Effect of Perturbations in Coriolis and Centrifugal Forces on the Non-Linear Stability of $L_4$ in the Photogravitational Restricted Three Body Problem

Kavita Chauhan¹, S. N. Rai², Rajiv Aggarwal³

¹Department of P. G. Department of Mathematics, V.K.S. University, Ara, India
²Department of Department of Mathematics, S.B. College, Ara, India
³Department of Mathematics, Sri Aurobindo College, University of Delhi, Delhi, India

Email: kavitachauhan908@gmail.com, rajiv_agg1973@yahoo.com

Received 16 July 2015; accepted 21 December 2015; published 24 December 2015

Copyright © 2015 by authors and Scientific Research Publishing Inc.
This work is licensed under the Creative Commons Attribution International License (CC BY).
http://creativecommons.org/licenses/by/4.0/

Abstract

Effect of perturbations in Coriolis and centrifugal forces on the non-linear stability of the libration point $L_4$ in the restricted three body problem is studied when both the primaries are axis symmetric bodies (triaxial rigid bodies) and the bigger primary is a source of radiation. Moser’s conditions are utilized in this study by employing the iterative scheme of Henrard for transforming the Hamiltonian to the Birkhoff’s normal form with the help of double D’Alembert’s series. It is found that $L_4$ is stable for all mass ratios in the range of linear stability except for the three mass ratios $\mu_1$, $\mu_2$, and $\mu_3$, which depend upon the perturbations $\varepsilon_1$ and $\varepsilon_2$ in the Coriolis and centrifugal forces respectively and the parameters $A_1$, $A_2$, $A_3$ and $A_4$ which depend upon the semi-axes $a_1, b_1, c_1; a_2, b_2, c_2$ of the triaxial rigid bodies and $p$, the radiation parameter.

Keywords

Restricted Three Body Problem, Axis Symmetric Bodies; Non-Linear Stability, Libration Point $L_4$, Double D’Alembert’s Series Method

1. Introduction

We propose to study the effect of perturbations in Coriolis ($\varepsilon_1$) and centrifugal forces ($\varepsilon_2$) on the non-linear
stability of libration point \((L_4)\) when both the primaries are axis symmetric bodies and the bigger primary is a source of radiation. We use Moser’s conditions by employing the iterative scheme of Henrard (Deprit and Deprit-Bartholome \([1]\)), for transforming the involved Hamiltonian to the Birkhoff’s normal form with the help of double D’Alembert’s series. In the year 1983, Bhatnagar and Hallan \([2]\) investigated the perturbation effects in Coriolis and centrifugal forces in the non-linear aspect of stability of \(L_4\). Rajiv Aggarwal \(\text{et al.}\) \([3]\) studied the non-linear stability of \(L_4\) in the restricted three body problem for radiated axes symmetric primaries with resonances. Mamta Jain and Rajiv Aggarwal \([4]\) investigated the existence of non-collinear libration points and their stability (in linear sense) in the circular restricted three body problem, in which they had considered the smaller primary as an oblate spheroid and the bigger one as a point mass including the effect of dissipative force especially Stokes drag. Bhavneet Kaur and Rajiv Aggarwal \([5]\) studied the Robe’s restricted problem of 2 + 2 bodies when the bigger primary was a Roche ellipsoid. Jagdish Singh \([6]\) investigated the combined effects of perturbations, radiation and oblateness on the non-linear stability of triangular points. We have extended this study by taking the primaries as axis symmetric bodies. In the present paper, our aim is to examine the effect of perturbations in Coriolis and centrifugal forces in the non-linear stability of the libration point \(L_4\) of the restricted three body problem when both the primaries are axis symmetric bodies and the bigger primary is a source of radiation with its equatorial plane coincident with the plane of motion.

2. Equations of Motions and Linear Stability

We shall use dimensionless variables and adopt the notation and terminology of Szebehely \([7]\). The equations of motion of the infinitesimal mass \(m_3\) in a synodic co-ordinate system \((x, y)\) are

\[
\ddot{x} - 2n\dot{y} = \Omega_x, \quad \ddot{y} + 2n\dot{x} = \Omega_y,
\]

where

\[
\Omega = \frac{n^2}{2}(x^2 + y^2) + (1 - p)\left[\frac{1 - \mu}{r_1} + \frac{1 - \mu}{2r_1^3}A_1 + \frac{3(1 - \mu)y^2}{2r_1^5}A_2 + \frac{\mu}{2r_2}A_3 + \frac{3\mu y^2}{2r_2^5}A_4\right]
\]

\[
r_1^2 = (x - \mu)^2 + y^2, \quad r_2^2 = (x + 1 - \mu)^2 + y^2, \quad \mu = \frac{m_2}{m_1 + m_2} \leq 1/2, \quad n = 1 + \frac{3}{4}A_1 + \frac{3}{4}A_4.
\]

\[
A_1 = \frac{2a_1^2 - c_1^2 - b_1^2}{5R^2}, \quad A_2 = \frac{b_1^2 - a_1^2}{5R^2}, \quad A_3 = \frac{2a_2^2 - c_2^2 - b_2^2}{5R^2}, \quad A_4 = \frac{b_2^2 - a_2^2}{5R^2}.
\]

\(R\) is the distance between the primaries, \(m_1\) and \(m_2\) \((m_i \geq m_j)\) being the masses of the primaries. \(a_i, b_i\) and \(c_i\) are the semi-axes of the axis symmetric body of mass \(m_i\) and \(a_2, b_2\) and \(c_2\) are the semi-axes of the axis symmetric body of mass \(m_2\). The configuration is given in \textit{Figure 1}. Since \(0 < (p, A_1, A_2, A_4) \ll 1\), we will reject second and higher order terms in \(p, A_1, A_2, A_4\) and \(A_4\).

We adopt the method used by Bhatnagar and Hallan \([2]\) and give perturbation in Coriolis and centrifugal forces with the help of the parameters \(\alpha\) and \(\beta\) respectively. The unperturbed value of each is unity. Consequently we take the equations of motion as

\[
\ddot{x} - 2na\dot{y} - n^2\beta x = \frac{\partial F}{\partial x}, \quad \ddot{y} + 2na\dot{x} - n^2\beta y = \frac{\partial F}{\partial y},
\]

where

\[
\alpha = 1 + \epsilon_1, \quad |\epsilon_1| \ll 1, \quad \beta = 1 + \epsilon_2, \quad |\epsilon_2| \ll 1.
\]

\[
F = (1 - p)\left[\frac{1 - \mu}{r_1} + \frac{1 - \mu}{2r_1^3}A_1 + \frac{3(1 - \mu)y^2}{2r_1^5}A_2 + \frac{\mu}{2r_2}A_3 + \frac{3\mu y^2}{2r_2^5}A_4\right] + \frac{\mu}{2r_2}A_3 + \frac{3\mu y^2}{2r_2^5}A_4.
\]

Equations of motion of mass \(m_3\) can be put in the form

\[
\ddot{x} - 2na\dot{y} = \Omega_x, \quad \ddot{y} + 2na\dot{x} = \Omega_y,
\]

where
Figure 1. Configuration of the photogravitational restricted problem with both the primaries axis symmetric bodies and the bigger primary is source of radiation.

\[ \Omega = \frac{\beta n^2}{2} \left( x^2 + y^2 \right) + \left( 1 - p \right) \left( \frac{1 - \mu}{r_1} + \frac{(1 - \mu) A_1}{2r_1^3} + \frac{3(1 - \mu) A_2 y^2}{2r_1^5} \right) + \left( \frac{\mu + \mu A_1}{2r_2^3} + \frac{3\mu A_2 y^2}{2r_2^5} \right). \]

3. Location of Libration Point of \( L_4 \)

At \( L_4 \), we have \( \Omega_x = 0, \Omega_y = 0 \) and \( y \neq 0 \).

On solving above equations, we get

\[ x = \frac{-y}{2} + \frac{1}{3} \left( 1 - A_1 - \frac{7}{8} + \frac{1}{2\mu} A_2 + \frac{11 - 7\mu}{8(1 + \mu)} A_4 \right) \]

\[ y = \frac{\sqrt{3}}{2} \left( \frac{1}{3\sqrt{3}} p - \frac{2}{3\sqrt{3}} \epsilon_2 \right) + \frac{1}{2\sqrt{3}} A_4 - \frac{1}{2\sqrt{3}} A_1 + \frac{-11 + 15\mu}{8\sqrt{3}(1 + \mu)} A_4. \] (2)

The Lagrangian \( (L) \) of the system of equations (1) is

\[ L = \frac{1}{2} \left( \dot{x}^2 + \dot{y}^2 \right) + n \left( 1 + \epsilon_1 \right) \left( xy \dot{y} - y \dot{x} \right) + \frac{n^2}{2} \left( 1 + \epsilon_2 \right) (x^2 + y^2) \]

\[ + \left( 1 - p \right) \left( \frac{1 - \mu}{r_1} + \frac{1 - \mu}{2r_1^3} A_1 + \frac{3(1 - \mu)}{2r_1^5} A_2 y^2 \right) + \frac{\mu A_1}{2r_2^3} + \frac{3\mu A_2 y^2}{2r_2^5}. \]

Shift the origin to \( L_4 \) and expanding in power series of \( x \) and \( y \), we get
\[ L = L_0 + L_1 + L_2 + L_3 + L_4 + \cdots \] 

where

\[ L_0 = \frac{11 + \gamma^2}{8} + \frac{1}{16} \left( 13 + 4\gamma + 3\gamma^2 \right) A_1 + \frac{9}{16} \left( 1 + \gamma \right) A_2 + \frac{1}{16} \left( 13 - 4\gamma + 3\gamma^2 \right) A_3 \]

\[ + \frac{9}{16} \left( 1 - \gamma \right) A_4 + \frac{e_2}{2} - \frac{1 + \gamma}{2} p \]

\[ L_1 = -\frac{\sqrt{5}}{24} \left( 12 + 12\varepsilon_1 - \frac{16}{3} \varepsilon_2 + 7 - 15\gamma \right) A_4 + \frac{1}{3} A_5 + \frac{5}{3} A_6 + \frac{7 + 15\gamma}{1 + \gamma} A_7 - \frac{8}{3} p \]

\[ \frac{1}{8} \left( 4\gamma + 4\varepsilon_1 - \frac{8}{3} p + (4 + 3\gamma) A_4 + \frac{15 - 7\gamma}{1 - \gamma} A_5 + (-4 + 3\gamma) A_6 - \frac{15 + 7\gamma}{1 + \gamma} A_7 \right) \]

\[ L_2 = \frac{1}{2} \left( \dot{x}^2 + \dot{y}^2 \right) + \frac{1}{4} \left( 4 + 4\varepsilon_1 + 3A_4 + 3A_1 \right) \left( xy - \dot{x}\dot{y} \right) \]

\[ + \frac{3}{16} \left( x^2 - 20 + 10\varepsilon_2 + 6 + 47(4\gamma) A_4 + \left( \frac{15}{4} + \frac{47}{4} \gamma + \frac{1}{2\mu} \left( 1 + 7\gamma \right) \right) A_2 \]

\[ + (5 - 4\gamma) A_3 + \left( \frac{15}{4} - \frac{47}{4} \gamma + \frac{1}{2\mu} \left( 1 - 7\gamma \right) \right) A_4 + \frac{2\left( 1 - 3\gamma \right)}{3} p \]

\[ - \frac{\sqrt{5} xy}{8} \left( 22 - 6\varepsilon_1 \gamma + (6 + 13\gamma) A_4 + \left( \frac{87}{4} + \frac{15\gamma}{4} + \frac{1}{2\mu} \left( 11\gamma - 3 \right) \right) A_2 \right) \]

\[ - (6 - 13\gamma) A_5 + \left( \frac{87}{4} - \frac{15\gamma}{4} - \frac{11\gamma + 3}{2(1 - \mu)} \right) A_4 - \frac{2}{3} \left( 3 - \gamma \right) p \]

\[ + \frac{3}{16} \gamma^2 \left( 6 + \frac{14\varepsilon_2}{3} + 11A_4 + \left( \frac{1}{4} - \frac{15\gamma}{4} + \frac{1}{2\mu} \left( 3 - 11\gamma \right) \right) A_2 \right) \]

\[ + 11A_3 + \left( \frac{1}{4} - \frac{15\gamma}{4} + \frac{3 + 11\gamma}{2(1 - \mu)} \right) A_4 - \frac{2}{3} \left( 3 - 3\gamma \right) p \]

\[ L_3 = -\frac{1}{32} \left( 14\gamma + 50\varepsilon_1 \right) - (6 + 25\gamma) A_4 + \left( 7 - \frac{15\gamma}{2} + \frac{25 - 37\gamma}{2\mu} \right) A_2 \]

\[ + (6 + 25\gamma) A_3 + \left( 7 - \frac{15\gamma}{2} + \frac{37 + 25\gamma}{2(1 - \mu)} \right) A_4 + \left( \frac{4}{3} (4 + 4\gamma) \right) p \]

\[ - \frac{\sqrt{5}}{32} \left( 6 + 82\varepsilon_2 \right) A_4 + \left( 75 + \frac{435\gamma}{2} + \frac{1}{2\mu} (41 + 75\gamma) \right) A_2 \]

\[ - \frac{4}{3} (-8 + 21\gamma) p + (43 - 60\gamma) A_5 + \left( \frac{75 - 435\gamma}{2} + \frac{1}{2(1 - \mu)} (41 - 75\gamma) \right) A_4 \]

\[ + \frac{3}{32} \left( 22\gamma + 30\varepsilon_2 + (22 + 65\gamma) A_4 + \left( 76 + \frac{55\gamma}{2} + \frac{45\gamma - 41}{2\mu} \right) A_2 \right) \]

\[ + \frac{4}{3} (2 + 3\gamma) p - (22 - 65\gamma) A_5 + \left( 76 + \frac{55\gamma}{2} + \frac{1}{2(1 - \mu)} (41 + 45\gamma) \right) A_4 \]

\[ - \frac{\sqrt{5}}{32} \left( 6 + \frac{2}{3} \varepsilon_2 + 23A_4 + \left( -\frac{75\gamma}{2} + \frac{1 - 45\gamma}{2\mu} \right) A_2 - \frac{4}{3} (2 - 9\gamma) p + 23A_3 + \left( 75\gamma + \frac{1 + 45\gamma}{2(1 - \mu)} \right) A_4 \right) \]

\[ x y^2 \]

\[ - \frac{\sqrt{5}}{32} \left( 6 + \frac{2}{3} \varepsilon_2 + 23A_4 + \left( -\frac{75\gamma}{2} + \frac{1 - 45\gamma}{2\mu} \right) A_2 - \frac{4}{3} (2 - 9\gamma) p + 23A_3 + \left( 75\gamma + \frac{1 + 45\gamma}{2(1 - \mu)} \right) A_4 \right) \]

\[ y^3 \]
$$L_4 = -\frac{1}{256} \left( 574 + 190 \epsilon x_2 + (285 + 200 \gamma) A_i + p_{27} A_2 + q_{2} + (285 - 200 \gamma) A_{4} + p_{27}^\prime A_{4} \right) x^4$$

$$+ \frac{5 \sqrt{3}}{192} \left( 30 \gamma + \frac{86 \gamma}{3} \epsilon x_2 + \left( -54 + 53 \gamma \right) A_4 + p_{28} A_2 + \left( 54 + 53 \gamma \right) A_{4} + p_{28}^\prime A_{4} + \frac{q_{28} \gamma}{3} \right) x^3 y$$

$$+ \frac{3}{128} \left( 82 + 230 \epsilon x_3 + \left( 405 + 340 \gamma \right) A_4 + p_{29} A_2 + \left( 405 - 340 \gamma \right) A_{4} + p_{29}^\prime A_{4} + q_{29} \gamma p \right) x^2 y^2$$

$$- \frac{5 \sqrt{3}}{64} \left( 18 \gamma + \frac{74 \gamma}{3} + (18 + 71 \gamma) A_{4} + p_{30} A_2 + \left( -18 + 71 \gamma \right) A_{4} + p_{30}^\prime A_{4} + q_{30} \gamma p \right) xy^3$$

$$+ \frac{3}{256} \left( 2 - \frac{110 \epsilon x_3}{3} + 65 A_{4} + p_{31} A_{2} + 65 A_{4} + p_{31}^\prime A_{4} + q_{31} \gamma p \right) y^4$$

Hamiltonian function $H$ corresponding to above Lagrangian is given by:

$$H = H_0 + H_1 + H_2 + H_3 + H_4 + \cdots$$

where

$$H_0 = -L_0, \quad H_1 = 0, \quad H_3 = -L_3, \quad H_4 = -L_4$$

and

$$H_2 = \frac{1}{2} \left( p_x^2 + p_y^2 \right) + n a \left( y p_x - x p_y \right) + E x^2 + F y^2 + G x y,$$

$$E = \frac{1}{8} + e_1 - \frac{5}{8} \epsilon x_2 - \frac{3}{16} \left( 1 + 4 \gamma \right) A_i - \frac{3}{64} \left( 15 + 47 \gamma + \frac{2 \left( 1 + 7 \gamma \right)}{\mu} \right) A_{4} - \frac{3 \left( 1 - 4 \gamma \right)}{16} A_{4}$$

$$- \frac{3}{64} \left( 15 + 47 \gamma + \frac{2 \left( 1 - 7 \gamma \right)}{1 - \mu} \right) A_{4} - \frac{1}{8} \left( 1 - 3 \gamma \right) p,$$

$$F = \frac{5}{8} + e_1 - \frac{7}{8} \epsilon x_3 - \frac{1}{16} A_i + \frac{3}{64} \left( -1 + 15 \gamma + \frac{2 \left( -3 + 11 \gamma \right)}{\mu} \right) A_{4}$$

$$+ \frac{1}{16} \left( 1 - 15 \gamma + \frac{2 \left( 3 + 11 \gamma \right)}{1 - \mu} \right) A_{4} + \frac{1}{8} \left( 1 - 3 \gamma \right) p,$$

$$G = \frac{y (9 + 11 \epsilon x_2)}{4 \sqrt{3}} + \frac{\sqrt{3}}{8} A_i + \frac{1}{4} \left( 87 + 15 \gamma + \frac{2 \left( 3 + 11 \gamma \right)}{\mu} \right) A_{4}$$

$$+ \left( -6 + 13 \gamma \right) A_{4} - \frac{1}{4} \left( 87 - 15 \gamma + \frac{2 \left( 3 + 11 \gamma \right)}{1 - \mu} \right) A_{4} - \frac{2 \left( 3 - \gamma \right)}{3} p.$$
\[ \lambda^4 + \frac{\lambda^2}{2} + \left(1 + 8\epsilon_1 - 3\epsilon_2 - \frac{3\gamma}{2} A_1 + \frac{3\gamma}{2} A_1 - 3(1 + \gamma) A_2 - 3(1 - \gamma) A_4 \right) \]
\[ + \frac{9}{16} \left(1 - \gamma^2\right) \left(3 + \frac{22}{3} \epsilon_2 + 13A_1 + 13A_3 + \frac{2}{3} p\right) + \frac{45}{64} \left(7 + 4\gamma - 3\gamma^2\right) A_2 \]
\[ + \frac{45}{64} \left(7 - 4\gamma - 3\gamma^2\right) A_4 = 0. \]

Let the discriminant of the characteristic Equation (6) be denoted by \( D \).

If \( D < 0 \), then \( \lambda^2 < 0 \), it is bounded, hence stable when \( 0 < \mu < \mu_c \) where
\[
\mu_c = 0.0385208965 \cdots + (0.642057883 \cdots) e_i - (0.338863882 \cdots) e_2 - (0.00891747 \cdots) p \\
-(0.285001784 \cdots) A_1 - (1.381268434 \cdots) A_2 - (0.06278 \cdots) A_3 - (0.10349 \cdots) A_4.
\]

Let the roots of characteristic Equation (6) be \( \pm \omega'_1 \) and \( \pm \omega'_2 \). These are long term and short term perturbed frequencies, which are given by
\[
\omega'^2_1 + \omega'^2_2 = 1 + 8\epsilon_1 - 3\epsilon_2 - \frac{3\gamma}{2} A_1 + \frac{3\gamma}{2} A_1 - 3(1 + \gamma) A_2 - 3(1 - \gamma) A_4 \\
\omega'^2_1 \omega'^2_2 = \frac{9}{16} \left(1 - \gamma^2\right) \left(3 + \frac{22}{3} \epsilon_2 + 13A_1 + 13A_3 + \frac{2}{3} p\right) \\
+ \frac{45}{64} \left(7 + 4\gamma - 3\gamma^2\right) A_2 + \frac{45}{64} \left(7 - 4\gamma - 3\gamma^2\right) A_4.
\]

Here \( \omega'_1, \omega'_2 \) represent the perturbed basic frequencies. The unperturbed basic frequencies \( \omega_1, \omega_2 \), are given by
\[
\omega_1^2 + \omega_2^2 = 1, \quad \omega_1 \omega_2 = \frac{27}{16} \left(1 - \gamma^2\right), \quad (0 < \omega_2 < 0.5 < \omega_1 < 1).
\]

We may write
\[
\omega'_1 = \omega_1 \left(1 + p_1 \epsilon_1 + p_2 \epsilon_2\right), \quad \omega'_2 = \omega_2 \left(1 + q_1 \epsilon_1 + q_2 \epsilon_2\right),
\]
by taking perturbations in the Coriolis and centrifugal forces. Here \( p_1, p_2, q_1, q_2 \) are to be determined so that Equations (8) are satisfied. Simple calculations give
\[
p_1 = -q_1 = \frac{4}{k'_1} = \frac{4}{k'_2}, \quad p_2 = \frac{22\omega^2_1 - 49}{18k'_1} = \frac{22\omega^2_2 - 49}{18k'_2}, \\
q_2 = \frac{22\omega^2_2 - 49}{18k'_1} = -\frac{22\omega^2_1 - 49}{18k'_2}
\]
where
\[
k'_1 = 2\omega^2_1 - 1; \quad k'_2 = 1 - 2\omega^2_2.
\]

4. Determination of the Normal Co-Ordinates

To express \( H_2 \) in normal form, we consider the set of linear Equation (5), the solution of which can be obtained as
\[
\frac{x}{2n\lambda - G} = \frac{y}{\lambda^2 - n^2 + 2E} = \frac{p_x}{n\lambda^2 - \lambda G - 2nE + n^3} = \frac{p_y}{\lambda^3 + n^2 \lambda + 2\lambda E - nG}.
\]

We use the canonical transformations from the phase space \((x, y, p_x, p_y)\) into the phase space of the angles \((\phi_1, \phi_2)\) and the action moment as \((I_1, I_2)\) i.e.
\[
X = JT
\]
where
Following the procedure of Bhatnagar and Hallan [2], we get the normal form of the Hamiltonian
\[ H_2 = \alpha_1 I_1 - \alpha_2 I_2. \]

Taking
\[ H = H_0 + H_2, \]

Equations of motion
\[ \frac{dI_i}{dt} = \frac{\partial H}{\partial \phi_i}, \quad \frac{d\phi_i}{dt} = -\frac{\partial H}{\partial I_i} \quad (i = 1, 2) \]
become
\[ \frac{dI_i}{dt} = 0; \quad (i = 1, 2), \]
\[ \frac{d\phi_i}{dt} = \alpha_i, \quad \frac{d\phi_i}{dt} = -\alpha_i. \]

The general solution of the equations of the motion is
\[ I_i = \text{constant} \quad (i = 1, 2), \]
\[ \phi_i = \alpha_i t + \text{constant}, \]
\[ \phi_2 = -\alpha_2 t + \text{constant}. \]

5. Second Order Normalization

Now, to perform Birkhoff’s normalization, the coordinates \((x, y)\) are to be expanded in double D’Alembert series:
\[ x = \sum_{m=0}^{\infty} B^{1.0}_n (\phi_1, \phi_2, I_1, I_2), \quad y = \sum_{m=0}^{\infty} B^{0.1}_n (\phi_1, \phi_2, I_1, I_2), \quad (11) \]

where \(B^{1.0}_n\) and \(B^{0.1}_n\) are homogenous functions of degree \(n\) in \(I_1^{1/2}, I_2^{1/2}\) and are in the form
\[ \sum_{m=0}^{\infty} \sum_{j} \sum_{m, n} \sum_{i, j} \left[ C_{n, m, m, j} \cos(i\phi_1 + j\phi_2) + S_{n, m, m, j} \sin(i\phi_1 + j\phi_2) \right]. \]
The double summation over the indices $i$ and $j$ is such that:

1) $i$ runs over those integers in the interval $0 \leq i \leq n - m$ that have the same parity as $n - m$
2) $j$ runs over those integers in the interval $-m \leq j \leq m$ that have the same parity as $m$.

$I_1$ and $I_2$ are to be regarded as constants of integration and $\phi_1, \phi_2$ are to be determined as linear functions of time ($t$) such that

$$
\dot{\phi}_1 = \omega'_1 + \sum_{n=1}^{\infty} f_{2n}(I_1, I_2), \quad \dot{\phi}_2 = -\omega'_2 + \sum_{n=1}^{\infty} g_{2n}(I_1, I_2).
$$

where $f_{2n}$ and $g_{2n}$ are of the form

$$
f_{2n} = \sum_{n=2m-2}^{\infty} f'_{2(n-w)2m} I_1^w I_2^m, \quad g_{2n} = \sum_{n=2m-2}^{\infty} g'_{2(n-w)2m} I_1^w I_2^m.
$$

According to Deprit and Deprit Bartholome [1], the canonical character of the transformation will be ensured formally by requesting that the double D’Alembert series satisfy the identities

$$
\{x, y\} = \{x, \dot{x}\} = 0, \quad \{y, \dot{y}\} = 0, \quad \{x, \dot{y}\} = 1, \quad \{\dot{x}, \dot{y}\} = 0.
$$

Where the left hand members stand for the Poisson’s brackets with respect to the phase variables $(\phi_1, \phi_2, I_1, I_2)$. The first order components $B^{1,0}_1$ and $B^{0,1}_2$ in $I_1^{1/2}$ and $I_2^{1/2}$ are the values of $x$ and $y$ given by Equation (10)

$$
B^{1,0}_1 = J_{13} \left( 2\omega'_1 \right)^{1/2} I_1^{1/2} \cos \phi_1 + J_{14} \left( 2\omega'_2 \right)^{1/2} I_2^{1/2} \cos \phi_2,
$$

$$
B^{0,1}_2 = J_{21} \left( \frac{2}{\omega'_2} \right)^{1/2} I_1^{1/2} \sin \phi_1 + J_{22} \left( \frac{2}{\omega'_1} \right)^{1/2} I_2^{1/2} \sin \phi_2 + J_{23} \left( 2\omega'_1 \right)^{1/2} I_1^{1/2} \cos \phi_1
$$

$$
+ J_{24} \left( 2\omega'_2 \right)^{1/2} I_2^{1/2} \cos \phi_2.
$$

The values of $J_{13}, J_{14}, J_{21}, J_{22}, J_{23}, J_{24}$ can be obtained from Appendix.

Proceeding as in Deprit and Deprit-Bartholome [1], it is observed that the second order components $B^{2,0}_1$ and $B^{0,2}_2$ are solutions of the partial differential equations

$$
\Delta_1 \Delta_2 B^{1,0}_1 = \Phi_2, \quad \Delta_1 \Delta_2 B^{0,1}_2 = -\Psi_2,
$$

where

$$
\Delta_i = D^2 + \omega'_i, \quad (i = 1, 2),
$$

$$
\Phi_2 = \left( D^2 - \frac{3}{8} \left( 6 + \frac{14}{3} \epsilon_2 + 11A_i + p_{10} A_2 + 11A_i + p'_{10} A_4 + q_{10} p \right) \right) X_2 + \left[ \frac{1}{2} \left( 4 + 4 \epsilon_1 + 3A_i + 3A_i \right) D \right.
$$

$$
- \frac{\sqrt{3}}{8} \left( 6\gamma + \frac{22\gamma_3}{3} \epsilon_2 + \left( 6 + 13\gamma \right) A_i + p_{10} A_2 + \left( -6 + 13\gamma \right) A_i + p'_{10} A_4 + q_{10} p \right) \right] Y_2.
$$

$$
\Psi_2 = - \left( D^2 - \frac{3}{8} \left( 2 + \frac{10}{3} \epsilon_2 + \left( 5 + 4\gamma \right) A_i + p_{10} A_2 + \left( 5 - 4\gamma \right) A_i + p'_{10} A_4 + q_{10} p \right) \right) X_2 + \left[ \frac{1}{2} \left( 4 + 4 \epsilon_1 + 3A_i + 3A_i \right) D \right.
$$

$$
+ \frac{\sqrt{3}}{8} \left( 6\gamma + \frac{22\gamma_3}{3} \epsilon_2 + \left( 6 + 13\gamma \right) A_i + p_{10} A_2 + \left( -6 + 13\gamma \right) A_i + p'_{10} A_4 + q_{10} p \right) \right] Y_2.
$$

$$
D = \omega'_i \frac{\partial}{\partial \phi_i} - \omega'_i \frac{\partial}{\partial \phi_i}
$$

$X_2$ and $Y_2$ are obtained by

$$
X_2 = \left( \frac{\partial L_2}{\partial \dot{X}} \right)_{j \sum \phi_{1j}^{1,0}}, \quad Y_2 = \left( \frac{\partial L_2}{\partial \dot{Y}} \right)_{j \sum \phi_{1j}^{1,0}}.
$$

Now
$B_{2,0}^i = r_i I_{1} + r_I I_{2} + r_i I_{3} \cos 2\phi_i + r_i I_{4} \cos 2\phi_i + r_i I_{5} I_{6} \cos (\phi_i - \phi_1) + r_I I_{7} I_{8} I_{9} I_{10} \cos (\phi_i + \phi_1),$

$B_{2,0}^j = s_j I_{1} + s_j I_{2} + s_j I_{3} \cos 2\phi_i + s_j I_{4} \cos 2\phi_i + s_j I_{5} I_{6} \cos (\phi_i - \phi_1) + s_I I_{7} I_{8} I_{9} I_{10} \cos (\phi_i + \phi_1),$

where

$r_i = r_{i,1} (1 + \alpha_i \epsilon_i + \alpha_i \epsilon_i^2) + \left( r_{i,2} + r_{i,3} \gamma^2 \right) A_i + r_{i,4} A_i + r_{i,5} p$

$r_j = r_{j,1} (1 + \alpha_i \epsilon_i + \alpha_i \epsilon_i^2) + \left( r_{j,2} + r_{j,3} \gamma^2 \right) A_i + r_{j,4} A_i + r_{j,5} p$

$s_i = s_{i,1} (1 + \beta_i \epsilon_i + \beta_i \epsilon_i^2) + \left( s_{i,2} + s_{i,3} \gamma^2 \right) A_i + s_{i,4} A_i + s_{i,5} p$

$s_j = s_{j,1} (1 + \beta_i \epsilon_i + \beta_i \epsilon_i^2) + \left( s_{j,2} + s_{j,3} \gamma^2 \right) A_i + s_{j,4} A_i + s_{j,5} p$

The values of all $r_i, r_j, s_i, s_j$ for $i = 1, 2, \ldots, 6; j = 7, 8, \ldots, 10$ can be obtained from the authors on request as the expressions are very long and contained in large number of pages.

6. Third-Order Terms in $H$

Following the procedure of Bhatnagar and Hallan [2], Hamiltonian $H$ given by Equation (4) transforms to the Hamiltonian in which the 3rd order term in $I_{1,2}$ and $I_{3,4}$ is zero. That is $H_3 = 0$.

7. Second Order Coefficient in the Frequencies

Following the iterative procedure of Henrard, the third order homogeneous components $B_{1,0}^i$ and $B_{1,0}^j$ in Equation (11) can be obtained by partial differential equations

$\Delta \Delta B_{1,0} = \Phi - 2 f_2 P - 2 g_2 Q,$

$\Delta \Delta B_{0,1} = \Psi - 2 f_2 U - 2 g_2 V,$

where

$\Phi = \left( D^2 - \lambda_1 \right) X_1 + \left( \lambda_1 D + \lambda_3 \right) Y_1,$

$\Psi = \left( D^2 - \lambda_1 \right) Y_1 - \left( \lambda_1 D + \lambda_3 \right) X_1,$

$P = \frac{\partial}{\partial \phi_1} \left[ \left( \omega_0^2 - \frac{\partial^2}{\partial \phi_i^2} \right) - \lambda_1 \left( \omega_0^2 - \frac{\partial^2}{\partial \phi_i^2} \right) + \left( \lambda_1 \omega_0^2 - \frac{\partial^2}{\partial \phi_i^2} \right) \left( \omega_0^2 - \frac{\partial^2}{\partial \phi_i^2} \right) \right],$

$Q = \frac{\partial}{\partial \phi_1} \left[ \left( \omega_0^2 - \frac{\partial^2}{\partial \phi_i^2} \right) - \lambda_1 \left( \omega_0^2 - \frac{\partial^2}{\partial \phi_i^2} \right) + \left( \lambda_1 \omega_0^2 - \frac{\partial^2}{\partial \phi_i^2} \right) \left( \omega_0^2 - \frac{\partial^2}{\partial \phi_i^2} \right) \right],$

$U = \frac{\partial}{\partial \phi_1} \left[ \left( \omega_0^2 - \frac{\partial^2}{\partial \phi_i^2} \right) + \lambda_1 \omega_0^2 \frac{\partial^2}{\partial \phi_i^2} \right] - \left( \lambda_1 \omega_0^2 \frac{\partial^2}{\partial \phi_i^2} \right) \left( \omega_0^2 - \frac{\partial^2}{\partial \phi_i^2} \right) \left( \omega_0^2 - \frac{\partial^2}{\partial \phi_i^2} \right) \right],$

$V = \frac{\partial}{\partial \phi_1} \left[ \left( \omega_0^2 - \frac{\partial^2}{\partial \phi_i^2} \right) - \omega_0^2 \frac{\partial^2}{\partial \phi_i^2} \right] - \left( \lambda_1 \omega_0^2 \frac{\partial^2}{\partial \phi_i^2} \right) \left( \omega_0^2 - \frac{\partial^2}{\partial \phi_i^2} \right) \left( \omega_0^2 - \frac{\partial^2}{\partial \phi_i^2} \right) \right].$
The values of $p_{ij}, p_{ij}', q_{ij}$ are given in Appendix.

The partial derivatives in the last two equations have been obtained by substituting $x = B_{1,0}^{1.0} + B_{2,0}^{1.0}$ and $y = B_{1,0}^{3.0} + B_{2,0}^{3.0}$ in $L_1$ and $L_2$. Now choosing

$$f_2 = f_{2,0}'I_1 + f_{0,2}'I_2, \quad g_2 = g_{2,0}'I_1 + g_{0,2}'I_2.$$  

We find that

$$A = f_{2,0}' = \frac{\text{coefficient of } \cos \Phi_1}{2(\text{coefficient of } \cos \Phi_1 \text{ in } P)}$$

$$B = f_{0,2}' = g_{2,0}' = \frac{\text{coefficient of } \cos \Phi_2}{2(\text{coefficient of } \cos \Phi_2 \text{ in } Q)}.$$
After simplification the values of $A$, $B$ and $C$ are given by:

\[
A = \frac{(\omega^2-1)(124\omega^5 - 696\omega^2 + 81)}{72k^4(5\omega^2-1)} \left[ 1 + (\chi - \kappa)\varepsilon_1 + (\chi' - \kappa')\varepsilon_2 \right] \\
+ \frac{1696\omega^6 - 20320\omega^4 + 14547\omega^2 - 1107}{6(\omega^2-1)(124\omega^4 - 696\omega^2 + 81)} A_i \\
+ \frac{1696\omega^6 - 20320\omega^4 + 14547\omega^2 - 1107}{6(\omega^2-1)(124\omega^4 - 696\omega^2 + 81)} A_i \\
- \frac{3(1208\omega^8 + 2914\omega^6 + 725\omega^4 - 624\omega^2 + 45)}{2k^2(5\omega^2-1)(\omega^2-1)(124\omega^4 - 696\omega^2 + 81)} A_{i,\gamma} \\
+ (\xi - \eta) A_i + \frac{3(1208\omega^8 + 2914\omega^6 + 725\omega^4 - 624\omega^2 + 45)}{2k^2(5\omega^2-1)(\omega^2-1)(124\omega^4 - 696\omega^2 + 81)} A_{i,\gamma} \\
+ (\xi_i - \eta_i) A_i + (\sigma - \rho) p \\
\]

\[
B = \frac{u(64u^2 + 43)}{6k^2k^2(5\omega^2-1)(5\omega^2-1)} \left[ 1 + (X_1 - \kappa)\varepsilon_1 + (X' - \kappa')\varepsilon_2 - \frac{6719u^2 - 2319}{6(64u^2 + 43)} A_i - \frac{6719u^2 - 2319}{6(64u^2 + 43)} A_i \\
+ \frac{3(1116800u^8 + 1504808u^6 - 10165353u^4 + 1972620u^2 - 93312)}{16u^2k^2(1 - 5\omega^2)(1 - 5\omega^2)(64u^2 + 43)} A_{i,\gamma} \\
- \frac{3(1116800u^8 + 1504808u^6 - 10165353u^4 + 1972620u^2 - 93312)}{16u^2k^2(1 - 5\omega^2)(1 - 5\omega^2)(64u^2 + 43)} A_{i,\gamma} \\
+ \frac{15(1 + \gamma)}{(1 - 5\omega^2)(1 - 5\omega^2)} A_i + \frac{15(1 - \gamma)}{(1 - 5\omega^2)(1 - 5\omega^2)} A_i \\
+ (\tau - \rho) p \right],
\]

\[
C = \frac{(\omega^2-1)(124\omega^5 - 696\omega^2 + 81)}{72k^4(5\omega^2-1)} \left[ 1 + (\chi_2 - \kappa_2)\varepsilon_1 + (\chi'_2 - \kappa'_2)\varepsilon_2 \right] \\
+ \frac{1696\omega^6 - 20320\omega^4 + 14547\omega^2 - 1107}{6(\omega^2-1)(124\omega^4 - 696\omega^2 + 81)} A_i \\
+ \frac{1696\omega^6 - 20320\omega^4 + 14547\omega^2 - 1107}{6(\omega^2-1)(124\omega^4 - 696\omega^2 + 81)} A_i \\
- \frac{3(1208\omega^8 + 2914\omega^6 + 725\omega^4 - 624\omega^2 + 45)}{2k^2(5\omega^2-1)(\omega^2-1)(124\omega^4 - 696\omega^2 + 81)} A_{i,\gamma} \\
+ (\xi - \eta) A_i + \frac{3(1208\omega^8 + 2914\omega^6 + 725\omega^4 - 624\omega^2 + 45)}{2k^2(5\omega^2-1)(\omega^2-1)(124\omega^4 - 696\omega^2 + 81)} A_{i,\gamma} \\
+ (\xi_i - \eta_i) A_i + (\sigma - \rho) p \right].
\]
The values of $\xi, \zeta, \eta, \xi', \zeta', \eta'$, $\xi, \zeta, \eta, \xi', \zeta', \eta', \sigma, \sigma', \rho, \rho'$ and $\tau$ can be obtained from the author on request as the expressions are again very long and contained in large number of pages. Coefficients of $\varepsilon_1$ and $\varepsilon_2$ can be obtained by Bhatnagar and Hallan [2].

8. Stability

While evaluating $B_{1,0}^1, B_{0,1}^1, B_{1,0}^0$ and $B_{0,1}^0$ the condition (i) of Moser’s theorem as in Moser [9] is assumed. Now we verify that this condition is satisfied. The condition is $k_1\varepsilon_1 + k_2\varepsilon_2 \neq 0$ for all pairs $(k_1, k_2)$ of rational integers such that

$$|k_1| + |k_2| \leq 4 \quad (13)$$

We note that the inequalities (13) are violated when $\varepsilon_1 = 2\varepsilon_2$ and $\varepsilon_1 = 3\varepsilon_2$

Case (i) $\varepsilon_1 = 2\varepsilon_2$.

We get

$$\varepsilon_1 = \frac{2}{3 \sqrt{2}} \left(1 + \frac{3}{2} \varepsilon_2 - \frac{3\gamma}{4} A_2 + \frac{3\gamma}{4} A_4 - \frac{3}{2} (1 - \gamma) A_2 - \frac{3}{2} (1 - \gamma) A_4 \right),$$

$$\varepsilon_2 = \frac{1}{3 \sqrt{5}} \left(1 + \frac{3}{2} \varepsilon_2 - \frac{3\gamma}{4} A_2 + \frac{3\gamma}{4} A_4 - \frac{3}{2} (1 - \gamma) A_2 - \frac{3}{2} (1 - \gamma) A_4 \right).$$

Putting these values in second of Equations (8), we get

$$\gamma^2 = \frac{611}{675} + \frac{4864}{6075} \varepsilon_2 - \frac{1024}{675} \varepsilon_1 + \frac{128 \rho}{6075} + \left(\frac{23835 + 503\sqrt{1833}}{10125}\right) A_4 + \frac{64(65 + \sqrt{1833})}{10125} A_3$$

$$+ \left(\frac{23835 - 503\sqrt{1833}}{10125}\right) A_2 + \frac{64(65 - \sqrt{1833})}{10125} A_1.$$ 

Putting $\gamma = 1 - 2\mu$ and solving for $\mu$, denoting this value by $\mu_{c1}$, we get

$$\mu_{c1} = 0.02429\ldots + \frac{64}{135\sqrt{1833}} (36\varepsilon_1 - 19\varepsilon_2) - 0.17907\ldots A_1 - 1.17746\ldots A_2$$

$$- 0.03685\ldots A_1 - 0.05968\ldots A_1 - 0.005536495\ldots p \quad (15)$$

Case (ii) $\varepsilon_1 = 3\varepsilon_2$.

Proceeding as in case (i), we get $\mu = \mu_{c2}$, where

$$\mu_{c2} = 0.01352\ldots + \frac{4}{45\sqrt{213}} (36\varepsilon_1 - 19\varepsilon_2) - 0.09938\ldots A_1 - 2.15996\ldots A_2$$

$$- 0.01938\ldots A_1 - 0.03093\ldots A_1 - 0.003045283\ldots p. \quad (16)$$

Hence for the value $\mu_{c1}$ and $\mu_{c2}$ of mass ratios, condition (i) of Moser’s theorem is not satisfied. The normalized Hamiltonian up to fourth order is

$$H = \varepsilon_1 I_1 - \varepsilon_2 I_2 + \frac{1}{2} \left( A I_i^2 + 2B I_i I_2 + C I_2^2 \right) +$$

where $A, B, C$ are given by Equation (12).

Now after simplification, the determinant $D$ occurring in condition (ii) of Moser’s theorem is given by:

$$D = \det (b_{ij}) \quad (i, j = 1, 2, 3), \quad b_{ij} = \left( \frac{\partial^2 H}{\partial I_i \partial I_j} \right)_{I_i, I_j = 0}$$

$$b_{i3} = b_{3j} = \left( \frac{\partial H}{\partial I_i} \right)_{I_i, I_j = 0} \quad (i, j = 1, 2), \quad b_{33} = 0.$$
That is

$$D = \begin{vmatrix} A & B & \omega' \\ B & C & -\omega' \\ \omega' & -\omega' & 0 \end{vmatrix} = -(A\omega'^2 + 2B\omega'\omega + C\omega'^2).$$

Substituting the values of $A, B, C$ from Equation (12) and $\omega', \omega''$ using the Equation (8) and Equation (9), we obtain

$$D = \frac{9(36 - 541u^2 + 644u^4)}{2(4u^2 - 1)(25u^2 - 4)}$$

$R, R', R'', R_1$ and $R'_1$ are given in the Appendix. Values of $m_1$ and $m_2$ can be obtained from Bhatnagar and Hallan [2]. It is seen that the condition (ii) of Moser’s theorem is satisfied i.e. $D \neq 0$ if in the interval $0 < \mu < \mu_*$, mass ratio does not take the value

$$(17)$$

$$\mu = 0.010936677 \ldots, \alpha = -0.02942 \ldots, \beta = 772.85704 \ldots, \alpha' = -0.10408 \ldots,$$

$$= -16.46591 \ldots, \beta'' = -166.304 \ldots, \beta = 17.63703 \ldots, \alpha'' = 250.922 \ldots.$$  

9. Conclusions

The abscissa of $L_4$ is independent of the perturbation in Coriolis ($\epsilon_1$) and centrifugal forces ($\epsilon_2$) and ordinate of $L_4$ is affected by perturbation in centrifugal force (Equation (2)).

With the increase of perturbation in Coriolis force, the range of linear stability increases whereas if we increase perturbation in centrifugal force, the range of stability decreases (Equation (7)).

Values of second order coefficients $(A, B, C)$ in the polynomials $(f_2, g_2)$ occurring in the frequencies $\phi_1$ and $\phi_2$ are affected by the perturbations in Coriolis and centrifugal forces. It is observed that if perturbation in Coriolis and centrifugal forces increase then values of second order coefficients $(A, B, C)$ increase (Equation (12)).

$$\mu_{41}, \mu_{42}$ corresponds to the resonance cases $\omega' = 2\omega'$ and $\omega'' = 3\omega'$. Their values are given in Equation (13).

Values of $\mu_{41}, \mu_{42}, \mu_{43}$ (values of $\mu$ at which Moser’s theorem is not applicable) increase if perturbation in Coriolis force increases and decrease if perturbation in centrifugal force increases (Equations (15)-(17)).

It may be observed that values of $\mu_{41}, \mu_{42}$ decrease if parameters of axis symmetric bodies $(A_1, A_2, A_3, A_4)$ and radiation pressure $(p)$ increase (Equations (15) and (16)).

Moser’s second condition is violated for unperturbed problem (i.e. for $\epsilon_1 = \epsilon_2 = A_1 = A_2 = A_3 = A_4 = p = 0$) when $\mu = \mu' = 0.010936677 \ldots$ (Equation (17)).

It may also be observed that value of $\mu_{43}$ increases if $A_1$ of the bigger primary and $p$ increase. If $A_1, A_2, A_3$ increase, value of $\mu_{43}$ decreases (Equation (17)).

By taking both the primaries as axis symmetric bodies and the bigger mass as a source of radiation, the triangular point $L_4$ is stable in the range of linear stability except for the three mass ratios given in Equations (15)-(17) at which Moser’s theorem does not apply.

The results of Jagadish Singh [6] can be deduced by taking $a_1 = b_1$ and $a_2 = b_2$.

All the results of Bhatnagar and Hallan [2] can be deduced by taking $A_1 = A_2 = A_3 = A_4 = p = 0$.

References


Appendix

\[ p_{27} = \frac{1}{4} \left[ 1485 + 4025\gamma + \frac{2(285 + 115\gamma)}{\mu} \right], \quad q_{27} = -\frac{2}{3}(87 + 113\gamma), \quad q_{28} = \frac{2}{3}(63 - \gamma), \]

\[ p_{28} = -\frac{1}{4} \left[ 567 + 291\gamma + \frac{2(171 - 43\gamma)}{\mu} \right], \quad q_{29} = \frac{2}{3}(111 - 169\gamma), \quad q_{30} = \frac{2}{3}(21 + 5\gamma), \]

\[ p_{29} = \frac{1}{4} \left[ 2325 + 5665\gamma + \frac{2(345 + 215\gamma)}{\mu} \right], \quad q_{31} = \frac{2}{3}(-29 + 91\gamma), \]

\[ p_{30} = \frac{1}{4} \left[ 147 + 111\gamma + \frac{2(-69 + 37\gamma)}{\mu} \right], \quad p'_{30} = \frac{1}{4} \left[ 1485 - 4025\gamma + \frac{2}{1-\mu}(285 - 115\gamma) \right], \]

\[ p_{31} = \frac{1}{4} \left[ 175 + 1235\gamma + \frac{2(55 + 185\gamma)}{\mu} \right], \quad p_{32} = \frac{1}{4} \left[ 567 - 291\gamma + \frac{2}{1-\mu}(171 + 43\gamma) \right], \]

\[ p'_{30} = -\frac{1}{4} \left[ 147 - 111\gamma + \frac{2}{1-\mu}(-69 - 37\gamma) \right], \quad p'_{30} = -\frac{1}{4} \left[ 147 - 111\gamma + \frac{2}{1-\mu}(-69 - 37\gamma) \right], \quad p_8 = -\frac{16}{3} p_5, \]

\[ p'_{31} = -\frac{1}{4} \left[ 175 - 1235\gamma + \frac{2}{1-\mu}(55 - 185\gamma) \right], \quad p_5 = -\frac{3}{64} \left[ 15 + 47\gamma + \frac{2}{\mu}(1 + 7\gamma) \right], \quad p_9 = p_7, \]

\[ p_6 = \frac{3}{64} \left[ -1 + 15\gamma + \frac{2}{\mu}(-3 + 11\gamma) \right], \quad p_7 = \frac{1}{4} \left[ 87 + 15\gamma + \frac{2}{\mu}(-3 + 11\gamma) \right], \]

\[ p_{14} = \frac{3}{64} \left[ 75 - \frac{1}{2} \gamma + \frac{1}{2\mu}(1 - 45\gamma) \right], \quad p_{10} = -\frac{16}{3} p_6, \quad p'_6 = -\frac{3}{64} \left[ 15 - 47\gamma + \frac{2}{1-\mu}(1 - 7\gamma) \right], \]

\[ p'_6 = \frac{3}{64} \left[ 1 - 15\gamma - \frac{2}{\mu}(3 + 11\gamma) \right], \quad p'_7 = \frac{1}{4} \left[ 87 - 15\gamma - \frac{2}{1-\mu}(3 + 11\gamma) \right], \]

\[ p'_6 = -\frac{16}{3} p'_6, \quad p'_7 = p'_7, \quad p'_9 = -\frac{16}{3} p'_8, \quad q_5 = -\frac{1}{8}(1 - 3\gamma), \]

\[ q_6 = \frac{1}{8}(1 - 3\gamma), \quad q_7 = -\frac{2}{3}(3 - \gamma), \quad \]

\[ q_8 = -\frac{16}{3} q_5, \quad p_{11} = \frac{7}{2} - \frac{15}{2} \gamma + \frac{1}{2\mu}(-37 + 25\gamma), \]

\[ p_{14} = \left[ -\frac{75}{2} \gamma + \frac{1}{2\mu}(1 - 45\gamma) \right], \quad p'_{14} = \left[ \frac{75}{2} \gamma + \frac{1}{2\mu}(1 + 45\gamma) \right], \quad q_{14} = -\frac{4}{3}(2 - 9\gamma), \]

\[ q_9 = q_7, \quad q_{10} = -\frac{16}{3} q_8, \quad p_{12} = \frac{75 + 435}{2} \gamma + \frac{1}{2\mu}(41 + 75\gamma), \quad p_{13} = \frac{76 + 55}{2} \gamma + \frac{1}{2\mu}(-41 + 45\gamma), \]

\[ p'_{14} = \left[ \frac{75}{2} \gamma + \frac{1}{2\mu}(37 + 25\gamma) \right], \quad p'_{12} = \left[ 75 - \frac{435}{2} \gamma + \frac{1}{2\mu}(41 - 75\gamma) \right], \]

\[ p'_{13} = \left[ -76 + \frac{55}{2} \gamma + \frac{1}{2\mu}(41 + 45\gamma) \right], \quad q_{11} = \frac{4}{3}(4 + \gamma), \quad q_{12} = -\frac{4}{3}(-8 + 21\gamma), \quad q_{13} = \frac{4}{3}(2 + 3\gamma), \]

\[ R = \frac{3}{2} \left[ 3917 u^4 - 14357 u^2 + 492 \right] \]

\[ -27 \gamma \left[ \frac{1593600 u^0 + 21222096 u^6 - 1305200 u^8 + 5408175 u^4 - 840076 u^2 + 23616}{2(4 u^2 - 1)(25 u^2 - 4)(16 u^2 + 117)} \right] \]
\[ R' = \frac{81(1 + \gamma)(7 - 40u^2)(36 - 541u^2 + 644u^4)}{(4u^2 - 1)(25u^2 - 4)} \]
\[ + \frac{(1 + \gamma)(4u^2 - 1)(25u^2 - 4)(2025 - 26081u^2 + 55552u^4 - 2480u^6)}{k_1^4k_2^2(1 - 5\omega_1^2)(1 - 5\omega_2^2)} \]
\[ + (4u^2 - 1)(25u^2 - 4) \left[ \frac{\alpha_4^4(124\alpha_4^4 - 696\alpha_2^4 + 81)(\zeta - \eta)}{k_1^4(1 - 5\omega_1^2)} + \frac{24u^4(64u^2 + 43)}{k_1^4k_2^2(1 - 5\omega_1^2)(1 - 5\omega_2^2)} \right] \]
\[ \times \left\{ \zeta - \eta + \frac{15(1 + \gamma)}{(1 - 5\omega_1^2)(1 - 5\omega_2^2)} + \frac{\alpha_4^4(124\alpha_4^4 - 696\alpha_2^4 + 81)(\zeta' - \eta')}{k_1^4(1 - 5\omega_1^2)} \right\} \]
\[ R'' = \frac{4u^2 - 1)(25u^2 - 4) \left[ \frac{\alpha_4^4(124\alpha_4^4 - 696\alpha_2^4 + 81)(\sigma - \rho)}{k_1^4(1 - 5\omega_1^2)} + \frac{24u^4(64u^2 + 43)(\tau - \rho)}{k_1^4k_2^2(1 - 5\omega_1^2)(1 - 5\omega_2^2)} \right] \]
\[ \left. + \frac{24u^4(64u^2 + 43)(\tau - \rho)}{k_1^4k_2^2(1 - 5\omega_1^2)(1 - 5\omega_2^2)} \right] \]

\[ R \] and \[ R' \] can be obtained from \[ R \] and \[ R' \] respectively by replacing \( \gamma \) by \(-\gamma\), \( \xi \) by \( \zeta \), \( \eta \) by \( \eta \), \( \zeta \) by \( \zeta' \), \( \xi' \) by \( \xi' \) and \( \eta' \) by \( \eta' \).

\[ J_{13} = \frac{l_1}{2\alpha_1k_1} \left[ 1 + \alpha_1e_1 + \alpha_1' e_2 + \left( -\frac{3\gamma}{4k_1} + \frac{33}{4l_1^2} \right) A_1 + p_2 A_1 + \left( \frac{3\gamma}{4k_1} + \frac{33}{4l_1^2} \right) A_1 + p_2 A_1 + q_2 p \right], \]
\[ J_{14} = \frac{l_2}{2\alpha_1k_1} \left[ 1 + \alpha_1e_1 + \alpha_1' e_2 + \left( \frac{3\gamma}{4k_1} + \frac{33}{4l_1^2} \right) A_1 + p_2 A_1 + \left( \frac{3\gamma}{4k_1} + \frac{33}{4l_1^2} \right) A_1 + p_2 A_1 + q_2 p \right], \]
\[ J_{21} = -\frac{4\alpha_1}{l_1k_1} \left[ 1 + \alpha_1e_1 + \alpha_1' e_2 + \left( \frac{3\gamma}{4k_1} - \frac{33}{4l_1^2} \right) A_1 + p_2 A_1 + \left( \frac{3\gamma}{4k_1} - \frac{33}{4l_1^2} \right) A_1 + p_2 A_1 + q_2 p \right], \]
\[ J_{23} = \frac{3\sqrt{3\gamma}}{2\alpha_1k_1} \left[ 1 + \alpha_2e_1 + \alpha_2' e_2 + \left( \frac{6 + 13 \gamma}{6 \gamma} - \frac{33}{4l_1^2} \right) A_1 \right] \]
\[ + p_2 A_1 + \left( \frac{-6 + 13 \gamma}{6 \gamma} + \frac{33}{4l_1^2} \right) A_1 + p_2 A_1 + q_2 p \right], \]
\[ J_{24} = \frac{3\sqrt{3\gamma}}{2\alpha_1l_2k_2} \left[ 1 + \alpha_2e_1 + \alpha_2' e_2 + \left( \frac{6 + 13 \gamma}{6 \gamma} + \frac{33}{4l_1^2} \right) A_1 \right] \]
\[ + p_2 A_1 + \left( \frac{-6 + 13 \gamma}{6 \gamma} - \frac{33}{4l_1^2} \right) A_1 + p_2 A_1 + q_2 p \right]. \]

Values of \( \alpha_{13}, \alpha_{14}, \alpha_{21}, \alpha_{22}, \alpha_{23}, \alpha_{24}, \alpha_{25}, \alpha_{26}, \alpha_{27}, \alpha_{28} \) and \( \alpha_{24} \) can be obtained from Hallan and Bhatnagar (983).

Values of \( \xi, \eta, \zeta, \xi, \eta, \zeta, \xi', \eta', \sigma, \rho, \tau, p_21, p_22, p_23, p_24, p_25, p_26, p_27, p_28, p_29, p_21, p_22, p_23, p_24, p_25, p_26, p_27, p_28, p_29 \) and \( p_{26} \) can be obtained from the author on request as the expressions are very long and contained in large number of pages.