Necessity of Oversampling Theorem for Affine Frames

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ABSTRACT
Let \( a, n \geq 2 \) be two natural numbers. C. K. Chui and X. L. Shi proved that for any affine frame
\[
\psi_{b,j,k}(x) = a^{i/2}\psi(a^{j}x - kb), \quad j, k \in \mathbb{Z},
\]
of \( L^2(\mathbb{R}) \), and the family \( \{n^{-1/2}\psi_{b,a,j,k}, j, k \in \mathbb{Z}\} \) is also a frame with the same bounds if \( n \) is relatively prime to \( a \). In this paper we prove that \( n \) is relatively prime to \( a \) which is also necessary.

KEYWORDS
Affine Frame; Oversampling

1. Introduction
Let \( L^2 = L^2(\mathbb{R}) \) denote, as usual, the space of all complex-valued square integrable functions on the real line with inner product \( \langle \cdot, \cdot \rangle \) and norm \( \| \cdot \| \). For any \( \psi \in L^2 = L^2(\mathbb{R}) \), we will use the notation
\[
\psi_{b,j,k}(x) = a^{i/2}\psi(a^{j}x - kb), \quad j, k \in \mathbb{Z},
\]
where \( a > 1 \) and \( b > 0 \). A function \( \psi \in L^2 \) is said to generate an affine frame
\[
\{\psi_{b,j,k} : j, k \in \mathbb{Z}\}
\]
of \( L^2 \), with frame bounds \( A \) and \( B \), where \( 0 < A \leq B < \infty \), if it satisfies
\[
A\|f\|^2 \leq \sum_{j,k \in \mathbb{Z}} \left| \langle f, \psi_{b,j,k} \rangle \right|^2 \leq B\|f\|^2, \quad \forall f \in L^2.
\]
The frame (2) of \( L^2 \) is called a tight frame, if (3) holds with \( A = B \), see [1] and [2]. In 1993, C. K. Chui and X. L. Shi [3] proved the following oversampling theorem:

Theorem A. Let \( a \geq 2 \) be any positive integer and \( b > 0 \). Also, let \( \psi \in L^2 \) generate a frame
\[
\{\psi_{b,j,k} : j, k \in \mathbb{Z}\}
\]
with frame bounds \( A \) and \( B \) as given by (3). Then for any positive integer \( n \) which is relatively prime to \( a \), the family
\[
\{n^{-1/2}\psi_{b,a,j,k} : j, k \in \mathbb{Z}\}
\]
remains a frame of \( L^2 \) with the same bounds. If \( (n,a) \neq 1 \), this result does not hold. But they only gave a countexample for the case where \( a = 2, b = 1, n = 2 \) as in [4]. For other positive integer \( n \) and \( a \) which satisfy \( (n,a) \neq 1 \), they did not prove. The aim of this paper is to establish the inverse proposition of Theorem A, and then we following:
Theorem 1.1. Let $a \geq 2$ be any positive integer and $b > 0$. Also, let $\{\psi_{b,j,k} : j, k \in \mathbb{Z}\}$ be any affine frame of $L^2$ with frame bounds $A$ and $B$. The family (4) remains a frame of $L^2$ with the same bounds: that is,

$$nA\|f\|^2 \leq \sum_{j,k \in \mathbb{Z}} \left| \left\langle f, \psi_{b,j,k} \right\rangle \right|^2 \leq nB\|f\|^2, f \in L^2,$$

if and only if $n$ and $a$ are relatively prime.

2. Proofs

The sufficiency has been included in the theorem 4 of [3]. In the following we will prove the necessary part of the theorem.

Suppose for any affine frame (2) of $L^2$ with frame bounds $A$ and $B$, the family (4) is also a frame of $L^2$ with the same bounds. Then when (1) forms an orthonormal basis, the family (4) forms a tight frame with frame bound 1. So we just need to prove that there exists a function $\psi$ such that the family (1) forms the orthonormal basis, but for any two positive integers $n$ and $a$ which satisfy $(n,a) \geq 2$, there exist two functions $f_1$ and $f_2$ such that

$$S(f_1) = \sum_{j,k \in \mathbb{Z}} \left| \left\langle f_1, \psi_{b,j,k} \right\rangle \right|^2$$

doesn’t equal

$$S(f_2) = \sum_{j,k \in \mathbb{Z}} \left| \left\langle f_2, \psi_{b,j,k} \right\rangle \right|^2.$$

Let $\psi(x) = \psi_{H} (x) = \chi_{(0,1/2)} (x) \text{sgn} \left( \frac{1}{2} - x \right)$, then $\{\psi_{j,k} : j,k \in \mathbb{Z}\}$ forms an orthonormal basis, which is called Haar basis. Set

$$f_1(x) = \psi_{H} \left( x + \frac{1}{2} \right) \quad \text{and} \quad f_2(x) = \chi_{(1/2,1/2)} (x).$$

We prove that if $(n,a) = m \geq 2, m \in \mathbb{N}$, then

$$S(f_1) - S(f_2) \neq 0.$$

$$S(f_1) = \sum_{j,k \in \mathbb{Z}} \left| \left\langle f_1, \psi_{H,j,k} \right\rangle \right|^2$$

$$= \sum_{k=0}^{n/2} \left( \frac{1}{2a^n} \right)^2 + \sum_{k=1}^{n/2} \left( \frac{3k}{n} + \frac{5}{2} \right)^2 + \sum_{k=1}^{n/2} \left( \frac{3k}{n} + \frac{5}{2} \right)^2 + \sum_{k=1}^{n/2} \left( \frac{3k}{n} + \frac{5}{2} \right)^2$$

$$+ \sum_{k=0}^{n/2} \left( \frac{1}{2a^n} \right)^2 + \sum_{k=1}^{n/2} \left( \frac{3k}{n} + \frac{5}{2} \right)^2 + \sum_{k=1}^{n/2} \left( \frac{3k}{n} + \frac{5}{2} \right)^2 + \sum_{k=1}^{n/2} \left( \frac{3k}{n} + \frac{5}{2} \right)^2$$

$$+ \sum_{j=1}^{n/2} \frac{1}{2a^n} \left( \frac{1}{2a^n} \right)^2 + \sum_{j=1}^{n/2} \frac{1}{2a^n} \left( \frac{1}{2a^n} \right)^2 + \sum_{j=1}^{n/2} \frac{1}{2a^n} \left( \frac{1}{2a^n} \right)^2 + \sum_{j=1}^{n/2} \frac{1}{2a^n} \left( \frac{1}{2a^n} \right)^2$$

$$+ \sum_{j=1}^{n/2} \frac{1}{2a^n} \left( \frac{1}{2a^n} \right)^2 + \sum_{j=1}^{n/2} \frac{1}{2a^n} \left( \frac{1}{2a^n} \right)^2 + \sum_{j=1}^{n/2} \frac{1}{2a^n} \left( \frac{1}{2a^n} \right)^2 + \sum_{j=1}^{n/2} \frac{1}{2a^n} \left( \frac{1}{2a^n} \right)^2$$

and

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\[ S(f_2) = \sum_{j,k=2} \left[(f_2, \psi_{j,k}^n)\right]^2 = \sum_{k=0}^{n/2-1} \left(\frac{1}{2} - k \right)^2 n + \sum_{j=0}^{n/2-1} \left(\frac{1}{2} + k \right)^2 n \]

\[ + \sum_{j=0}^{n/2-1} \left(\frac{3}{2} + k \right)^2 n + \sum_{k=0}^{n/2-1} a^j \left(\frac{1}{2} - \frac{k}{a^j} \right)^2 n + \sum_{j=0}^{n/2-1} \left(\frac{3}{2} + \frac{k}{a^j} \right)^2 n \]

\[ + \sum_{j=0}^{n/2-1} \sum_{k=0}^{n/2-1} \left(\frac{1}{2} - \frac{k}{a^j} \right)^2 n + \sum_{j=0}^{n/2-1} a^j \left(\frac{1}{2} + \frac{k}{a^j} \right)^2 n \]

\[ + \sum_{j=0}^{n/2-1} a^j \left(\frac{1}{2} + \frac{k}{a^j} \right)^2 n + \sum_{j=0}^{n/2-1} a^j \left(\frac{1}{2} + \frac{k}{a^j} \right)^2 n \]

Denote \( \Delta = S(f_1) - S(f_2) \). We have

\[ \Delta = A_1 + A_2 + A_3 + A_4 + A_5 + A_6, \]

where

\[ A_1 = \sum_{j=0}^{n/2-1} \left(\frac{3k}{n} + \frac{3}{2} \right)^2 n + \sum_{j=0}^{n/2-1} \left(\frac{3k}{n} + \frac{5}{2} \right)^2 n - \sum_{k=0}^{n/2-1} \left(\frac{3k}{n} + \frac{1}{2} \right)^2 n, \]

\[ A_2 = \sum_{k=0}^{n/2-1} a^j \left(\frac{1}{2} + \frac{k}{a^j} \right)^2 n + \sum_{k=0}^{n/2-1} a^j \left(\frac{1}{2} - \frac{k}{a^j} \right)^2 n, \]

\[ A_3 = \sum_{k=0}^{n/2-1} a^j \left(\frac{1}{2} + \frac{k}{a^j} \right)^2 n + \sum_{k=0}^{n/2-1} a^j \left(\frac{1}{2} - \frac{k}{a^j} \right)^2 n \]

\[ A_4 = \sum_{k=0}^{n/2-1} a^j \left(1 - \frac{1}{a^j} - \frac{2k}{a^j} \right)^2 n + \sum_{k=0}^{n/2-1} a^j \left(1 + \frac{1}{a^j} + \frac{2k}{a^j} \right)^2 n \]

\[ - \sum_{k=0}^{n/2-1} a^j \left(\frac{1}{a^j} + \frac{2k}{a^j} \right)^2 n - \sum_{k=0}^{n/2-1} a^j \left(\frac{1}{a^j} + \frac{2k}{a^j} \right)^2 n, \]

\[ A_5 = - \sum_{k=0}^{n/2-1} a^j + \sum_{k=0}^{n/2-1} a^j, \]

and

\[ A_6 = \sum_{k=0}^{n/2-1} a^j \left(\frac{1}{2} - \frac{1}{a^j} - \frac{k}{a^j} \right)^2 n + \sum_{k=0}^{n/2-1} a^j \left(\frac{1}{2} + \frac{1}{a^j} + \frac{k}{a^j} \right)^2 n. \]

In order to prove the theorem, we have three cases.
Case 1. When $a = n$.

We have $\left\lfloor a'n/2 \right\rfloor = 0$ if $j \leq -1$. Thus, if $n$ is an even integer, we can get

$$A_1 = \frac{n}{6} + \frac{4}{3n}; A_2 = \frac{n}{3(a - 1)} - \frac{2}{3a(a - 1)}; A_3 = 0; A_4 = \sum_{j=1}^{n} a'^j; A_5 = \sum_{j=1}^{n-1} (2-n)a'^j; A_6 = 0$$

So, we have

$$\Delta = \frac{n^2 - 5n + 18}{6(n - 1)} + \frac{4}{3n} + \frac{2}{3n(n - 1)} > 0.$$ 

If $n$ is an odd integer, we have

$$A_1 = \frac{n}{6} - \frac{1}{6n}; A_2 = \frac{n}{3(a - 1)} - \frac{1}{3a(a - 1)}; A_3 = 0; A_4 = \frac{1}{n}; A_5 = \sum_{j=2}^{n} (1-n)a'^j + \frac{2}{n} = \frac{1}{n}; A_6 = 0.$$ 

So, we have

$$\Delta = \frac{n^2 - 6n + 9}{6n} + \frac{n^2 - 5n + 4}{3n(n - 1)} \neq 0.$$ 

Case 2. When $a \geq n + 1$.

If $n$ is an even integer, we have

$$A_1 = \frac{n}{6} + \frac{4}{3n}; A_2 = \frac{n}{3(a - 1)} + \frac{2}{5n(a - 1)}; A_3 = 0; A_4 = \sum_{j=1}^{n} a'^j; A_5 = \sum_{j=1}^{n} (2-n)a'^j; A_6 = 0.$$ 

Thus

$$\Delta = \frac{an^2 - 5n^2 + 18n + 8a - 4}{6n(a - 1)} > \frac{n^2 - 5n + 7 + 19n - 4}{6n(a - 1)} > 0.$$ 

If $n$ is an odd integer, we can get $n \geq 3$ because of $(n,a) \geq 3$. As in the case 1, we also have

$$A_1 = \frac{n}{6} - \frac{1}{6n}; A_2 = \frac{n}{3(a - 1)} - \frac{1}{3a(a - 1)}; A_3 = 0; A_4 = 0; A_5 = \frac{1-n}{a-1}; A_6 = 0.$$ 

So, we get

$$\Delta = \frac{an^2 - 5n^2 + 6n - a - 1}{6n(a - 1)} \geq \frac{n(n - 2)^2 + n - 2}{6n(a - 1)} > 0.$$ 

Case 3. When $a < n$.

If $n$ is an even integer. Let

$$\Lambda_1 = \{ j \leq -1; j \in Z \text{ and } a'n/2 \text{ is a positive integer} \}$$

and

$$\Lambda_2 = \{ j \leq -1; j \in Z \text{ and } a'^j/2 \text{ is a not positive integer} \}$$

When $\Lambda_1 \neq \emptyset$, there exists an integer $J$ satisfying $J \in \Lambda_1, J-1 \in \Lambda_2$. Therefore we have

$$A_1 + A_4 + A_5 + A_6 = \sum_{j=J-1}^{3} (3a^2/2 - a'^j) + \sum_{j=J}^{n} f_j(x_j) + \sum_{j=J}^{n-1} (3-n)a'^j,$$

where $x_j = \left\lfloor a'^j/2 \right\rfloor, f_j(x_j) = -\frac{6}{n}x_j^2 - \left(\frac{a'n-1}{a'n}\right)x_j$. When $\frac{1}{2}a'n < 1$, we have $\frac{1}{2}a'n > a'n - 1$ and $f_j\left(\frac{1}{2}a'n\right) < 0$. Thus we have
\[
\sum_{j \leq -1} f_j(x_j) = \sum_{j \leq -1} f_j(x_j) \geq \sum_{j \leq -1} f_j \left( \frac{1}{2} a^j a^j \right) > \sum_{j \leq -1} f_j \left( \frac{1}{2} a^j a^j \right) = \sum_{j \leq -1} -3a^j + \frac{3}{2} \ n \ a^j.
\]

Therefore
\[
\Delta \geq \frac{n^2 (a - 2)^2 + (a + 2)^2 + 7a^2 - 8}{6n (a^2 - 1)} > 0.
\]

When \( \Lambda_1 = \emptyset \), similar to the case \( \Lambda_1 \neq \emptyset \), we also have
\[
\sum_{j \leq -1} f_j(x_j) = \sum_{j \leq -1} f_j(x_j) \geq \sum_{j \leq -1} f_j \left( \frac{1}{2} a^j a^j \right) > \sum_{j \leq -1} f_j \left( \frac{1}{2} a^j a^j \right) = \sum_{j \leq -1} -3a^j + \frac{3}{2} \ n \ a^j.
\]

So we have
\[
\Delta \geq \frac{n (a - 2)^2}{6(a^2 - 1)} + \frac{4}{3n} + \frac{2}{3n (a - 1)} > 0.
\]

If \( n \) is an odd integer. We have
\[
A_1 + A_4 + A_2 + A_6 = \sum_{j \leq -1} g_j(x_j) + h_j(y_j) + a^j - na^j
\]

where
\[
\begin{align*}
x_j &= \left[ a^j n/2 \right], g_j(x_j) = -\frac{2}{n} \left[ x^j_j - (a^j n - 1) x^j_j \right], \\
h_j &= \left[ a^j n/2 + 1/2 \right], h_j(y_j) = -\frac{4}{n} \left( y^j_j - a^j n \ y^j_j \right).
\end{align*}
\]

A familiar calculation shows
\[
\Delta > -\frac{1}{n} \log n + \frac{n}{48} - \frac{7}{n} > 0.
\]

Since \( (a, n) \geq 3 \) and \( a < n \), we have \( n \geq 9, a \geq 3 \). Also when \( n = 9 \) and \( a = 3 \), we have
\[
\Delta = \frac{14}{27} > 0.
\]

When \( n = 9 \) and \( a = 6 \), obviously we have
\[
\Delta \geq \frac{10}{27} > 0.
\]

When \( n \geq 15 \), \( -\frac{1}{n} \log n + \frac{n}{48} - \frac{7}{6n} > \frac{5}{144} \). So we have \( \Delta > 0 \) in this case. This completes the proof of the theorem.

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