u,v) = \infty \), take \( \xi > 0 \) satisfying \( \xi \nu_v \geq 8 \), such that \( f_{20}(u,v) \geq \left( \xi (u+v) \right)^{p-1} \). Similarly, we get \( \|A_2(u,v)\| \geq \|u,v\| \), then \( \|A(u,v)\|_0 = \|A_1(u,v)\| + \|A_2(u,v)\| \geq \|u,v\| \), \( \forall (u,v) \in \partial \Omega_{R_1} \cap K \). In either case, we always may set \( \|A(u,v)\|_0 \geq \|u,v\| \), \( \forall (u,v) \in \partial \Omega_{R_1} \cap K \). By Theorem A, \( i(A,\Omega_{R_1} \cap K, K) = 0 \).

On the other hand, from \( (H_3) \), there exists \( R''_i > r'_i \) such that \( f_i(u,v) \leq (\theta_i(u+v))^{p-1} \), when \( u+v \geq R_i \), where \( \theta_i > 0 (i = 1, 2) \) satisfies \( \max \{\theta_1, \mu_1, \mu_2\} \leq 3 \). There are two cases to consider.

Case (i). Suppose that \( \max_{1 \leq i \leq 3} f_i(u,v) \) is bounded, then there exists \( M_i > 0 \) satisfying \( f_i(u,v) \leq M_i^{p-1} \), \( i = 1, 2 \). Taking \( R''_i > \max \{R''_1, \frac{M_i}{3} \mu_1, \frac{M_i}{3} \mu_2\} \), let \( \Omega_{R''_i} = \{(u,v) \in K; \|u,v\| < R''_i \} \), when \( (u,v) \in \partial \Omega_{R''_i} \cap K \), we get

\[ A_i(u(t), v(t)) \leq \frac{M_i}{6} \phi_i \left( \int_0^1 G(t,\tau) a_i(\tau) d\tau \right) \leq \frac{\|u,v\|}{2} \]

Hence, \( \|A_i(u,v)\| \leq \frac{\|u,v\|}{2} \). Similarly, we have \( A_2(u(t), v(t)) \leq \frac{\|u,v\|}{2} \), hence \( \|A_i(u,v)\| \leq \frac{\|u,v\|}{2} \), then \( \|A(u,v)\|_0 = \|A_1(u,v)\| + \|A_2(u,v)\| \leq \|u,v\| \), \( \forall (u,v) \in \partial \Omega_{R''_i} \cap K \).

Case (ii). Suppose that \( \max_{1 \leq i \leq 3} f_i(u,v) \) is unbounded, since \( f_i(u,v) \) is continuous in \( [0,\infty) \times [0,\infty) \), so there exists constant \( R_i \geq R''_i \) and two points \( (u,v) \) in \( [0,\infty) \times [0,\infty) \) such that \( R''_i \leq R_i \), and \( f_i(u,v) \leq f_i(u,v) \). Then we get \( f_i(u,v) \leq f_i(u,v) \leq (\theta_i(u+v))^{p-1} \leq (\theta_i R_i)^{p-1}, i = 1, 2 \). Let \( \Omega_{R''_i} = \{(u,v) \in K; \|u,v\| < R''_i \} \), when \( (u,v) \in \partial \Omega_{R''_i} \cap K \), we get

\[ A_i(u(t), v(t)) \leq \frac{\theta_i R''_i}{6} \phi_i \left( \int_0^1 G(t,\tau) a_i(\tau) d\tau \right) \leq \frac{R''_i}{2} = \frac{\|u,v\|}{2} \]
Hence, \( \| A_i(u, v) \| \leq \left\| \frac{(u, v)}{2} \right\| \). Similarly, we have \( A_i(u(t), v(t)) \leq \frac{R^*_i}{2} = \left\| \frac{(u, v)}{2} \right\| \), then \( \| A_i(u, v) \| \leq \frac{\| (u, v) \|}{2} \), so \( \| A(u, v) \| = \| A_i(u, v) \| + \| A_i(u, v) \| \leq \| (u, v) \| , \forall (u, v) \in \Omega_{\eta_i} \cap K \). In either case, we always may set \( \| A(u, v) \| \leq \| (u, v) \| , \forall (u, v) \in \Omega_{\eta_i} \cap K \). By Theorem A, \( i (A, \Omega_{\eta_i} \cap K, K) = 1 \). Through the additivity of the fixed point index we know that

\[
i \left( A, \Omega_{\eta_i} \cap K \right) \left( \Omega_{\eta_i} \cap K, K \right) = i \left( A, \Omega_{\eta_i} \cap K, K \right) - i \left( A, \Omega_{\eta_i} \cap K, K \right) = 0 - 1 = -1
\]

Therefore it follows from the fixed-point theorem that \( A \) has a fixed point \( (u, v) \in \Omega_{\eta_i} \cap K \) and thus \( (u(t), v(t)), t \in (0, 1) \) is a positive solution of BVP (1.1). This completes the proof.

**Remark 3.1** Note that if \( f \) is superlinear or sublinear, our conclusions hold. Limit conditions of \( A \) such that \( (u, v) \) is a positive solution of BVP (1.1). This completes the proof.

**Theorem 3.3** Suppose that \( H \) holds. If the following conditions are satisfied:

\[
\begin{align*}
(H_1) & \quad f^{(i)}_i(u, v) = \rho_i \in [0, +\infty), i = 1, 2 \quad \text{satisfies } \max \left\{ \rho_i^{-1} \mu_i, \rho_i^{-1} \mu_i \right\} \leq 3; \\
(H_2) & \quad f_{2\infty}(u, v) = \lambda_2 \in [0, +\infty) \quad \text{or} \quad f_{2\infty}(u, v) = \lambda_2 \in (0, +\infty), \text{ where } \lambda_2 \text{ satisfies } \lambda_2^{x^i} v_i \geq 8 \quad (i = 1 \text{ or } i = 2)
\end{align*}
\]

then the system (1.1) has at least one positive solution \( (u(t), v(t)), t \in (0, 1) \).

**Proof.** Choosing \( \varepsilon_i > 0 (i = 1, 2) \) such that \( \max \left\{ \rho_i^{-1} \mu_i, \rho_i^{-1} \mu_i \right\} \leq 3 \) and \( (\lambda_i - \varepsilon_i)^{x^i} v_i \geq 8 \), \( i = 1 \) or \( i = 2 \). From \( (H_2) \), there exists \( R_i^* > 0 \) such that \( f_{1\infty}(u, v) \leq (\rho_i + \varepsilon_i)(u, v)^{x^i} \) \( (i = 1, 2) \) when \( 0 \leq u + v \leq R_i^* \). Let \( \Omega_{\eta_i} = \left\{ (u, v) \in K; \| (u, v) \| < R_i^* \right\} \). Then \( (u, v) \in \hat{\Omega}_{\eta_i} \cap K \), we get

\[
A_i(u(t), v(t)) \leq \left( \frac{\rho_i + \varepsilon_i}{6} \right) \| u + v \| \| G(t, u(t), v(t)) \| \| a_i(t) \| dt \leq \left( \frac{\rho_i + \varepsilon_i}{6} \right) \| (u, v) \| \| a_i \| \leq \left( \frac{\rho_i + \varepsilon_i}{6} \right) \| (u, v) \|
\]

Hence, \( \| A_i(u, v) \| \leq \left( \frac{\rho_i + \varepsilon_i}{6} \right) \| (u, v) \| \). Similarly, we have \( A_i(u(t), v(t)) \| A_i(u, v) \| \leq \left( \frac{\rho_i + \varepsilon_i}{6} \right) \| (u, v) \| \), then \( \| A(u, v) \| = \| A_i(u, v) \| + \| A_i(u, v) \| \leq \| (u, v) \| , \forall (u, v) \in \Omega_{\eta_i} \cap K \). By Theorem A, \( i (A, \Omega_{\eta_i} \cap K, K) = 1 \).

On the other hand, From \( (H_2) \), if \( f_{2\infty}(u, v) = \lambda_2 \), there exists \( R_i^* > r_i^* \) such that \( f_{2\infty}(u, v) \geq (\lambda_2 - \varepsilon_i)(u, v)^{x^i} \) when \( u(t) + v(t) \geq R_i^* \). Let \( R_i^* > R_i^* \) such that \( R_i^* \leq \| u \| + \| v \| \leq R_i^* \), let \( \Omega_{\eta_i} = \left\{ (u, v) \in K; \| (u, v) \| < R_i^* \right\} \). Then \( (u, v) \in \hat{\Omega}_{\eta_i} \cap K, \quad t \in \left[ \frac{1}{4}, \frac{1}{2} \right] \), we get

\[
u + \nu \geq (t - (1 - t)) \| (u, v) \| \| \geq \frac{1}{8} \| (u, v) \| ,
\]

\[
A_i(u(t), v(t)) \geq \left( \frac{\lambda_2 - \varepsilon_i}{8} \right) \| (u, v) \| \| G(t, u(t), v(t)) \| \| a_i(t) \| dt \| \| \geq \left( \frac{\lambda_2 - \varepsilon_i}{8} \right) \| (u, v) \| v_i \geq \| (u, v) \| \| a_i \|
\]

Hence, \( \| A_i(u, v) \| \geq \| (u, v) \| \), then \( \| A(u, v) \| = \| A_i(u, v) \| + \| A_i(u, v) \| \geq \| (u, v) \| , \forall (u, v) \in \Omega_{\eta_i} \cap K \).

If \( f_{2\infty}(u, v) = \lambda_2 \), by \( (\lambda_2 - \varepsilon_i)^{x^i} v_i \geq 8 \), with the similar proofs of the condition \( f_{2\infty}(u, v) = \lambda_2 \), we get \( \| A_i(u, v) \| \geq \| (u, v) \| \). Then \( \| A(u, v) \| = \| A_i(u, v) \| + \| A_i(u, v) \| \geq \| (u, v) \| , \forall (u, v) \in \Omega_{\eta_i} \cap K \). In either case, we always may set \( \| A(u, v) \| = \| (u, v) \| , \forall (u, v) \in \Omega_{\eta_i} \cap K \). By Theorem A, \( i (A, \Omega_{\eta_i} \cap K, K) = 0 \). Through the additivity of the fixed point index we know that

\[
i \left( A, \Omega_{\eta_i} \cap K \right) \left( \Omega_{\eta_i} \cap K, K \right) = i \left( A, \Omega_{\eta_i} \cap K, K \right) - i \left( A, \Omega_{\eta_i} \cap K, K \right) = 0 - 1 = -1
\]

Therefore it follows from the fixed-point theorem that \( A \) has a fixed point \( (u, v) \in \Omega_{\eta_i} \cap K \) and thus \( (u(t), v(t)), t \in (0, 1) \) is a positive solution of BVP (1.1). This completes the proof.

**Theorem 3.4** Suppose that \( H \) holds. If the following conditions are satisfied:
Theorem 3.5 Assume that $H$, $H_2$ holds. If the following conditions are satisfied:

\[(H_1) \quad f_{i0}(u,v) = 0, \quad i = 1, 2; \quad (H_{i0}) \quad f_{i}^{\infty}(u,v) = \infty \quad \text{or} \quad f_{2}^{\infty}(u,v) = \infty,
\]

Then the system (1.1) has at least two positive solutions \((u(t), v_1(t))\) and \((u(t), v_2(t))\) satisfying

\[0 < \| (u_1, v_1) \| \leq \eta_1 \leq \| (u_2, v_2) \|.
\]

Theorem 3.6 Assume that $H$, $H_3$, $H_{y}$, $H_{\theta}$ hold. Then the system (1.1) has at least two positive solutions \((u_i(t), v_i(t))\) and \((u_2(t), v_2(t))\) satisfying

\[0 < \| (u_1, v_1) \| \leq \eta_1 \leq \| (u_2, v_2) \|.
\]

Remark 3.3 Under suitable weak conditions, the multiplicity results for fourth-order singular boundary value problem with $p$-Laplacian are established. Our results extend and improve the results of [5]-[8].

References


