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Periodic Orbits of the First Kind in the CR4BP When the Second Primary Is a Triaxial Rigid Body

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Abstract

The present paper deals with the existence of periodic orbits in the Circular Restricted Four-Body Problem (CR4BP) in two-dimensional co-ordinate system when the second primary is a triaxial rigid body and the third primary of inferior mass (in comparison of the other primaries) is placed at triangular libration point L_4 of the Circular Restricted Three-Body Problem (CR3BP). With the help of generating solutions, we formed a basis for the existence of periodic orbits, then an analytical approach given by Hassan *et al.* [1], was applied to our model of equilateral triangular configuration. It is found that in general solution also; the character of periodic orbits is conserved. For verification of the existence of periodic orbits, we have applied the criterion of Duboshin [2] and found satisfied.

Keywords

Autonomous Four-Body Problem, CR4BP, Triaxial Rigid Body, Regularization, Generating and General Solution, Periodicity

1. Introduction

Giacaglia [3] applied the method of analytic continuation to examine the existence of periodic orbits of collision of the first kind in the CR3BP. Bhatnagar [4] generalized the problem in elliptic case. Further Bhatnagar [5] extended the work of Giacaglia [3] in the CR4BP by considering three primaries at the vertices of an equilateral triangle. In last three decades, a series of works have been performed by different authors with different perturbations in the circular and elliptic restricted three-body and four-body problem but nobody established the proper mathematical model of the Restricted Four-Body Problem (R4BP).

Recently Ceccaroni and Biggs [6] studied the autonomous coplanar CR4BP with an extension to low-thrust propulsion for application to the future science mission. In their problem, they also studied the stability region of the artificial and natural equilibrium points in the Sun-Jupiter Trojan Asteroid-Spacecraft system. Using the concept of Ceccaroni and Biggs [6] and the method of Hassan *et al.* [1], we have proposed to study the existence of periodic orbits of the first kind in the autonomous R4BP by considering the second primary as a triaxial rigid body.

2. Equations of Motion of the Infinitesimal Mass

Let $P_i(i=1,2,3)$ be the three primaries of masses $m_j(j=1,2,3)$ respectively, where $m_1 \ge m_2 > m_3$ and the fourth body P_4 of infinitesimal mass m be assumed so small that it can't influence the motion of the primaries but the motion of $P_4(m)$ is influenced by them. Moreover, we assumed that the mass m_3 (mass of the third primary placed at L_4 of the R3BP) is small enough so that it can't influence the motion of the two dominating primaries P_1 and P_2 but can influence the motion of the infinitesimal body $P_4(m)$.

Thus the centre of mass (*i.e.* the bary-centre) *i.e.* the centre of rotation of the system remains at the bary-centre O of the two primaries P_1 and P_2. Also, all the primaries P_1, P_2 and P_3 are moving in the same plane of motion in different circular orbits of radii OP_1, OP_2 and OP_3 respectively around the bary-centre O with the same angular velocity $\boldsymbol{\omega}$. Considering (O, XY) as an inertial frame in such a way that the XY-plane coincides with the plane of motion of the primaries and origin coincides with O. Initially let the principal axes of the second primary P_2 are parallel to the synodic axes $(O, \xi\eta)$ and its axis of symmetry is perpendicular to the plane of motion. Since the primaries are revolving without rotation about O with the same angular velocity as that of the synodic axes, hence, the principal axes of P_2 will remain parallel to the co-ordinate axes throughout the motion.

Let at any time t, $P_1(\xi_1, 0)$ and $P_2(\xi_2, 0)$ be the positions of two dominating primaries on the x-axis of the rotating (synodic) co-ordinate system and $P_3(\xi_3, \eta_3)$ be the third primary placed at the equilibrium point L_4 of P_1 and P_2. Let $\boldsymbol{r}_1, \boldsymbol{r}_2$ and $\boldsymbol{r}_3$ be the displacements of P_1, P_2 and P_3 relative to P_4 as shown in **Figure 1** and $\boldsymbol{r}$ be the position vector of $P_4(x, y)$, then

$$\begin{aligned} \boldsymbol{r} &= x\hat{i} + y\hat{j} = \boldsymbol{OP}_4, \\ \boldsymbol{r}_1 &= (x-\xi_1)\hat{i} + y\hat{j} = \boldsymbol{P}_1\boldsymbol{P}_4, \\ \boldsymbol{r}_2 &= (x-\xi_2)\hat{i} + y\hat{j} = \boldsymbol{P}_2\boldsymbol{P}_4, \\ \boldsymbol{r}_3 &= (x-\xi_3)\hat{i} + (y-\eta_3)\hat{j} = \boldsymbol{P}_3\boldsymbol{P}_4. \end{aligned} \tag{1}$$

Let $\boldsymbol{F}_1, \boldsymbol{F}_2$ and $\boldsymbol{F}_3$ be the gravitational forces exerted by the primaries P_1, P_2 and P_3 respectively on the infinitesimal mass m at $P_4(x, y)$, then

$$\boldsymbol{F}_1 = -\frac{\gamma m m_1}{r_1^3}\left\{(x-\xi_1)\hat{i} + y\hat{j}\right\}. \tag{2}$$

where γ is the gravitational constant.

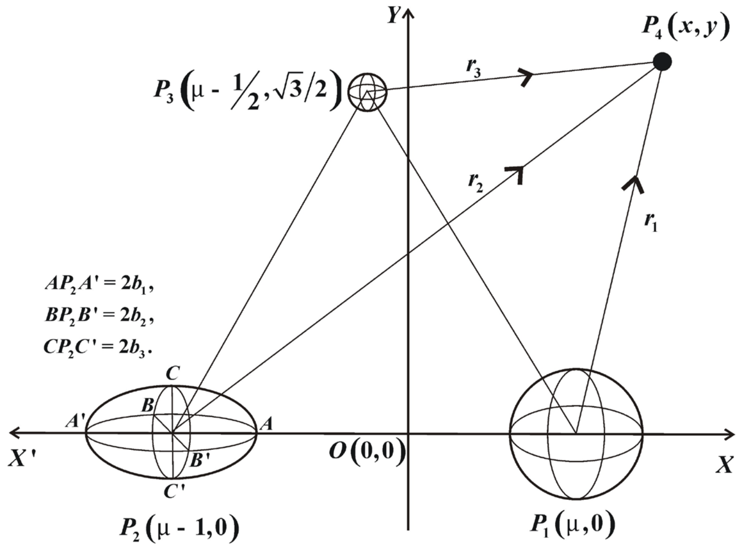

Figure 1. Configuration of CR4BP when the second primary is a Triaxial rigid body.

Let b_1, b_2 and b_3 be the lengths of the semi-axes of the second primary $P_2(\xi_2, 0)$ then the gravitational force exerted by $P_2(\xi_2, 0)$ on $P_4(x, y)$ is given by McCuskey [7]

$$\boldsymbol{F}_2 = -\frac{\gamma m m_2}{r_2^3}\hat{r}_2 - \frac{3\gamma m m_2}{2r_2^4}\left(\frac{2b_1^2 - b_2^2 - b_3^2}{5R^2}\right)\hat{r}_2 + \frac{15\gamma m m_2}{2r_2^6}\left(\frac{b_1^2 - b_2^2}{5R^2}\right)y^2\hat{r}_2.$$

Let $\sigma_1 = \frac{b_1^2 - b_3^2}{5R^2}, \sigma_2 = \frac{b_2^2 - b_3^2}{5R^2}$, then

where $\hat{r}_2$ is the unit vector along r_2 *i.e.*, $\hat{r}_2 = \frac{\boldsymbol{r}_2}{|\boldsymbol{r}_2|} = \frac{(x - \xi_2)\hat{i} + y\hat{j}}{r_2}$

$$\begin{aligned}\boldsymbol{F}_2 = -\gamma m m_2 \Bigg[&\left\{ \frac{x - \xi_2}{r_2^3} + \frac{3(2\sigma_1 - \sigma_2)(x - \xi_2)}{2r_2^5} - \frac{15(\sigma_1 - \sigma_2)(x - \xi_2)}{2r_2^7} y^2 \right\}\hat{i} \\ &+ \left\{ \frac{y}{r_2^3} + \frac{3(2\sigma_1 - \sigma_2)}{2r_2^5} y - \frac{15(\sigma_1 - \sigma_2)}{2r_2^7} y^3 \right\}\hat{j} \Bigg]\end{aligned} \quad (3)$$

and

$$\boldsymbol{F}_3 = -\frac{\gamma m m_3}{r_3^3}\left[(x - \xi_3)\hat{i} + (y - \eta_3)\hat{j}\right]. \quad (4)$$

Total gravitational force exerted by the three primaries on the infinitesimal mass is given by

$$\begin{aligned}\boldsymbol{F} &= \boldsymbol{F}_1 + \boldsymbol{F}_2 + \boldsymbol{F}_3 \\ &= -\gamma m \Bigg[\Bigg\{ \frac{m_1(x - \xi_1)}{r_1^3} + \frac{m_2(x - \xi_2)}{r_2^3} + \frac{m_3(x - \xi_3)}{r_3^3} + \frac{3m_2(2\sigma_1 - \sigma_2)(x - \xi_2)}{2r_2^5} \\ &\quad - \frac{15m_2(\sigma_1 - \sigma_2)(x - \xi_2)}{2r_2^7} y^2 \Bigg\}\hat{i} + \Bigg\{ \frac{m_1 y}{r_1^3} + \frac{m_2 y}{r_2^3} + \frac{m_3(y - \eta_3)}{r_3^3} + \frac{3m_2(2\sigma_1 - \sigma_2)}{2r_2^5} y \\ &\quad - \frac{15m_2(\sigma_1 - \sigma_2)}{2r_2^7} y^3 \Bigg\}\hat{j} \Bigg].\end{aligned} \quad (5)$$

The equation of motion of infinitesimal mass in the gravitational field of the three primaries P_1, P_2 and P_3 is given by

$$m\left[\frac{\partial^2 \boldsymbol{r}}{\partial t^2}+2\boldsymbol{\omega}\times\frac{\partial \boldsymbol{r}}{\partial t}+\frac{\partial \boldsymbol{\omega}}{\partial t}\times\boldsymbol{r}+\boldsymbol{\omega}\times(\boldsymbol{\omega}\times\boldsymbol{r})\right]=\boldsymbol{F}, \tag{6}$$

where

$$\frac{\partial^2 \boldsymbol{r}}{\partial t^2}=\ddot{x}\hat{i}+\ddot{y}\hat{j}=\text{relative acceleration},$$

$$\boldsymbol{\omega}\times\frac{\partial \boldsymbol{r}}{\partial t}=n\left(-\dot{y}\hat{i}+\dot{x}\hat{j}\right)=\text{coriolis acceleration},$$

$$\frac{\partial \boldsymbol{\omega}}{\partial t}\times\boldsymbol{r}=\text{Euler's acceleration},$$

and $\boldsymbol{\omega}\times(\boldsymbol{\omega}\times\boldsymbol{r})=-n^2\left(x\hat{i}+y\hat{j}\right)=$ centrifugal acceleration,

$\left(\text{as } \boldsymbol{\omega}=n\hat{k} \text{ is a constant vector}\right)$.

From Equations ((5) and (6)), we get

$$m\left[\left(\ddot{x}-2n\dot{y}-n^2x\right)\hat{i}+\left(\ddot{y}+2n\dot{x}-n^2y\right)\hat{j}\right]=-\gamma m\left[\left\{\frac{m_1(x-\xi_1)}{r_1^3}+\frac{m_2(x-\xi_2)}{r_2^3}+\frac{m_3(x-\xi_3)}{r_3^3}+\frac{3m_2(2\sigma_1-\sigma_2)(x-\xi_2)}{2r_2^5}-\frac{15m_2(\sigma_1-\sigma_2)(x-\xi_2)}{2r_2^7}y^2\right\}\hat{i}+\left\{\frac{m_1y}{r_1^3}+\frac{m_2y}{r_2^3}+\frac{m_3(y-\eta_3)}{r_3^3}+\frac{3m_2(2\sigma_1-\sigma_2)}{2r_2^5}y-\frac{15m_2(\sigma_1-\sigma_2)}{2r_2^7}y^3\right\}\hat{j}\right]$$

By equating the coefficients of $\hat{i}$ and $\hat{j}$ from both sides, we get the equations of motion of the infinitesimal mass as

$$\ddot{x}-2n\dot{y}-n^2x=-\gamma\left[\frac{m_1(x-\xi_1)}{r_1^3}+\frac{m_2(x-\xi_2)}{r_2^3}+\frac{m_3(x-\xi_3)}{r_3^3}+\frac{3m_2(2\sigma_1-\sigma_2)(x-\xi_2)}{2r_2^5}-\frac{15m_2(\sigma_1-\sigma_2)(x-\xi_2)}{2r_2^7}y^2\right], \tag{7}$$

$$\ddot{y}+2n\dot{x}-n^2y=-\gamma\left[\frac{m_1y}{r_1^3}+\frac{m_2y}{r_2^3}+\frac{m_3(y-\eta_3)}{r_3^3}+\frac{3m_2(2\sigma_1-\sigma_2)}{2r_2^5}y-\frac{15m_2(\sigma_1-\sigma_2)}{2r_2^7}y^3\right] \tag{8}$$

Let $\boldsymbol{v}=v_1\hat{i}+v_2\hat{j}$ be the linear velocity of the infinitesimal mass at $P_4(x,y)$ then

$$\boldsymbol{v}=\frac{\mathrm{d}\boldsymbol{r}}{\mathrm{d}t}=\frac{\partial \boldsymbol{r}}{\partial t}+\boldsymbol{\omega}\times\boldsymbol{r}, \qquad \left[\text{as } \frac{\mathrm{d}}{\mathrm{d}t}=\frac{\partial}{\partial t}+\boldsymbol{\omega}\times\right]$$

$$v_1\hat{i}+v_2\hat{j}=\left(\dot{x}-ny\right)\hat{i}+\left(\dot{y}+nx\right)\hat{j},$$

$\Rightarrow v_1=\dot{x}-ny, v_2=\dot{y}+nx$.

$\therefore$ Kinetic energy of the infinitesimal mass is given by

$T = \frac{1}{2}|\boldsymbol{v}|^2$ for unit mass of the infinitesimal body.

$$T = \frac{1}{2}\left(\dot{x}^2 + \dot{y}^2\right) + n\left(x\dot{y} - \dot{x}y\right) + \frac{n^2}{2}\left(x^2 + y^2\right). \tag{9}$$

where the mean motion of the synodic frame is given by

$$n^2 = 1 + \frac{3}{2}\left(2\sigma_1 - \sigma_2\right). \tag{10}$$

Let p_1 and p_2 be the momenta corresponding to the co-ordinates x and y respectively then $p_1 = \frac{\partial T}{\partial \dot{x}}, p_2 = \frac{\partial T}{\partial \dot{y}}$

$$\Rightarrow p_1 = \dot{x} - ny = v_1 \text{ and } p_2 = \dot{y} + nx = v_2$$

Thus

$$T = \frac{1}{2}\left(p_1^2 + p_2^2\right) \tag{11}$$

Let $V_i = (i = 1,2,3)$ be the gravitational potential of the primaries of masses $m_i\,(i = 1,2,3)$ at any point outside of the infinitesimal mass, then

$$V_1 = -\frac{\gamma m_1}{r_1}, \quad V_2 = -\frac{\gamma m_2}{r_2} - \frac{\gamma m_2\left(2\sigma_1 - \sigma_2\right)}{2r_2^3} + \frac{3\gamma m_2\left(\sigma_1 - \sigma_2\right)}{2r_2^5}y^2, \quad V_3 = -\frac{\gamma m_3}{r_3}. \tag{12}$$

$\therefore$ Total potential at any point outside of the infinitesimal mass due to three primaries is given by

$$V = \sum_{i=1}^{3} V_i = -\gamma\left(\frac{m_1}{r_1} + \frac{m_2}{r_2} + \frac{m_3}{r_3}\right) - \frac{\gamma m_2\left(2\sigma_1 - \sigma_2\right)}{2r_2^3} + \frac{3\gamma m_2\left(\sigma_1 - \sigma_2\right)}{2r_2^5}y^2. \tag{13}$$

The Hamiltonian of the infinitesimal body of unit mass is given by

$$H = \sum p\dot{x} - (T - V) = \left(p_1\dot{x} + p_2\dot{y}\right) - (T - V) \tag{14}$$

$$H = \frac{1}{2}\left(p_1^2 + p_2^2\right) + n\left(p_1 y - p_2 x\right) - \gamma\left(\frac{m_1}{r_1} + \frac{m_2}{r_2} + \frac{m_3}{r_3}\right) - \frac{\gamma m_2\left(2\sigma_1 - \sigma_2\right)}{2r_2^3} + \frac{3\gamma m_2\left(\sigma_1 - \sigma_2\right)}{2r_2^5}y^2 = C = \text{Constant} \tag{15}$$

Assuming μ as the mass ratio of m_2 and ε as the mass ratio of m_3 to the total mass of the dominating primaries P_1 and P_2 then $\mu = \frac{m_2}{m_1 + m_2}$ and $\varepsilon = \frac{m_3}{m_1 + m_2}$. Also assuming $m_1 + m_2 = 1$ then $m_2 = \mu, m_1 = 1 - \mu$ and $m_3 = \varepsilon$. From the definition of the centre of mass of m_1 and m_2 we have $m_1\xi_1 + m_2\xi_2 = 0$ which implies $\xi_1 = \mu$, $\xi_2 = \mu - 1$, $\xi_3 = \mu - \frac{1}{2}$ and $\eta_3 = \frac{\sqrt{3}}{2}$. Thus the co-ordinates of the three primaries P_1, P_2 and P_3 are $(\mu, 0), (\mu - 1, 0)$ and $\left(\mu - \frac{1}{2}, \frac{\sqrt{3}}{2}\right)$ respectively, which confirm $|\boldsymbol{P}_1\boldsymbol{P}_2| = |\boldsymbol{P}_2\boldsymbol{P}_3| = |\boldsymbol{P}_3\boldsymbol{P}_1| = 1$ *i.e.*, $P_1P_2P_3$ is an equilateral triangle of sides of unit length.

Now choosing unit of time in such a way that $\gamma = 1$ and taking $x = x_1$ and $y = x_2$, then the reduced Hamiltonian is given by

$$\begin{aligned} H &= \frac{1}{2}\left(p_1^2 + p_2^2\right) + n\left(p_1 x_2 - p_2 x_1\right) - \frac{1-\mu}{r_1} - \frac{\mu}{r_2} - \frac{\varepsilon}{r_3} \\ &\quad - \frac{\mu\left(2\sigma_1 - \sigma_2\right)}{2r_2^3} + \frac{3\mu\left(\sigma_1 - \sigma_2\right)}{2r_2^5} x^2 = C. \end{aligned} \tag{16}$$

The Hamiltonian-canonical equations are

$$\frac{\mathrm{d}x_i}{\mathrm{d}t} = \frac{\partial H}{\partial p_i}, \quad \frac{\mathrm{d}p_i}{\mathrm{d}t} = -\frac{\partial H}{\partial x_i}. \quad (i = 1, 2) \tag{17}$$

The energy integral of the infinitesimal mass is

$$\begin{aligned} &\frac{1}{2}\left(\dot{x}_1^2 + \dot{x}_2^2\right) \\ &= \frac{1}{2} n^2 \left(x_1^2 + x_2^2\right) + \frac{1-\mu}{r_1} + \frac{\mu}{r_2} + \frac{\varepsilon}{r_3} + \frac{\mu\left(2\sigma_1 - \sigma_2\right)}{2r_2^3} - \frac{3\mu\left(\sigma_1 - \sigma_2\right)}{2r_2^5} x_2^2 + C. \end{aligned} \tag{18}$$

where C is a constant.

3. Regularization

In our Hamiltonian given in Equation (16), there are three singularities $r_1 = r_2 = r_3 = 0$, so to examine the existence of periodic orbits around the first primary; we have to eliminate the singularity $r_1 = 0$ from the Hamiltonian in Equation (16). For this, let us define an extended generating function S by

$$S = \left(\mu + q_1^2 - q_2^2\right) p_1 + 2 q_1 q_2 p_2 \tag{19}$$

where $Q_i\left(i = 1, 2\right)$ are momenta associated with new co-ordinates $q_i\left(i = 1, 2\right)$ and $x_i = \frac{\partial S}{\partial p_i}, Q_i = \frac{\partial S}{\partial q_i}$.

Clearly,

$$x_1 = \frac{\partial S}{\partial p_1} = \mu + q_1^2 - q_2^2, \quad x_2 = \frac{\partial S}{\partial p_2} = 2 q_1 q_2 \tag{20}$$

$$Q_1 = 2\left(p_1 q_1 + p_2 q_2\right), \quad Q_2 = 2\left(p_2 q_1 - p_1 q_2\right) \tag{21}$$

$$r_1^2 = \left(x_1 - \mu\right)^2 + x_2^2 = \left(q_1^2 - q_2^2\right)^2 + 4 q_1^2 q_2^2 = \left(q_1^2 + q_2^2\right)^2$$

$$\begin{aligned} &r_1 = q_1^2 + q_2^2, \quad r_2^2 = 1 + r_1^2 + 2\left(q_1^2 - q_2^2\right) + \left(q_1^2 + q_2^2\right)^2, \\ &r_3^2 = 1 + r_1^2 + \left(q_1^2 - q_2^2\right) - 2\sqrt{3} q_1 q_2 + \left(q_1^2 + q_2^2\right)^2. \end{aligned} \tag{22}$$

From Equation (21), we have

$$p_1 = \frac{1}{2r_1}\left(Q_1 q_1 - Q_2 q_2\right), \quad p_2 = \frac{1}{2r_1}\left(Q_1 q_2 + Q_2 q_1\right), \tag{23}$$

$$\therefore p_1^2 + p_2^2 = \frac{1}{4r_1}\left(Q_1^2 + Q_2^2\right), \tag{24}$$

$$n\left(p_1x_2 - p_2x_1\right) = \frac{n}{2}\left(Q_1q_2 - Q_2q_1\right) - \frac{n\mu}{2r_1}\left(Q_1q_2 + Q_2q_1\right). \tag{25}$$

The combination of Equations ((15), (24) and (25)) gives the Hamiltonian H in terms of new variables $q_i, Q_i\left(i = 1,2\right)$ as

$$\begin{aligned} H = {} & \frac{1}{8r_1}\left(Q_1^2 + Q_2^2\right) + \frac{1}{2}n\left(Q_1q_2 - Q_2q_1\right) - \frac{n\mu}{2r_1}\left(Q_1q_2 + Q_2q_1\right) \\ & - \frac{1-\mu}{r_1} - \frac{\mu}{r_2} - \frac{\varepsilon}{r_3} - \frac{\mu\left(2\sigma_1 - \sigma_2\right)}{2r_2^3} + \frac{6\mu\left(\sigma_1 - \sigma_2\right)}{r_2^5}q_1^2q_2^2 = C. \end{aligned} \tag{26}$$

Let us introduce pseudo time τ by the equation

$$\mathrm{d}t = r_1\mathrm{d}\tau \quad \left(\tau = 0 \text{ when } t = 0\right) \tag{27}$$

The Canonical equations of motion corresponding to the regularized Hamiltonian K are given by

$$\frac{\mathrm{d}q_{i'}}{\mathrm{d}\tau} = \frac{\partial K}{\partial Q_i}, \quad \frac{\mathrm{d}Q_{i'}}{\mathrm{d}\tau} = -\frac{\partial K}{\partial q_i}, \quad \left(i = 1,2\right) \tag{28}$$

where the regularized Hamiltonian is given by $K = r_1\left(H - C\right) = 0$

$$\begin{aligned} \text{i.e., } K = {} & \frac{1}{8}\left(Q_1^2 + Q_2^2\right) + \frac{1}{2}nr_1\left(Q_1q_2 - Q_2q_1\right) - \frac{n\mu}{2}\left(Q_1q_2 + Q_2q_1\right) - \left(1-\mu\right) \\ & - \frac{\mu r_1}{r_2} - \frac{\varepsilon r_1}{r_3} - \frac{\mu r_1\left(2\sigma_1 - \sigma_2\right)}{2r_2^3} + \frac{6\mu r_1\left(\sigma_1 - \sigma_2\right)}{r_2^5}q_1^2q_2^2 - r_1C = 0. \end{aligned} \tag{29}$$

Since ε is very-very small in comparison of the masses of the dominating primaries hence $\forall \varepsilon \in \left]0, \mu\right[$, we can take $\varepsilon = \mu\varepsilon_0$ and $C = C_0 + \mu C_1 + \mu^2 C_2 + \mu^3 C_3 + \cdots$.

Let us write $K = K_0 + \mu K_1 = 0$ then from Equation (29), we have

$$K_0 = \frac{1}{8}\left(Q_1^2 + Q_2^2\right) + \frac{1}{2}r_1\left[n\left(Q_1q_2 - Q_2q_1\right) - 2C_0\right] - 1 = -\lambda\left(\text{say}\right) \tag{30}$$

$$K_1 = 1 - \frac{n}{2}\left(Q_1q_2 + Q_2q_1\right) - r_1\left[C_1 + \frac{1}{r_2} + \frac{\varepsilon_0}{r_3} + \frac{A}{r_2^3} - \frac{Bq_1^2q_2^2}{r_2^5}\right] \tag{31}$$

where

$$A = \frac{1}{2}\left(2\sigma_1 - \sigma_2\right), B = 3\left(\sigma_1 - \sigma_2\right).$$

4. Generating Solution (*i.e.*, Solution When $\mu = 0$)

For generating solution, we shall choose K_0 for our Hamiltonian function, so in order to solve the Hamilton-Jacobi equation associated with K_0, let us write $Q_i = \frac{\partial W}{\partial q_i}\left(i = 1,2\right)$ and $1 - \lambda = \alpha > 0$ arbitrary constant.

Since t is not involved explicitly in K_0 hence the Hamilton-Jacobi equation may be written as

$$\frac{1}{8}\left[\left(\frac{\partial W}{\partial q_1}\right)^2 + \left(\frac{\partial W}{\partial q_2}\right)^2\right] + \frac{1}{2}r_1\left[n\left(q_2\frac{\partial W}{\partial q_1} - q_1\frac{\partial W}{\partial q_2}\right) - 2C_0\right] = \alpha. \tag{32}$$

Putting $q_1 = \rho\cos\varphi, q_2 = \rho\sin\varphi$

$$\rho^2 = q_1^2 + q_2^2 = r_1 \text{ and } \varphi = \tan^{-1}\left(\frac{q_2}{q_1}\right). \tag{33}$$

Now $W = W(q_1, q_2) = W(\rho, \varphi)$,

$$\begin{aligned} &\Rightarrow Q_1 = \frac{\partial W}{\partial q_1} = \frac{\partial W}{\partial \rho}\cos\varphi - \frac{\partial W}{\partial \varphi}\cdot\frac{\sin\varphi}{\rho} \\ &\text{and } Q_2 = \frac{\partial W}{\partial q_2} = \frac{\partial W}{\partial \rho}\sin\varphi + \frac{\partial W}{\partial \varphi}\cdot\frac{\cos\varphi}{\rho} \end{aligned}. \tag{34}$$

$$\therefore \left(\frac{\partial W}{\partial q_1}\right)^2 + \left(\frac{\partial W}{\partial q_2}\right)^2 = \left(\frac{\partial W}{\partial \rho}\right)^2 + \frac{1}{\rho^2}\left(\frac{\partial W}{\partial \varphi}\right)^2 \text{ and } q_2\frac{\partial W}{\partial q_1} - q_1\frac{\partial W}{\partial q_2} = -\frac{\partial W}{\partial \varphi}.$$

Thus the Equation (32) reduces to

$$\frac{1}{8}\left[\left(\frac{\partial W}{\partial \rho}\right)^2 + \frac{1}{\rho^2}\left(\frac{\partial W}{\partial \varphi}\right)^2\right] + \frac{1}{2}\rho^2\left[-n\frac{\partial W}{\partial \varphi} - 2C_0\right] = \alpha \tag{35}$$

This is a partial differential equation of second degree, so by the method of variable separable, the solution may be written as

$$W = U(\rho) + 2G\phi, \tag{36}$$

where G is an arbitrary constant.

Now introducing a new variable z by $r_1 = \rho^2 = z$ then $\frac{\mathrm{d}z}{\mathrm{d}\rho} = 2\rho$

$$\therefore \frac{\partial W}{\partial \rho} = \frac{\partial U}{\partial \rho} = \frac{\mathrm{d}U}{\mathrm{d}\rho} = \frac{\mathrm{d}U}{\mathrm{d}z}\cdot\frac{\mathrm{d}z}{\mathrm{d}\rho} = 2\rho\frac{\mathrm{d}U}{\mathrm{d}z}$$

$$\text{i.e., } \frac{\partial W}{\partial \rho} = 2\rho\frac{\mathrm{d}U}{\mathrm{d}z} \text{ and } \frac{\partial W}{\partial \varphi} = 2G \tag{37}$$

Introducing Equation (37) in Equation (36), we get

$$\frac{1}{8}\left[\left(2\rho\frac{\mathrm{d}U}{\mathrm{d}z}\right)^2 + \frac{1}{\rho^2}(2G)^2\right] + \frac{1}{2}\rho^2\left[-n\cdot 2G - 2C_0\right] = \alpha,$$

$$z\left(\frac{\mathrm{d}U}{\mathrm{d}z}\right)^2 = -\frac{G^2}{z} + 2z\left[n\cdot G + C_0\right] + 2\alpha,$$

$$\begin{aligned} \left(\frac{\mathrm{d}U}{\mathrm{d}z}\right)^2 &= -\frac{G^2}{z^2} + 2\left[n\cdot G + C_0\right] + \frac{2\alpha}{z}, \\ &= -\frac{2(nG + C_0)}{z^2}\left[-z^2 - \frac{\alpha z}{nG + C_0} + \frac{G^2}{2(nG + C_0)}\right], \\ &= -\frac{2(nG + C_0)}{z^2}F(z), \end{aligned}$$

where

$$F(z) = -z^2 - \frac{\alpha z}{nG + C_0} + \frac{G^2}{2(nG + C_0)} \text{ is a quadratic expression in } z. \tag{38}$$

Thus

$$\frac{\mathrm{d}U}{\mathrm{d}z} = \sqrt{-2(nG+C_0)}\frac{\sqrt{F(z)}}{z}, \tag{39}$$

$$\text{i.e., } U(z,G,\alpha) = \sqrt{-2(nG+C_0)}\int_{z_1}^{z}\frac{\sqrt{F(z)}}{z}\mathrm{d}z, \tag{40}$$

where z_1 is the smaller root of the equation $F(z)=0$.

From Equation (40), we conclude that for general solution, we need only two arbitrary constants assigned as α and G. Therefore the solution (40) may be regarded as a general solution. Following Giacaglia [3] and Bhatnagar [5], let us introduce the parameters n,a,e,l by the relations

$$z_1 = na(1-e),\ \ z_2 = na(1+e)\ \text{ and }\ z = z_1\cos^2\frac{l}{2} + z_2\sin^2\frac{l}{2} = na(1-e\cos l). \tag{41}$$

where z_1 and z_2 are the two roots of the equation $F(z)=0$, a is the semi-major axis, e is the eccentricity and l is the semi-latus rectum of the elliptic orbit of the infinitesimal mass around the first primary. It may be noted that for $z = z_1, l = 0$.

From Equation (41),

$$z_1 + z_2 = 2na,\quad z_1z_2 = n^2a^2(1-e^2) \tag{42}$$

Again since z_1 and z_2 are the roots of the equation $F(z)=0$, hence we have,

$$z_1 + z_2 = -\frac{\alpha}{nG+C_0}\ \text{ and }\ z_1z_2 = -\frac{G^2}{2(nG+C_0)}. \tag{43}$$

From Equations ((42) and (43)),

$$2na = -\frac{\alpha}{nG+C_0},\quad n^2a^2(1-e^2) = -\frac{G^2}{2(nG+C_0)},$$

$$\Rightarrow a = -\frac{\alpha}{2n(nG+C_0)} = \frac{\alpha}{n[-2(nG+C_0)]}.$$

Introducing a new parameter L by the relation

$$\alpha = L[-2(nG+C_0)]^{\frac{1}{2}} > 0 \tag{44}$$

Then

$$a = \frac{L}{n[-2(nG+C_0)]^{\frac{1}{2}}} > 0 \tag{45}$$

Also $n^2a^2(1-e^2) = -\dfrac{G^2}{2(nG+C_0)}$

$$n^2(1-e^2)\frac{L^2}{n^2[-2(nG+C_0)]} = \frac{G^2}{[-2(nG+C_0)]}$$

$$\Rightarrow e^2 = \left(1 - \frac{G^2}{L^2}\right)^{\frac{1}{2}} \le 1 \tag{46}$$

From Equation (38),

$$\begin{aligned}
F(z) &= -z^2 - \frac{\alpha z}{(nG + C_0)} + \frac{G^2}{2(nG + C_0)}, \\
&= -z^2 + \frac{2zL\left[-2(nG + C_0)\right]^{\frac{1}{2}}}{\left[-2(nG + C_0)\right]} + \frac{G^2}{2(nG + C_0)}, \\
&= -z^2 + 2z\frac{L}{\left[-2(nG + C_0)\right]^{\frac{1}{2}}} + \frac{G^2}{2(nG + C_0)}, \qquad \left[\text{using Equation (44)}\right] \\
&= -z^2 + 2zan - z_1 z_2, \qquad \left[\text{using Equation (45)}\right] \\
&= n^2 a^2 e^2 - (z - na)^2, \\
F(z) &= n^2 a^2 e^2 \sin^2 l,
\end{aligned}$$

$$i.e.,\ F(z) = -z^2 - \frac{\alpha z}{n(G + C_0)} + \frac{G^2}{2(nG + C_0)} = n^2 a^2 e^2 \sin^2 l \tag{47}$$

The Hamilton-Canonical equation of motion corresponding to the Hamiltonian K_0 are given by

$$\begin{aligned}
\frac{\mathrm{d}q_1}{\mathrm{d}\tau} &= \frac{\partial K_0}{\partial Q_1}, \qquad \frac{\mathrm{d}q_2}{\mathrm{d}\tau} = \frac{\partial K_0}{\partial Q_2}, \\
\frac{\mathrm{d}Q_1}{\mathrm{d}\tau} &= -\frac{\partial K_0}{\partial q_1}, \qquad \frac{\mathrm{d}Q_2}{\mathrm{d}\tau} = -\frac{\partial K_0}{\partial q_2},
\end{aligned} \tag{48}$$

where $K_0 = \frac{1}{8}\left(Q_1^2 + Q_2^2\right) + \frac{1}{2}\rho^2\left[n\left(Q_1 q_2 - Q_2 q_1\right) - 2C_0\right] - 1$.

$$\Rightarrow \frac{\partial K_0}{\partial Q_1} = \frac{1}{4}Q_1 + \frac{1}{2}\rho^2 n q_2, \quad \frac{\partial K_0}{\partial Q_2} = \frac{1}{4}Q_2 - \frac{1}{2}\rho^2 n q_1.$$

Thus

$$q_1' = \frac{1}{4}Q_1 + \frac{1}{2}\rho^2 n q_2 \text{ and } q_2' = \frac{1}{4}Q_2 - \frac{1}{2}\rho^2 n q_1. \tag{49}$$

where $(')$ primes denote the differentiation with respect to τ.

Now $\rho^2 = q_1^2 + q_2^2 = z$

$$\Rightarrow 2\rho\frac{\mathrm{d}\rho}{\mathrm{d}\tau} = 2q_1\frac{\mathrm{d}q_1}{\mathrm{d}\tau} + 2q_2\frac{\mathrm{d}q_2}{\mathrm{d}\tau} = \frac{\mathrm{d}z}{\mathrm{d}\tau},$$

$$\Rightarrow 2\rho\rho' = 2\left(q_1 q_1' + q_2 q_2'\right) = \frac{\mathrm{d}z}{\mathrm{d}\tau}. \tag{50}$$

But

$$\begin{aligned}
q_1 q_1' + q_2 q_2' &= q_1\left(\frac{1}{4}Q_1 + \frac{1}{2}\rho^2 n q_2\right) + q_2\left(\frac{1}{4}Q_2 - \frac{1}{2}\rho^2 n q_1\right) \quad \left[\text{using Equation (49)}\right] \\
&= \frac{1}{4}\left(q_1 Q_1 + q_2 Q_2\right)
\end{aligned}$$

Thus

$$2\rho\rho' = 2\sum_{i=1}^{2} q_i q_i' = \frac{1}{2}\sum_{i=1}^{2} q_i Q_i = \frac{\mathrm{d}z}{\mathrm{d}\tau} \tag{51}$$

Also

$$\begin{aligned}\sum_{i=1}^{2} q_i Q_i &= q_1 Q_1 + q_2 Q_2 \\ &= q_1\left(\frac{\partial W}{\partial q_1}\right) + q_2\left(\frac{\partial W}{\partial q_2}\right) \\ &= \rho\cos\varphi\left(\cos\varphi\frac{\partial W}{\partial\rho} - \frac{\sin\varphi}{\rho}\frac{\partial W}{\partial\varphi}\right) + \rho\sin\varphi\left(\sin\varphi\frac{\partial W}{\partial\rho} + \frac{\cos\varphi}{\rho}\frac{\partial W}{\partial\varphi}\right) \\ &= \rho\frac{\partial W}{\partial\rho} \qquad [\text{using Equation (34)}] \\ &= 2\rho^2\frac{\mathrm{d}U}{\mathrm{d}z} \qquad [\text{using Equation (37)}]\end{aligned}$$

$$\Rightarrow \sum_{i=1}^{2} q_i Q_i = 2z\left(\frac{\mathrm{d}U}{\mathrm{d}z}\right) \tag{52}$$

Also from Equations ((39), (51) and (52)),

$$\frac{1}{2}\rho\frac{\partial W}{\partial\rho} = z\rho\rho' = 2\sum_{i=1}^{2} q_i q_i' = \frac{1}{2}\sum_{i=1}^{2} q_i Q_i = z\frac{\mathrm{d}U}{\mathrm{d}z} = \sqrt{-2(nG+C_0)F(z)} = \frac{\mathrm{d}z}{\mathrm{d}\tau} \tag{53}$$

From the last relation of Equation (53), we have

$$\Rightarrow \frac{\mathrm{d}z}{\mathrm{d}\tau} = \sqrt{-2(nG+C_0)}\sqrt{F(z)}$$

$$\Rightarrow \frac{\mathrm{d}z}{\sqrt{F(z)}} = \sqrt{-2(nG+C_0)}\,\mathrm{d}\tau$$

$$\Rightarrow \int_{z_1}^{z}\frac{\mathrm{d}z}{\sqrt{F(z)}} = \sqrt{-2(nG+C_0)}\int_{\tau_0}^{\tau}\mathrm{d}\tau \quad \text{where } \tau = z_1 \Rightarrow l = 0, z = \tau_0$$

$$\Rightarrow \int_0^l \frac{nal\sin l\mathrm{d}l}{nal\sin l} = \sqrt{-2(nG+C_0)}(\tau-\tau_0) \qquad [\text{using Equations (41) and (48)}]$$

$$\Rightarrow l = [-2(nG+C_0)]^{\frac{1}{2}}(\tau-\tau_0)$$

$$\Rightarrow l = \int_{z_1}^{z}\frac{\mathrm{d}z}{\sqrt{F(z)}} = [-2(nG+C_0)]^{\frac{1}{2}}(\tau-\tau_0) \tag{54}$$

Again from Equation (53),

$$\frac{\mathrm{d}z}{\mathrm{d}t}\cdot\frac{\mathrm{d}t}{\mathrm{d}\tau} = \sqrt{-2(nG+C_0)}\sqrt{F(z)}$$

$$\frac{\mathrm{d}z}{\mathrm{d}t}r_1 = \sqrt{-2(nG+C_0)}\sqrt{F(z)}$$

$$\Rightarrow z\frac{\mathrm{d}z}{\mathrm{d}t} = \sqrt{-2(nG+C_0)}\sqrt{F(z)}$$

$$\Rightarrow \mathrm{d}t = \frac{1}{[-2(nG+C_0)]^{\frac{1}{2}}}\frac{z\mathrm{d}z}{\sqrt{F(z)}}$$

$$\Rightarrow \int_{t_0}^{t} \mathrm{d}t = \frac{1}{\left[-2\left(nG+C_0\right)\right]^{\frac{1}{2}}} \int_0^l \frac{an\left(1-e\cos l\right) ane \sin l \mathrm{d}l}{ane \sin l}$$

$$t - t_0 = \frac{an}{\left[-2\left(nG+C_0\right)\right]^{\frac{1}{2}}}\left(l - e\sin l\right),$$

where t_0 is a constant. (55)

Now taking L and G as arbitrary constants in lieu of α and G and the solutions may be given by the relations

$$\frac{\partial W}{\partial L} = \frac{\partial U}{\partial L} = \int_{z_1}^{z} \frac{\mathrm{d}z}{\sqrt{F(z)}} = l, \quad \frac{\partial W}{\partial G} = \frac{\partial U}{\partial G} + 2\varphi = g. \tag{56}$$

From Equation (40),

$$U(z,G,L) = \left[-2\left(nG+C_0\right)\right]^{\frac{1}{2}} \int_{z_1}^{z} \sqrt{F(z)} \frac{\mathrm{d}z}{z}.$$

Differentiating partially with respect to G, we get

$$\frac{\partial U}{\partial G} = \frac{\partial}{\partial G}\int_{z_1}^{z} \sqrt{-2\left(nG+C_0\right)F(z)}\frac{\mathrm{d}z}{z},$$

$$= \int_{z_1}^{z} \frac{\partial}{\partial G}\sqrt{-2\left(nG+C_0\right)F(z)}\frac{\mathrm{d}z}{z},$$

$$= \int_{z_1}^{z} \frac{1}{2\sqrt{-2\left(nG+C_0\right)F(z)}} \frac{\partial}{\partial G}\left[-2\left(nG+C_0\right)F(z)\right]\frac{\mathrm{d}z}{z},$$

$$= \int_{z_1}^{z} \frac{1}{2\sqrt{-2\left(nG+C_0\right)F(z)}} \frac{\partial}{\partial G}\left[2z^2\left(nG+C_0\right) + 2zL\left\{-2\left(nG+C_0\right)\right\}^{\frac{1}{2}} - G^2\right]\frac{\mathrm{d}z}{z},$$

$$= \int_{z_1}^{z} \frac{1}{2\sqrt{-2\left(nG+C_0\right)F(z)}}\left[nz^2 - n^2az - G\right]\frac{\mathrm{d}z}{z},$$

$$= \frac{1}{\left[-2\left(nG+C_0\right)\right]^{\frac{1}{2}}}\left[n\int_{z_1}^{z}\frac{z\mathrm{d}z}{\sqrt{F(z)}} - n^2a\int_{z_1}^{z}\frac{\mathrm{d}z}{\sqrt{F(z)}} - G\int_{z_1}^{z}\frac{\mathrm{d}z}{z\sqrt{F(z)}}\right],$$

$$= \frac{1}{\left[-2\left(nG+C_0\right)\right]^{\frac{1}{2}}}\left[n^2a\left(l - e\sin l\right) - n^2al - \frac{G}{an}\int_0^l \frac{\mathrm{d}l}{\left(1-e\cos l\right)}\right],$$

$$= -\frac{n^2ae\sin l}{\left[-2\left(nG+C_0\right)\right]^{\frac{1}{2}}} - \frac{G}{na\left[-2\left(nG+C_0\right)\right]^{\frac{1}{2}}}\int_0^l \frac{\mathrm{d}l}{\left(1-e\cos l\right)},$$

$$= -n\frac{L}{\left[-2\left(nG+C_0\right)\right]^{\frac{1}{2}}}\frac{e\sin l}{\left[-2\left(nG+C_0\right)\right]^{\frac{1}{2}}} - \frac{G}{L}\int_0^l \frac{\mathrm{d}l}{\left(1-e\cos l\right)}, \quad \text{[using Equation (45)]}$$

$$= \frac{-nL\sin l\sqrt{1-\frac{G^2}{L^2}}}{-2\left(nG+C_0\right)} - \sqrt{1-e^2}\int_0^l \frac{\mathrm{d}l}{\left(1-e\cos l\right)}, \quad \text{[using Equation (46)]}$$

$$= \frac{n\sqrt{L^2-G^2}\sin l}{2\left(nG+C_0\right)} - \sqrt{1-e^2}\int_0^l \frac{\mathrm{d}l}{\left(1-e\cos l\right)},$$

$$\Rightarrow \frac{\partial U}{\partial G} = \frac{n\sqrt{L^2 - G^2}\sin l}{2(nG + C_0)} - h,$$

where

$$h = \sqrt{1-e^2}\int_0^l \frac{\mathrm{d}l}{(1 - e\cos l)} \quad (e \neq 1). \tag{57}$$

From Equation (56),

$$g = \frac{\partial U}{\partial G} + 2\varphi,$$

$$\Rightarrow g = 2\varphi + \frac{n\sqrt{L^2 - G^2 \sin l}}{2(nG + C_0)} - h, \tag{58}$$

$$\Rightarrow \varphi = \frac{1}{2}(g + h) - \frac{n\sqrt{L^2 - G^2}}{4(nG + C_0)}\sin l, \quad \text{where } (e \neq 1, G \neq 0, h \neq 0)$$
$$\text{and } \varphi = \frac{1}{2}g - \frac{nL}{4C_0}\sin l, \quad \text{where } (e = 1, G = 0, h = 0) \tag{59}$$

Now let us find the value of K_0 in terms of l, g, L, G. For this, we have

$$K_0 = \frac{1}{8}\left(Q_1^2 + Q_2^2\right) + \frac{1}{2}\rho^2\left[n\left(Q_1 q_2 - Q_2 q_1\right) - 2C_0\right] - 1,$$
$$= \frac{1}{8}\left[\left(\frac{\partial W}{\partial \rho}\right)^2 + \left(\frac{\partial W}{\partial \varphi}\right)^2\right] + \frac{1}{2}\rho^2\left[-n\frac{\partial W}{\partial \varphi} - 2C_0\right] - 1$$
$$= \frac{1}{8}\left[\left(2\rho\frac{\mathrm{d}U}{\mathrm{d}z}\right)^2 + \frac{(2G)^2}{\rho^2}\right] + \frac{1}{2}\rho^2\left[-2Gn - 2C_0\right] - 1,$$
$$= \frac{1}{2z}\left[\left(z\frac{\mathrm{d}U}{\mathrm{d}z}\right)^2 + G^2\right] - z(nG + C_0) - 1,$$
$$= \frac{1}{2z}\left[-2(nG + C_0)F(z) + G^2\right] - z(nG + C_0) - 1, \quad [\text{using Equation (53)}]$$
$$= \frac{1}{2z}\left[2z^2(nG + C_0) + 2\alpha z - G^2 + G^2\right] - z(nG + C_0) - 1, \quad [\text{using Equation (38)}]$$
$$= z(nG + C_0) + \alpha - z(nG + C_0) - 1,$$
$$= \alpha - 1,$$

$$\Rightarrow K_0 = L\left[-2(nG + C_0)\right]^{\frac{1}{2}} - 1. \tag{60}$$

Therefore, for the problem generated by the Hamiltonian K_0, the equations of motion are

$$\frac{\mathrm{d}L}{\mathrm{d}\tau} = \frac{\partial K_0}{\partial l} = 0 \Rightarrow L = \text{constant} = L_0,$$
$$\frac{\mathrm{d}G}{\mathrm{d}\tau} = \frac{\partial K_0}{\partial g} = 0 \Rightarrow G = \text{constant} = G_0,$$
$$\frac{\mathrm{d}l}{\mathrm{d}\tau} = -\frac{\partial K_0}{\partial L} = \left[-2(nG + C_0)\right]^{\frac{1}{2}} = \eta_l\,(\text{say}) \quad \Rightarrow l = \eta_l\tau + l_0, \tag{61}$$
$$\frac{\mathrm{d}g}{\mathrm{d}\tau} = -\frac{\partial K_0}{\partial G} = -\frac{L}{\left[-2(nG + C_0)\right]^{\frac{1}{2}}} = \eta_g\,(\text{say}) \quad \Rightarrow g = \eta_g\tau + g_0.$$

Further we are to express q_i and $Q_i\ (i=1,2)$ in terms of canonical elements l,g,L,G.

From Equation (34),

$$Q_1=\frac{\partial W}{\partial q_i}=\cos\varphi\frac{\partial W}{\partial\rho}-\frac{\sin\varphi}{\rho}\frac{\partial W}{\partial\varphi}$$
$$=\cos\varphi 2\rho\frac{\mathrm{d}U}{\mathrm{d}z}-\frac{\sin\varphi}{\rho}\frac{\partial W}{\partial\varphi}$$
$$=\pm\frac{2}{\sqrt{z}}\left[\left\{-2\left(nG+C_0\right)\right\}^{\frac{1}{2}}\sqrt{F(z)}\cos\varphi-G\sin\varphi\right],$$

$$\text{i.e., } Q_1=2\left[\frac{eL\sin l\cos\varphi-G\sin\varphi}{\pm\sqrt{na(1-e\cos l)}}\right].$$

Thus,

$$\begin{aligned}&Q_1=\pm\frac{2\left[eL\sin l\cos\varphi-G\sin\varphi\right]}{\sqrt{na(1-e\cos l)}},\quad Q_2=\pm\frac{2\left[eL\sin l\cos\varphi+G\sin\varphi\right]}{\sqrt{na(1-e\cos l)}},\\&q_1=\pm\left[na(1-e\cos l)\right]^{\frac{1}{2}}\cos\varphi,\qquad q_2=\pm\left[na(1-e\cos l)\right]^{\frac{1}{2}}\sin\varphi,\end{aligned}\tag{62}$$

where φ is given by the first equation of system (59).

When $e=1,G=0,h=0,$ then the variables $q_i,Q_i\ (i=1,2)$ can be expressed in terms of canonical elements (l,g,L,G) as

$$\begin{aligned}&Q_1=\pm\frac{4L}{\sqrt{2an}}\cos\frac{l}{2}\cos\varphi,\quad Q_2=\pm\frac{4L}{\sqrt{2an}}\cos\frac{l}{2}\sin\varphi,\\&q_1=\pm\sqrt{2an}\sin\frac{l}{2}\cos\varphi,\qquad q_2=\pm\sqrt{2an}\sin\frac{l}{2}\sin\varphi,\end{aligned}\tag{63}$$

where φ is given by the second equation of (59).

The original synodic cartesian co-ordinates in a uniformly rotating (synodic) system are obtained from the Equations ((20) and (23)) when $\mu=0$, as

$$\begin{aligned}&x_1=q_1^2-q_2^2,\qquad\qquad x_2=2q_1q_2,\\&p_1=\frac{1}{2z}\left(Q_1q_1-Q_2q_2\right),\quad p_2=\frac{1}{2z}\left(Q_2q_1-Q_1q_2\right).\end{aligned}\tag{64}$$

The sidereal cartesian co-ordinates are obtained by considering the transformation

$$\begin{aligned}&X_1=x_1\cos nt-x_2\sin nt,\quad X_2=x_1\sin nt+x_2\cos nt,\\&\dot{X}_1=p_1\cos nt-p_2\sin nt,\quad \dot{X}_2=p_1\sin nt+p_2\cos nt,\end{aligned}\tag{65}$$

where t is given by the Equation (55).

Now let us express K_1 in terms of the canonical elements l,g,L,G.

From Equation (31),

$$K_1=1-\frac{1}{2}n\left(Q_1q_2+Q_2q_1\right)-r_1\left[C_1+\frac{1}{r_2}+\frac{\varepsilon_0}{r_3}+\frac{A}{r_2^3}-\frac{Bq_1^2q_2^2}{r_2^5}\right]$$

Now,

$$\begin{aligned}
Q_1q_2+Q_2q_1 &= \rho\sin\varphi\frac{\partial W}{\partial q_1}+\rho\cos\varphi\frac{\partial W}{\partial q_2}\\
&= \rho\sin\varphi\left[\cos\varphi\frac{\partial W}{\partial\rho}-\frac{\sin\varphi}{\rho}\frac{\partial W}{\partial\varphi}\right]+\rho\cos\varphi\left[\sin\varphi\frac{\partial W}{\partial\rho}+\frac{\cos\varphi}{\rho}\frac{\partial W}{\partial\varphi}\right]\\
&= \rho\frac{\partial W}{\partial\rho}\sin2\varphi+\frac{\partial W}{\partial\varphi}\cos2\varphi \qquad [\text{using Equation (34)}]\\
&= 2\sqrt{-2(nG+C_0)F(z)}\sin2\varphi+2G\cos2\varphi \qquad [\text{using Equation (53)}]\\
&= 2ane\left[-2(nG+C_0)\right]^{\frac{1}{2}}e\sin l\sin2\varphi+2G\cos\varphi \qquad [\text{using Equation (47)}]\\
&= 2eL\sin l\sin2\varphi+2G\cos2\varphi
\end{aligned}$$

$$Q_1q_2+Q_2q_1=2\left[eL\sin l\sin2\varphi+G\cos2\varphi\right]$$

$$\Rightarrow\frac{n}{2}(Q_1q_2+Q_2q_1)=n\left[eL\sin l\sin2\varphi+2G\cos2\varphi\right]$$

$$q_1^2q_2^2=\rho^2\cos^2\varphi\rho^2\sin^2\varphi=\rho^4(\sin\varphi\cos\varphi)^2=\frac{z^2}{4}\sin^2 2\varphi$$

Thus,

$$K_1=1-n(eL\sin l\sin2\varphi+G\cos2\varphi)-z\left[C_1+\frac{1}{r_2}+\frac{\varepsilon_0}{r_3}+\frac{A}{r_2^3}-\frac{Bz^2\sin^2 2\varphi}{4r_2^5}\right] \tag{66}$$

where

$$\begin{aligned}
r_1&=na(1-e\cos l)=z,\\
r_2^2&=1+z^2+2z\cos2\varphi,\\
r_3^2&=1+z^2+z\cos2\varphi-\sqrt{3}z\sin2\varphi
\end{aligned}$$

where a is given by Equation (45), e is given by Equation (46) and φ is given by the first equation of (58).

By neglecting the higher order terms of e, let the co-efficient of μ be denoted by R then the complete Hamiltonian in terms of canonical variables l,g,L,G is given by

$$K=L\left[-2(nG+C_0)\right]^{\frac{1}{2}}-1+\mu R$$

$\therefore$ The equations of motion for the complete Hamiltonian are

$$\begin{aligned}
&\frac{\mathrm{d}L}{\mathrm{d}\tau}=\frac{\mathrm{d}K}{\mathrm{d}l}=\mu\frac{\partial R}{\partial l},\quad \frac{\mathrm{d}G}{\mathrm{d}\tau}=\frac{\mathrm{d}K}{\mathrm{d}g}=\mu\frac{\partial R}{\partial g},\\
&\frac{\mathrm{d}l}{\mathrm{d}\tau}=-\frac{\mathrm{d}K}{\mathrm{d}L}=-\left[-2(nG+C_0)\right]^{\frac{1}{2}}-\mu\frac{\partial R}{\partial L},\\
&\frac{\mathrm{d}g}{\mathrm{d}\tau}=-\frac{\mathrm{d}K}{\mathrm{d}G}=\frac{nL}{\left[-2(nG+C_0)\right]^{\frac{1}{2}}}-\mu\frac{\partial R}{\partial G}.
\end{aligned} \tag{67}$$

where

$$\begin{aligned}
R=&1-n(eL\sin l\sin2\varphi+G\cos2\varphi)\\
&-z\left[G+\frac{1}{r_2}+\frac{\varepsilon_0}{r_3}+\frac{A}{r_2^3}-\frac{Ba^2n^2(1-2e\cos l)\sin^2 2\varphi}{4r_2^5}\right]
\end{aligned}$$

The Equation (67) forms the basis of a general perturbation theory for the problem in question. The solution given in Equations ((62) and (63)) are periodic if l and g have commensurable frequencies that is, if

$$\left|\frac{\eta_l}{\eta_g}\right| = \frac{2\left|nG + C_0\right|}{L} = \frac{p}{q} \quad (68)$$

where p and q are integers.

The periods of q_i, Q_i are $\frac{4\pi}{\eta_l}$ and $\frac{4\pi}{\eta_g}$, so that in case of commensurability, the period of the solution is $\frac{4\pi p}{\eta_l}$ and $\frac{4\pi q}{\eta_g}$.

5. Existence of Periodic Orbits When $\mu \neq 0$

Here we shall follow the method used by Choudhary [8] to prove the existence of periodic orbits when $\mu \neq 0$.

From Equations (67) when $\mu = 0$, we have for

$$\begin{aligned}
&\frac{\mathrm{d}L}{\mathrm{d}\tau} = \frac{\partial K_0}{\partial l} = 0, \quad \frac{\mathrm{d}G}{\mathrm{d}\tau} = \frac{\partial G_0}{\partial g} = 0, \\
&\frac{\mathrm{d}l}{\mathrm{d}\tau} = -\frac{\partial K_0}{\partial L} = -\left[-2\left(nG + C_0\right)\right]^{-\frac{1}{2}} = \eta_1(o) \quad \text{say} \\
&\frac{\mathrm{d}g}{\mathrm{d}\tau} = -\frac{\partial K_0}{\partial G} = \frac{nL}{\left[-2\left(nG + C_0\right)\right]^{\frac{1}{2}}} = \eta_2(o) \quad \text{say}
\end{aligned} \quad (69)$$

Let $x_1 = L, x_2 = G, y_1 = l$ and $y_2 = g$ then

$$\frac{\mathrm{d}x_1}{\mathrm{d}\tau} = \frac{\mathrm{d}x_2}{\mathrm{d}\tau} = 0, \quad \frac{\mathrm{d}y_1}{\mathrm{d}\tau} = \eta_1(o), \quad \frac{\mathrm{d}y_2}{\mathrm{d}\tau} = \eta_2(o)$$

Thus the Equation (69) can be written as

$$\frac{\mathrm{d}x_{i'}}{\mathrm{d}\tau} = 0 \text{ and } \frac{\mathrm{d}y_{i'}}{\mathrm{d}\tau} = \eta_i(o)$$

$$\Rightarrow x_i = a_i, y_i = \eta_i(o)\tau + \omega_i \quad (i = 1, 2) \quad (70)$$

These are generating solutions of the two-body problem.

Here a_i, η_i are constants given by

$$\eta_1(o) = \left[-\frac{\partial K_0}{\partial x_1}\right]_{x_1 = a_1}, \quad \eta_2(o) = \left[-\frac{\partial K_0}{\partial x_2}\right]_{x_2 = a_2} \quad (71)$$

The generating solutions will be periodic with the period τ_0 if

$$x_i(\tau_0) - x_i(o) = 0, \quad y_i(\tau_0) - y_i(o) = \eta_i(o)\tau = 2\pi\kappa_i \quad (i = 1, 2) \quad (72)$$

Here $\kappa_i (i = 1, 2)$ are integers, so that $\eta_i(o)$ are commensurable.

Let the general solution in the neighbourhood of the generating solution be periodic with the period $\tau_0 + \alpha\tau_0 = (1 + \alpha)\tau_0$, α is negligible quantity of the order of μ. Let us introduce new independent variable ς by the equation

$\varsigma = \frac{\tau}{1+\alpha}$. The period of the general solution will be

$\varsigma_0 + \alpha\varsigma_0 = (1+\alpha)\varsigma_0 = (1+\alpha)\frac{\tau_0}{1+\alpha} = \tau_0$ which is same as the period of the generating solution. The Equation (67) now can be written as

$$\frac{\mathrm{d}x_i}{\mathrm{d}\varsigma} = (1+\alpha)\frac{\partial K}{\partial y_i}, \quad \frac{\mathrm{d}y_i}{\mathrm{d}\varsigma} = -(1+\alpha)\frac{\partial K}{\partial x_i} \tag{73}$$

Following Poincare [9], the general solutions in the neighbourhood of the generating solutions may be written as

$$x_i = a_i + \beta_i + q_i(\varsigma), \quad y_i = \eta_i(o)\varsigma + \omega_i + \gamma_i + \eta_i(\varsigma) \tag{74}$$

Following the method of Hassan *et al.* [1], Bhatnagar [4] [5] and Choudhary [8], the Duboshin's conditions [2] for the existence of periodic orbits are given by

(i) $$\frac{\partial[K_1]}{\partial\omega_i} = 0 \quad (i = 1,2) \tag{75}$$

(ii) $$\frac{\partial[K_1]}{\partial a_i} = 0 \quad (i = 1,2) \tag{76}$$

(iii) $$J = \frac{\partial(q_2,\eta_1,\eta_2)}{\partial(\gamma_2,\beta_1,\beta_2)} \neq 0 \quad (\text{with } \mu = \beta_i = \gamma_i = 0) \tag{77}$$

where $[K_1]$ is the first degree term of K_1 given in Equation (31).

Here the Equations ((75) and (76)) together justify the Equation (77).

Following Bhatnagar [5], Hassan *et al.* [1], Equation (77) can be written as

$$J = \begin{vmatrix} \frac{\partial^2[K_1]}{\partial\omega_2^2} & 0 & 0 \\ \frac{\partial^2[K_1]}{\partial\omega_2\partial a_1} & -\tau_0\frac{\partial^2 K_0}{\partial a_1^2} & -\tau_0\frac{\partial^2 K_0}{\partial a_2\partial a_1} \\ \frac{\partial^2[K_1]}{\partial\omega_2\partial a_2} & -\tau_0\frac{\partial^2 K_0}{\partial a_1\partial a_2} & -\tau_0\frac{\partial^2 K_0}{\partial a_2^2} \end{vmatrix} = \tau_0^2\frac{\partial^2[K_1]}{\partial\omega_2^2}\begin{vmatrix} \frac{\partial^2 K_0}{\partial a_1^2} & \frac{\partial^2 K_0}{\partial a_2\partial a_1} \\ \frac{\partial^2 K_0}{\partial a_1\partial a_2} & \frac{\partial^2 K_0}{\partial a_2^2} \end{vmatrix}$$

$$\text{i.e., } J = \tau_0^2\frac{\partial^2[K_1]}{\partial\omega_2^2}\left[\frac{\partial^2 K_0}{\partial a_1^2}\cdot\frac{\partial^2 K_0}{\partial a_2^2} - \left(\frac{\partial^2 K_0}{\partial a_1\partial a_2}\right)^2\right] \tag{78}$$

From Equation (60),

$$K_0 = a_1\left[-2(na_2 + C_0)\right]^{\frac{1}{2}} - 1,$$

$$\frac{\partial K_0}{\partial a_1} = \left[-2(na_2 + C_0)\right]^{\frac{1}{2}}, \quad \frac{\partial^2 K_0}{\partial a_1^2} = 0$$

$$\text{and } \frac{\partial^2 K_0}{\partial a_1\partial a_2} = \frac{-n}{\left[-2(na_2 + C_0)\right]^{\frac{1}{2}}}.$$

$$\therefore J = \frac{n^2\tau_0^2}{2(na_2 + C_0)}\cdot\frac{\partial^2[K_1]}{\partial\omega_2^2} \tag{79}$$

Taking only zero degree terms (*i.e.*, for $e=0, h=l=y_1$)

$$r_1 = na = z, \quad r_2^2 = 1 + n^2a^2 + 2na\cos 2\varphi$$

$$r_3^2 = 1 + n^2a^2 + na\cos 2\varphi - \sqrt{3}na\sin 2\varphi = 1 + n^2a^2 + 2na\cos\left(2\varphi + \frac{\pi}{3}\right),$$

$$2\varphi = y_1 + y_2 - \frac{n\sqrt{x_1^2 - x_2^2}}{2\left(nx_2 + C_0\right)}\sin y_1,$$

$$y_1 = \eta_1^{(o)}\varsigma + \omega_1 + \gamma_1 + \eta_1(\varsigma), \qquad y_2 = \eta_2^{(o)}\varsigma + \omega_2 + \gamma_2 + \eta_2(\varsigma),$$
$$x_1 = a_1 + \beta_1 + \xi_1(\varsigma), \qquad x_2 = a_2 + \beta_2 + \xi_2(\varsigma),$$

$$\frac{\partial r_2}{\partial \omega_i} = -\frac{na\sin 2\varphi}{r_2}\left(\frac{2\partial\varphi}{\partial\omega_i}\right), \quad \frac{\partial r_3}{\partial \omega_i} = -\frac{na\sin\left(2\varphi + \frac{\pi}{3}\right)}{r_3}\left(\frac{2\partial\varphi}{\partial\omega_i}\right) \quad (i=1,2).$$

$$\left[K_1\right] = 1 - nG\cos 2\varphi - na\left[C_1 + \frac{1}{r_2} + \frac{\varepsilon_0}{r_3} + \frac{A}{r_2^3} - \frac{Ba^2n^2\sin^2 2\varphi}{4r_2^5}\right]. \tag{80}$$

Now,

$$\frac{\partial\left[K_1\right]}{\partial\omega_i} = nG\sin 2\varphi\left(2\frac{\partial\varphi}{\partial\omega_i}\right) - na\left[-\frac{1}{r_2^2}\frac{\partial r_2}{\partial\omega_i} - \frac{\varepsilon_0}{r_3^2}\frac{\partial r_3}{\partial\omega_i} - \frac{3A}{r_2^4}\frac{\partial r_2}{\partial\omega_i} - \frac{Bn^2a^2}{4}\left(\frac{r_2^5\, 2\sin 2\varphi\cdot\cos 2\varphi\, 2\frac{\partial\varphi}{\partial\omega_i} - \sin^2 2\varphi\cdot 5r_2^4\frac{\partial r_2}{\partial\omega_i}}{r_2^{10}}\right)\right],$$

$$= nG\sin 2\varphi\left(2\frac{\partial\varphi}{\partial\omega_i}\right) + \frac{na}{r_2^2}\left(-\frac{na\sin 2\varphi}{r_2}2\frac{\partial\varphi}{\partial\omega_i}\right) + \frac{na\varepsilon_0}{r_3^2}\left(-\frac{na\sin\left(2\varphi+\frac{\pi}{3}\right)}{r_3}2\frac{\partial\varphi}{\partial\omega_i}\right)$$
$$+\frac{3Ana}{r_2^4}\left(-\frac{na\sin 2\varphi}{r_2}2\frac{\partial\varphi}{\partial\omega_i}\right) + \frac{Bn^3a^3\sin 2\varphi\cdot\cos 2\varphi}{2r_2^5}\left(2\frac{\partial\varphi}{\partial\omega_i}\right)$$
$$-\frac{5Bn^2a^2}{4r_2^6}\left(-\frac{na\sin 2\varphi}{r_2}2\frac{\partial\varphi}{\partial\omega_i}\right)\sin^2 2\varphi,$$

$$= \left(2\frac{\partial\varphi}{\partial\omega_i}\right)\left[nG\sin 2\varphi - \frac{n^2a^2\sin 2\varphi}{r_2^3} - \frac{n^2a^2\varepsilon_0\sin 2\varphi}{2r_3^3} - \frac{\sqrt{3}n^2a^2\varepsilon_0\cos 2\varphi}{2r_3^3}\right.$$
$$\left. - \frac{3An^2a^2\sin 2\varphi}{r_2^5} + \frac{Bn^3a^3\sin 2\varphi\cdot\cos 2\varphi}{2r_2^5} + \frac{5Bn^3a^3\sin^3 2\varphi}{4r_2^7}\right],$$

$$\frac{\partial\left[K_1\right]}{\partial\omega_i} = \left(2\frac{\partial\varphi}{\partial\omega_i}\right)\left[\left(nG - \frac{n^2a^2}{r_2^3} - \frac{n^2a^2\varepsilon_0}{2r_3^3} - \frac{3An^2a^2}{r_2^5} + \frac{5Bn^4a^4\sin^2 2\varphi}{4r_2^7}\right)\sin 2\varphi + \left(\frac{Bn^3a^3\sin 2\varphi}{2r_2^5} - \frac{\sqrt{3}n^2a^2\varepsilon_0}{2r_3^3}\right)\cos 2\varphi\right]. \tag{81}$$

$$\Rightarrow \frac{\partial\left[K_1\right]}{\partial\omega_i} = \left(2\frac{\partial\varphi}{\partial\omega_i}\right)N, \quad \text{Similarly } \frac{\partial\left[K_1\right]}{\partial a_i} = \left(2\frac{\partial\varphi}{\partial a_i}\right)N \quad (i=1,2) \tag{82}$$

where

$$N=\left(nG-\frac{n^2a^2}{r_2^3}-\frac{n^2a^2\varepsilon_0}{2r_3^3}-\frac{3An^2a^2}{r_2^5}+\frac{5Bn^4a^4\sin^2 2\varphi}{4r_2^7}\right)\sin 2\varphi$$
$$+\left(\frac{Bn^3a^3\sin 2\varphi}{2r_2^5}-\frac{\sqrt{3}n^2a^2\varepsilon_0}{2r_3^3}\right)\cos 2\varphi$$

Here $\frac{\partial[K_1]}{\partial\omega_i}=\frac{\partial[K_1]}{\partial a_i}=0$ if and only if $N=0$ because $\frac{\partial\varphi}{\partial\omega_i},\frac{\partial\varphi}{\partial a_i}(i=1,2)$ are not necessarily zero. For this, putting $\cos 2\varphi=0$ then $\sin 2\varphi=1$,

$$\Rightarrow 2\varphi=\frac{\pi}{2}\Rightarrow\varphi=\frac{\pi}{4}$$

Thus from $N=0$, we get

$$nG-\frac{n^2a^2}{r_2^3}-\frac{n^2a^2\varepsilon_0}{2r_3^3}-\frac{3An^2a^2}{r_2^5}+\frac{5Bn^4a^4}{4r_2^7}=0 \tag{83}$$

where $r_2^2=1+n^2a^2, r_3^2=1+n^2a^2-\sqrt{3}na.$

From Equation (83),

$$G=n^2a^2\left[\frac{1}{r_2^3}+\frac{\varepsilon_0}{2r_3^3}+\frac{3A}{r_2^5}-\frac{5Bn^2a^2}{4r_2^7}\right],$$

where A and B are given in Equation (31).

Now from Equation (81),

$$\frac{\partial[K_1]}{\partial\omega_2}=2\frac{\partial\varphi}{\partial\omega_2}N$$
$$\frac{\partial^2[K_1]}{\partial\omega_2^2}=2\left[\frac{\partial^2\varphi}{\partial\omega_2^2}N+\frac{\partial\varphi}{\partial\omega_2}\frac{\partial N}{\partial\omega_2}\right]$$
$$2\varphi=y_1+y_2-\frac{n\sqrt{x_1^2-x_2^2}}{2(nx_2+C_0)}\sin y_1$$
$$2\frac{\partial\varphi}{\partial\omega_1}=1-\frac{n\sqrt{x_1^2-x_2^2}}{2(nx_2+C_0)}\cos y_1$$
$$2\frac{\partial\varphi}{\partial\omega_2}=1\quad\Rightarrow 2\frac{\partial^2\varphi}{\partial\omega_2^2}=0$$
$$\Rightarrow\frac{\partial^2[K_1]}{\partial\omega_2^2}=\frac{\partial N}{\partial\omega_2},$$

where

$$N=\left(nG-\frac{n^2a^2}{r_2^3}-\frac{n^2a^2\varepsilon_0}{2r_3^3}-\frac{3An^2a^2}{r_2^5}\right)\sin 2\varphi+\frac{5Bn^4a^4}{4}\left(\frac{\sin^3 2\varphi}{r_2^7}\right)$$
$$+\frac{Bn^3a^3}{4}\left(\frac{\sin 4\varphi}{r_2^5}\right)-\frac{\sqrt{3}n^2a^2\varepsilon_0}{2}\left(\frac{\cos 2\varphi}{r_3^3}\right)$$

Differentiating N partially with respect to ω_2, we get

$$\frac{\partial N}{\partial\omega_2}=-\frac{3n^3a^3}{2r_2^5}(2+\sigma_1-\sigma_2)+\frac{5n^3a^3}{4r_2^9}\left[3n^2a^2(3\sigma_1-5\sigma_2)-\frac{6}{\pi}(2\sigma_1-\sigma_2)\right]$$
$$-\frac{\sqrt{3}n^3a^3\varepsilon_0}{4r_3^5}\left[3\sqrt{3}na-2-2n^2a^2\right]$$

$$\frac{\partial N}{\partial \omega_2} = -\frac{3n^3a^3}{2r_2^5}\left(2+\sigma_1-\sigma_2\right) - \frac{5n^3a^3}{4r_2^9}\left[6\left(2\sigma_1-\sigma_2\right) - 3n^2a^2\left(3\sigma_1-5\sigma_2\right)\right] + \frac{\sqrt{3}n^2a^2\varepsilon_0}{4r_3^5}\left[2+2n^2a^2-3\sqrt{3}na\right] \quad (84)$$

where $\cos 2\varphi = 0, \sin 2\varphi = 1$.

If the collaboration of Equations ((83) and (84)) gives $\frac{\partial N}{\partial \omega_2} \neq 0$ and then $\frac{\partial^2 [K_1]}{\partial \omega_2^2} \neq 0$ which implies $J \neq 0$ *i.e.*, the orbit of the infinitesimal mass is periodic. This condition of periodic orbits given by Duboshin [2] may be confirmed by the regular trajectory of the Poincare surface of sections as in Hassan *et al.* [1] with suitable values of the parameters of μ, σ_1, σ_2.

6. Discussions and Conclusions

In order to prove the existence of periodic orbits of the first kind in the CR4BP, we have discussed the problem into five sections starting with introduction about the historical evolution of the topic. In the second section, we established the equations of motion of the infinitesimal mass under the perturbed gravitational field of the three primaries. In the present problem, the second primary is a tri-axial rigid body and other two are point masses *i.e.*, spheres. All the primaries are moving on their own circular orbits about the centre of mass of the dominant primaries P_1 and P_2. The primaries P_1 and P_2 are dominant in the sense that P_1 and P_2 have influence of attraction on the third primary P_3 and infinitesimal mass P_4 but P_3 and P_4 have no influence of attraction on the primaries P_1 and P_2 whereas P_3 has an influence of attraction on the infinitesimal mass P_4 only but not on P_1 and P_2. That's the reason for which the centre of mass of P_1 and P_2 didn't change. The second section ended with the energy integral of the infinitesimal mass at $P_4(x_1, x_2)$.

The energy function H contains three singularities $r_1 = 0, r_2 = 0$ and $r_3 = 0$ so in Hamiltonian mechanics, to keep the energy function $H = \text{constant}$, we need to eliminate any singularity for the case of collision with the corresponding primary. In the third section, we have introduced a suitable generating function for regularization of H to eliminate the singularity at $r_1 = 0$. After regularizing the Hamiltonian $H = C$, we have developed the canonical equations of motion corresponding to the regularized Hamiltonian $K = 0$.

In fourth section, we have established the generating solution *i.e.*, the solutions of the equations of motion of the infinitesimal mass by taking the first primary at the origin *i.e.*, at the centre of mass. On this consideration, we get $\mu = 0$ and the Hamiltonian becomes K_0. By taking K_0 as our Hamiltonian, we get the solution of the equations of motion, which is called generating solution. With the help of generating solution and the method of analytic continuation, we can find the general solution corresponding to the complete Hamiltonian $K = K_0 + \mu K_1$ where $\mu \neq 0$.

In fifth section, we have examined the existence of periodic orbits when $\mu \neq 0$ with the technique of Choudhary [8] applying to the conditions given by Duboshin [2]. Since our consideration satisfied all the conditions for periodic orbits given by Duboshin [2], hence we conclude that the periodic orbits of the infinitesimal mass around the first primary exist when suitable values of μ, σ_1, σ_2 are taken. By shifting the origin to the centre of the other primaries also, the existence of periodic orbits can be examined. Even by using "Mathematica", we can show the existence of periodic orbits of the infinitesimal mass around other primaries also, by taking suitable values of the parameters.

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