

The System Size Distribution for M/G/1 Queueing System under N-Policy with Startup/Closedown

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ABSTRACT

This paper develops a new method for calculating the system size distribution on two different M/G/1 queueing system under N-policy with general startup/closedown. Firstly, the stochastic decomposition property is used to derive the p.g.f. of the system size distribution. By the Leibniz formula of derivation, we investigate the additional system size distribution, and then, we get the recursion expression of system sizes distribution. Finally, several examples are given for illustrating the application of the recursion expression and sensitivity analysis is also performed.

Keywords: Queue-Length, N-Policy, Stochastic Decomposition, Recursion Expression, Leibniz Formula

1. Introduction

This paper presents a new approach for study M/G/1 queueing system under N-policy with general startup/closedown times. N-policy is an effective mechanism for cutting down operating cost, which has been widely applied in modeling queue-like production system. As soon as each service is completed, the server will be turned off for a random closedown time and a random startup time is needed before commencing his service, which reflects more actual situation.

Queueing system under N-policy first considered by Yadin and Naor [1] has been extensively expanded by many authors. Kella [2] discussed N-policy M/G/1 queueing system with server vacations. Bothakur *et al.* [3] extended Baker's model [4] with exponential startup time to the general startup time. Lee *et al.* [5-7] investigated the batch arrival N-policy M/G/1 queueing system with a single vacation and multiple vacations. The p.g.f. (probability generating function) of the system size distribution was derived, which shown the famous stochastic decomposition property proposed by Fuhrmann and Cooper [8] still came into existence. The concept of closedown time was introduced by Takagi [9]. Ke [10] developed an M/G/1 queueing system under varieties and extensions of NT queueing systems with breakdowns, startup and closedown.

It is difficult to get the analytic steady-state system size distribution based on the p.g.f. under the general distribution of service time. Wang and his co-authors [11-14] derived the analytic steady-state solutions of the N-policy M/M/1 queueing system, the N-policy M/H_k/1, the N-policy M/E_k/1, and the M/H₂/1 respectively. Recently, a maximum entropy approach was used to analyze the steady-state characteristics of M/G/1 queueing system [15,16]. Based on this method, Wang *et al.* [17-20] developed the approximate probability distribution of the system size for the N-policy M/G/1 queue with various cases. Also, the maximum entropy approach could be succeeded applied to batch arrival queueing system [21-23]. Tang [24] developed a total probability decomposition method for deriving the recursion expression of system size distribution in equilibrium. This method could be used to discuss batch arrival continuous-time or discrete-time queueing system [25,26], while the analysis procedure was very complicated.

This paper will present a new method for analyzing the N-policy queueing system. Our method can derive the analytical expressions of system size distribution which is different from the maximum entropy approach. It is to be noted that the analytic system size distribution under N-policy is very difficult to be derived by Tang's method. Our method is much simpler than Tang's in analyzing the N-policy queueing system.

The arrangement of this paper is as follows: First, we

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develop a new approach for derive the recursion expression of system size distribution in equilibrium. Second, some special cases are presented in this paper for our new method. Finally numerical examples are given for illustrating the accurate calculation of the system size distribution and then sensitivity analysis is investigated.

2. Model Description and Assumption

In this paper, we discuss two different kinds M/G/1 queueing system under N-policy with startup/closedown. The models are described in detailed as follows:

Assumption of the model 1.

1) Customers arrive according to a Poisson process with rate λ . Arriving customers form a single waiting line and are served according to the order of his arrivals. The server can process only one customer at a time.

2) The service time provided by a single server is an independent and identically distributed random variable (G) with a general distribution function ($G(t)$).

3) Whenever the system becomes empty, the sever shuts down with a random closedown time (D). When the number of arrivals in the queue reaches a predetermined threshold N , the server immediately reactivates and performs a startup time with random length (U) before starting service. When the server completes his startup, he starts serving the waiting customers until the system becomes empty.

4) If a customer arrives during a closedown time, the server is immediately started without operating N-policy and without a startup time.

Assumption of the model 2. The first three assumptions are the same as those in model 1. However the fourth assumption now is that even if a customer arrives during a closedown time, the server still be shut down. As soon as the closedown is over, the server immediately reactivates. Before starting service, the server still need a random startup time (U), however without satisfying the conditions of N-policy. In many literatures [9,10], they suppose that the server will resume shutting down and offers service when a customer arrives during a closedown time. However, in some real-world, the server (machine) restarts until the closedown is finished for protecting the machine when a customer arrives during the closedown time.

3. Preliminary Formula for the System Size Distribution for M/G/1 Queueing System

Before discussing our models, let us recall some results in the ordinary M/G/1 queueing system. For the sake of convenience, we define several items. We call a time interval when the server is working a *busy period*. A time interval when the server is unavailable (such as startup/closedown, or idle) an *idle period*. Let p_j^0 ($j = 0, 1, \dots$)

be the probability distribution of the system size in equilibrium. From Tang [24], we have

$$p_0^0 = 1 - \rho \quad (1)$$

$$p_j^0 = \lambda(1 - \rho)\theta_j, \quad j \geq 1 \quad (2)$$

where

$$\theta_j = \frac{1}{g(\lambda)} \int_0^\infty e^{-\lambda t} [1 - G(t)] \frac{(\lambda t)^{j-1}}{(j-1)!} dt$$

$$+ \frac{1}{g(\lambda)} \sum_{k=1}^{j-1} \theta_{j-k} \left\{ 1 - \sum_{i=0}^k \int_0^\infty e^{-\lambda t} \frac{(\lambda t)^i}{i!} dG(t) \right\}, \quad j \geq 1.$$

$$g(\lambda) = \int_0^\infty e^{-\lambda t} dG(t), \text{ and if } j \leq 0, \quad \sum_{k=1}^j = 0.$$

We denote θ_j by the probability distribution of system size during the busy period at stationary. Note that $\rho = \lambda E[G]$ is traffic intensity and it should to be supposed to be less than unity, which ensures the system reaching at equilibrium. The p.g.f. of the queue size of the M/G/1 queueing system is given by

$$\Pi_0(z) = \frac{(1 - \rho)(1 - z)g(\lambda(1 - z))}{g(\lambda(1 - z)) - z} \quad (3)$$

4. Analysis of Model 1

4.1. The p.g.f. of the System Size

In this section we derive the p.g.f. of the system size distribution following the similar arguments as in [10]. Model 1 is as an extension of the M/G/1 queueing system given in Section 3. At the beginning of the busy period for Model 1 can be described by

Case 1: no customers arrive when the server is shutting down, which occurs with probability $D^*(\lambda)$. For this system, the server will start up until the number of customer arrival reach N during the idle period. The p.g.f. of the system size at the beginning of the busy is given by $z^N U^*(\lambda - \lambda z)$, where $U^*(s)$ is LST of U .

Case 2: Some customers arrive when the server is shutting down, which occurs with probability $1 - D^*(\lambda)$. In this case, the service is started immediately without a startup time and without N-policy. There is only one customer at the beginning of the busy period.

By the stochastic decomposition results [8], the p.g.f. of the number of customers found at the beginning of the busy period is give by

$$\Phi_1(z) = D^*(\lambda) z^N U^*(\lambda - \lambda z) + (1 - D^*(\lambda)) z \quad (4)$$

where $U^*(\lambda - \lambda z)$ denotes the p.g.f. of the number of customers arrive during the startup time. Following the result of Medhi and Templeton [27], the stochastic de-

composition of the p.g.f. of number of customers in the N policy M/G/1 queueing system with startup/closedown still comes into existence. We obtain

$$\Pi_1(z) = \Pi_0(z) \cdot \frac{1 - \Phi_1(z)}{\Phi_1'(1)(1-z)} \tag{5}$$

where

$\Phi_1'(1) = D^*(\lambda)(N + \lambda E[U]) + 1 - D^*(\lambda)$ is the expected number of customers that arrive during the idle period plus startup time.

4.2. Additional System Size Distribution

From Equation (5) we see that the stationary system size is sum of two random variables, one of which is the system size of the general M/G/1 queue. The other is the additional system size distribution due to N-policy and startup/closedown. The p.g.f. of additional system size distribution is give by

$$\pi_a(z) = \frac{1 - \Phi_1(z)}{\Phi_1'(1)(1-z)} \tag{6}$$

Following the definition of p.g.f., the probability of additional system size distribution can be written by can be written by

$$p_a(n) = \frac{1}{n!} \cdot \frac{d^n(\pi_a(z))}{dz^n} \Big|_{z=0}, \quad n = 0, 1, \dots \tag{7}$$

Notice that, when $i < N$, we have

$$\frac{d^i(\Phi_1(z)-1)}{dz^i} \Big|_{z=0} = \begin{cases} -1, & i = 0 \\ 1 - D^*(\lambda), & i = 1 \\ 0, & i = 2, \dots, N-1 \end{cases} \tag{8}$$

And when $i \geq N$, we have

$$\begin{aligned} \frac{d^i(\Phi_1(z)-1)}{dz^i} \Big|_{z=0} &= D^*(\lambda) \sum_{k=0}^i \binom{i}{k} \frac{d^k(z^N)}{dz^k} \cdot \frac{d^{i-k}(U^*(\lambda - \lambda z))}{dz^{i-k}} \Big|_{z=0} \\ &= D^*(\lambda) \cdot (i!) \cdot \int_0^\infty e^{-\lambda t} \frac{(\lambda t)^{i-N}}{(i-N)!} dU(t) \end{aligned} \tag{9}$$

Substitute (8) and (9) into (7), we get the additional system size distribution

$$p_a(0) = \frac{1}{D^*(\lambda)(N + \lambda E[U]) + 1 - D^*(\lambda)} \tag{10}$$

$$p_a(n) = \frac{D^*(\lambda)}{D^*(\lambda)(N + \lambda E[U]) + 1 - D^*(\lambda)}, \quad n = 1, \dots, N-1 \tag{11}$$

$$p_a(n) = \frac{D^*(\lambda) \left(1 - \sum_{m=0}^{n-N} \int_0^\infty e^{-\lambda t} \frac{(\lambda t)^m}{m!} dU(t) \right)}{D^*(\lambda)(N + \lambda E[U]) + 1 - D^*(\lambda)}, \quad n = N, N+1, \dots \tag{12}$$

4.3. System Size Distribution

In this section we will derive the stationary-state system size distribution. The chain rule of differentiate is used as the main tool for calculation. Obviously, the probability of the number of customers in the queue which equals to zero is given by

$$p_0 = \Pi_1(z) \Big|_{z=0} = \frac{1 - \rho}{D^*(\lambda)(N + \lambda E[U]) + 1 - D^*(\lambda)} \tag{13}$$

When $n = 1, \dots, N-1$, by the Leibniz formula, from (5), the probability of the number of customers is n in the queue is also given by

$$\begin{aligned} p_n &= \frac{1}{\Phi_1'(1)} \cdot \frac{1}{n!} \sum_{k=0}^n \binom{n}{k} \cdot \frac{d^k(\Pi_0(z))}{dz^k} \cdot \frac{d^{n-k}(\Psi(z))}{dz^{n-k}} \Big|_{z=0} \\ &= \frac{1}{\Phi_1'(1)} \sum_{k=0}^n \frac{p_k^0}{(n-k)!} \cdot \frac{d^{n-k}(\Psi(z))}{dz^{n-k}} \Big|_{z=0} \end{aligned} \tag{14}$$

where, p_k^0 is given by Formulas (1) and (2). We denote $\Psi(z)$ by $\frac{\Phi_1(z)-1}{z-1}$, using the rule of chain of derivation again, conditioning on $n-k < N$, thus we have

$$\begin{aligned} \frac{d^{n-k}(\Psi(z))}{dz^{n-k}} \Big|_{z=0} &= \sum_{i=0}^{n-k} \binom{i}{n-k} \frac{d^i(\Phi_1(z)-1)}{dz^i} \cdot (-1)^{n-k-i} \cdot \frac{(n-k-i)!}{(z-1)^{n-k-i+1}} \Big|_{z=0} \\ &= (n-k)! - (n-k)(1 - D^*(\lambda)) \cdot (n-k-1)! \\ &= ((n-k)!) \cdot D^*(\lambda) \end{aligned} \tag{15}$$

Substitute (15) into (14) and combing (2), we get the system size distribution

$$p_n = \frac{(1-\rho) \cdot \left(\lambda \theta_n + D^*(\lambda) + \lambda D^*(\lambda) \sum_{k=1}^{n-1} \theta_k \right)}{D^*(\lambda)(N + \lambda E[U]) + 1 - D^*(\lambda)}, \quad n = 1, \dots, N-1 \tag{16}$$

When $n \geq N$, if $n-k \geq N$, that is $k \leq n-N$, we have

$$\frac{d^{n-k}(\Psi(z))}{dz^{n-k}} \Big|_{z=0} = ((n-k)! \cdot D^*(\lambda) \cdot (1-U(k, \lambda))), \quad (17)$$

From the (15), (17) and combing (2), we obtain the system size distribution.

$$p_n = \frac{(1-\rho) \left\{ D^*(\lambda)[1-U(0, \lambda)] + \lambda D^*(\lambda) \sum_{k=1}^{n-N} \theta_k [1-U(k, \lambda)] + D^*(\lambda) \lambda \sum_{k=n-N+1}^n \theta_k \right\}}{D^*(\lambda)(N + \lambda E[U]) + 1 - D^*(\lambda)}, \quad n \geq N \quad (18)$$

where

$$U(k, \lambda) = \sum_{m=0}^{n-k-N} \int_0^\infty e^{-\lambda t} \frac{(\lambda t)^m}{m!} dU(t),$$

$$U(0, \lambda) = \sum_{m=0}^{n-N} \int_0^\infty e^{-\lambda t} \frac{(\lambda t)^m}{m!} dU(t).$$

The recursion expressions of system size distribution are given by (13), (16) and (18), which can be used for calculating the probability of system size in equilibrium.

Remark 4.1 If the startup time equals to zero, which implies $E[U]=0$ and $U(0, \lambda)=1, U(k, \lambda)=1$, model 1 can be reduced to the M/G/1 queueing system with delayed N-policy. In this case, these results coincide with those of Tang’s system [28]. The approach discussed in this paper is much easier than the total probability decomposition method for the system size distribution used in [28].

5. Analysis of Model 2

5.1. The p.g.f. of the System Size

In this section, we consider another N-policy for M/G/1 queueing system with startup/closedown. In this system, the only difference from the model 1 is that if some customers arrive during the closedown period, the server will still be shut down in order to protecting the server. After the closedown, the server immediately commences a startup with a random time. The server will offer service without operating N-policy until the startup is finished.

For this system, from the similar arguments as for model 1, the p.g.f. for the number of customers found in the system at the beginning of the busy period is given by

$$\Phi_2(z) = D^*(\lambda)z^N U^*(\lambda - \lambda z) + (1 - D^*(\lambda))D^*(\lambda - \lambda z)U^*(\lambda - \lambda z), \quad (19)$$

where $D^*(\lambda - \lambda z)$ denotes the p.g.f. of the number of customers arrive during the closedown time. And the expected number of customers at the beginning of busy period is as follows

$$\Phi_2'(1) = D^*(\lambda)(N + \lambda E[U]) + (1 - D^*(\lambda))(E[D] + E[U]) \quad (20)$$

From (19) and (20), we have the distribution for the system size at an arbitrary time

$$\Pi_2(z) = \Pi_0(z) \cdot \frac{1 - \Phi_2(z)}{\Phi_2(1)(1-z)}. \quad (21)$$

5.2. System Size Distribution

In this section, we derive the system size distribution. Similarly, from (19), (20), we have the probability that the system is empty

$$p_0 = \frac{(1-\rho)[1 - (1 - D^*(\lambda))D^*(\lambda)U^*(\lambda)]}{\Phi_2'(1)}. \quad (22)$$

As in Section 4, when $n = 1, \dots, N-1$, the probability of the system size equals to n is given by

$$p_n = \frac{1}{\Phi_1'(1)} \sum_{k=0}^n \frac{p_k^0}{(n-k)!} \cdot \frac{d^{n-k}(\bar{\Psi}(z))}{dz^{n-k}} \Big|_{z=0}, \quad (23)$$

where, we denote $\bar{\Psi}(z)$ by $\frac{\Phi_2(z)-1}{z-1}$. From (19), we get

$$\begin{aligned} \frac{d^i(\Phi_2(z)-1)}{dz^i} \Big|_{z=0} &= \begin{cases} (1 - D^*(\lambda))D^*(\lambda)U^*(\lambda) - 1, & i = 0 \\ (1 - D^*(\lambda)) \sum_{l=0}^i \binom{l}{i} \Delta(i, l), & i = 1, \dots, N-1 \end{cases} \end{aligned} \quad (24)$$

where

$$\Delta(i, l) = \int_0^\infty e^{-\lambda t} (\lambda t)^l dD(t) \int_0^\infty e^{-\lambda t} (\lambda t)^{i-l} dU(t)$$

Thus, we get

$$\begin{aligned} \frac{d^{n-k}(\bar{\Psi}(z))}{dz^{n-k}} \Big|_{z=0} &= [1 - (1 - D^*(\lambda))D^*(\lambda)U^*(\lambda)]((n-k)!) \\ &- \sum_{i=1}^{n-k} \frac{(n-k)!}{i!} (1 - D^*(\lambda)) \sum_{l=0}^i \binom{l}{i} \Delta(i, l). \end{aligned} \quad (25)$$

Take (25) into (23), we get

$$p_n = \frac{1-\rho}{\Phi_1'(1)} \left\{ [1 - (1 - D^*(\lambda))D^*(\lambda)U^*(\lambda)] \right\}$$

$$\begin{aligned}
 & - (1 - D^*(\lambda)) \sum_{i=1}^n \frac{1}{i!} \sum_{l=0}^i \binom{l}{i} \Delta(i, l) \\
 & - (1 - D^*(\lambda)) \sum_{i=1}^{n-k} \frac{1}{i!} \sum_{l=0}^i \binom{l}{i} \Delta(i, l) \} \\
 & + \lambda \sum_{k=1}^n \theta_k \left[1 - (1 - D^*(\lambda)) \cdot D^*(\lambda) U^*(\lambda) \right], \\
 & n = 1, \dots, N - 1. \tag{26}
 \end{aligned}$$

When $n \geq N$, we analyze the following two cases.

1) If $n - k \geq N$, that is $k \leq n - N$, we have

$$\begin{aligned}
 & \frac{d^{n-k}(\bar{\Psi}(z))}{dz^{n-k}} \Big|_{z=0} = [1 - U(k, \lambda)] \cdot D^*(\lambda) \cdot ((n - k)!) \\
 & - (1 - D^*(\lambda)) \cdot \sum_{i=0}^{n-k} \frac{(n - k)!}{i!} \sum_{l=0}^i \binom{l}{i} \Delta(i, l), \tag{27}
 \end{aligned}$$

2) If $n - k < N$, which implies $k > n - N$, we get

$$\frac{d^{n-k}(\Psi(z))}{dz^{n-k}} \Big|_{z=0} = 1 - (1 - D^*(\lambda)) \cdot \sum_{i=0}^{n-k} \frac{(n - k)!}{i!} \sum_{l=0}^i \binom{l}{i} \Delta(i, l) \tag{28}$$

Substitute (27) and (28) into (23), we have

$$\begin{aligned}
 p_n &= \frac{1}{\Phi_1(1)} \sum_{k=0}^{n-N} \frac{p_k^0}{(n - k)!} \cdot \frac{d^{n-k}(\bar{\Psi}(z))}{dz^{n-k}} \Big|_{z=0} \\
 & + \frac{1}{\Phi_1(1)} \sum_{k=n-N+1}^n \frac{p_k^0}{(n - k)!} \cdot \frac{d^{n-k}(\bar{\Psi}(z))}{dz^{n-k}} \Big|_{z=0} \\
 &= \frac{1 - \rho}{\Phi_1(1)} \left\{ [1 - U(0, \lambda)] D^*(\lambda) - (1 - D^*(\lambda)) \sum_{i=0}^n \frac{1}{i!} \sum_{l=0}^i \binom{l}{i} \Delta(i, l) \right. \\
 & + \lambda \sum_{k=1}^{n-N} \theta_k \cdot [1 - U(k, \lambda)] D^*(\lambda) \\
 & - (1 - D^*(\lambda)) \cdot \sum_{i=0}^{n-k} \frac{1}{i!} \sum_{l=0}^i \binom{l}{i} \Delta(i, l) \} \\
 & + \lambda \sum_{k=n-N+1}^n \theta_k \left[1 - (1 - D^*(\lambda)) \cdot \sum_{i=0}^{n-k} \frac{1}{i!} \sum_{l=0}^i \binom{l}{i} \Delta(i, l) \right] \Big\}, \\
 & n \geq N \tag{29}
 \end{aligned}$$

We get the recursion expressions of the system size distribution for the model 2, which are shown by (22), (26) and (29).

Remark 5.1 Suppose that we let $P[U = 0] = 1$, which indicates $E[U] = 0$ and $U^*(\lambda) = 1$. In this case, our model 1 can be simplified to M/G/1 queueing system with general closedown time. It is different from Tang's system [28] because the server will be shut down, even if

some customer arrives during the closedown time. We get the system size distribution as follows:

$$p_0 = \frac{(1 - \rho) [1 - (1 - D^*(\lambda)) D^*(\lambda)]}{D^*(\lambda) N + (1 - D^*(\lambda)) E[D]} \tag{30}$$

$$\begin{aligned}
 p_n &= \frac{1 - \rho}{D^*(\lambda) N + (1 - D^*(\lambda)) E[D]} \\
 & \times \left\{ [1 - (1 - D^*(\lambda)) D^*(\lambda)] - \sum_{i=1}^n \frac{1}{i!} (1 - D^*(\lambda)) \sum_{l=0}^i \binom{l}{i} \bar{\Delta}(i, l) \right. \\
 & + \lambda \sum_{k=1}^n \theta_k \left\{ [1 - (1 - D^*(\lambda)) \cdot D^*(\lambda)] \right. \\
 & \left. \left. - (1 - D^*(\lambda)) \sum_{i=1}^{n-k} \frac{1}{i!} \sum_{l=0}^i \binom{l}{i} \bar{\Delta}(i, l) \right\} \right\}, \quad n = 1, \dots, N - 1 \tag{31}
 \end{aligned}$$

$$\begin{aligned}
 p_n &= \frac{1 - \rho}{D^*(\lambda) N + (1 - D^*(\lambda)) E[D]} \\
 & \times \left\{ - (1 - D^*(\lambda)) \sum_{i=0}^n \frac{1}{i!} \sum_{l=0}^i \binom{l}{i} \bar{\Delta}(i, l) \right. \\
 & \left. - \lambda (1 - D^*(\lambda)) \sum_{k=1}^{n-N} \theta_k \sum_{i=0}^{n-k} \frac{1}{i!} \sum_{l=0}^i \binom{l}{i} \bar{\Delta}(i, l) \right. \\
 & \left. + \lambda \sum_{k=n-N+1}^n \theta_k \left[1 - (1 - D^*(\lambda)) \sum_{i=0}^{n-k} \frac{1}{i!} \sum_{l=0}^i \binom{l}{i} \bar{\Delta}(i, l) \right] \right\} \\
 & n \geq N, \tag{32}
 \end{aligned}$$

where

$$\bar{\Delta}(i, l) = \begin{cases} 0, & i \neq l. \\ \int_0^\infty e^{-\lambda t} (\lambda t)^l dD(t), & i = l. \end{cases}$$

Remark 5.2 If we let the closedown time equal to zero, which implies $E[D] = 0$ and $D^*(\lambda) = 1$. In this case, our model 1 and 2 can describe the N-policy M/G/1 queueing system with general startup time. The system size distribution can be simplified to the following expressions:

$$p_0 = \frac{1 - \rho}{N + \lambda E[U]} \tag{33}$$

$$p_n = \frac{1 - \rho}{N + \lambda E[U]} \left\{ 1 + \lambda \sum_{k=1}^n \theta_k \right\}, \quad n = 1, \dots, N - 1 \tag{34}$$

$$\begin{aligned}
 p_n &= \frac{(1 - \rho) \left\{ 1 - U(0, \lambda) + \lambda \sum_{k=1}^{n-N} \theta_k [1 - U(k, \lambda)] + \lambda \sum_{k=n-N+1}^n \theta_k \right\}}{N + \lambda E[U]} \\
 & n \geq N. \tag{35}
 \end{aligned}$$

6. Numerical Example

To illustrate the implication of calculating the system size distribution, we consider the model of an M/G/1 queueing system under N-policy with general startup/closedown, which based on the model 1 discussed in Section 4. Here we assume that the service time follows the negative exponential with mean $E[G]=1/\mu$. The startup time is 3-stage Erlang distribution with mean $E[U]=3/\beta$ and the closedown time is 4-stage Erlang distribution with mean $E[D]=4/\gamma$. The numerical results are illustrated in **Table 1**.

Observing the **Table 1**, it is clear that as β increases, 1) the probability of the system being empty increases when other system parameters are constant; 2) when $n=1,\dots,N$, the probability of system size being n increases; 3) when $n=N+1,N+2,\dots$ the probability of system size being n decreases. The expected system size (EL) also decreases as β increases. Deserve to be mentioned is that the mean system size is not enough for the

system design and control. As we can see the expected system size is no more than 3 while the sum of probability of system size exceeding the mean system size presented in the last is about 30%, which can not be neglected.

7. Conclusions

In this paper, we derive the recursion expressions of system size distribution for two different N-policy M/G/1 queueing systems with general startup/closedown. We first utilize the derivation of the chain rule combing with the famous stochastic decomposition results for developing the probability distributions of system size in the system. We present several examples to illustrate the implement of calculating the system size distribution from these recursion expressions, and investigate the effects for different system parameters. We get the system size distribution characteristic, which is important in practice. The method in this paper can be used in N-policy batch arrival queueing system.

Table 1. The distribution of the system size.

	β							
	$(\lambda = 1, \mu = 2, \gamma = 2, N = 5)$							
	2	3	4	5	6	7	8	9
P ₀	0.2396	0.2516	0.2580	0.262	0.2647	0.2667	0.2682	0.2694
P ₁	0.1672	0.1755	0.1799	0.1827	0.1846	0.186	0.1871	0.1879
P ₂	0.1309	0.1374	0.1409	0.1431	0.1446	0.1457	0.1465	0.1472
P ₃	0.1128	0.1184	0.1214	0.1233	0.1246	0.1255	0.1262	0.1268
P ₄	0.1037	0.1089	0.1117	0.1134	0.1146	0.1154	0.1161	0.1166
P ₅	0.0932	0.0978	0.1003	0.1019	0.1029	0.1037	0.1043	0.1048
P ₆	0.0405	0.0386	0.0372	0.036	0.035	0.0342	0.0336	0.033
P ₇	0.0304	0.0259	0.0232	0.0214	0.0201	0.0192	0.0185	0.0179
P ₈	0.0206	0.0156	0.0131	0.0116	0.0107	0.01	0.0095	0.0092
P ₉	0.0131	0.0089	0.0070	0.0061	0.0055	0.0051	0.0048	0.0046
P ₁₀	0.0081	0.0049	0.0037	0.0031	0.0028	0.0026	0.0024	0.0023
P ₁₁	0.005	0.0026	0.0019	0.0016	0.0014	0.0013	0.0012	0.0012
P ₁₂	0.0032	0.0015	0.0010	0.0008	0.0007	0.0007	0.0006	0.0006
P ₁₃	0.0021	0.0009	0.0005	0.0004	0.0004	0.0003	0.0003	0.0003
P ₁₄	0.0016	0.0005	0.0003	0.0002	0.0002	0.0002	0.0002	0.0002
P ₁₅	0.0013	0.0004	0.0002	0.0001	0.0001	0.0001	0.0001	0.0001
P ₁₆	0.0011	0.0003	0.0001	0.0001	0.0001	0.0001	0	0
P ₁₇	0.001	0.0003	0.0001	0.0001	0	0	0	0
P ₁₈	0.001	0.0002	0.0001	0	0	0	0	0
P ₁₉	0.001	0.0002	0.0001	0	0	0	0	0
EL	2.7203	2.4915	2.4203	2.3885	2.3704	2.3594	2.3501	2.3438
Sum	0.3279	0.3077	0.3006	0.2698	0.2945	0.2929	0.2916	0.2908

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