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Complements to the Theory of Higher-Order Types of Asymptotic Variation for Differentiable Functions

Antonio Granata

Department of Mathematics and Computer Science, University of Calabria, Rende, Italy
Email: antoniogranata1973@gmail.com

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Abstract

The purpose of this paper is to add some complements to the general theory of higher-order types of asymptotic variation developed in two previous papers so as to complete our elementary (but not too much!) theory in view of applications to the theory of finite asymptotic expansions in the real domain, the asymptotic study of ordinary differential equations and the like. The main results concern: 1) a detailed study of the types of asymptotic variation of an infinite series so extending the results known for the sole power series; 2) the type of asymptotic variation of a Wronskian completing the many already-published results on the asymptotic behaviors of Wronskians; 3) a comparison between the two main standard approaches to the concept of "type of asymptotic variation": via an asymptotic differential equation or an asymptotic functional equation; 4) a discussion about the simple concept of logarithmic variation making explicit and completing the results which, in the literature, are hidden in a quite-complicated general theory.

Keywords

Higher-Order Regularly-Varying Functions, Higher-Order Rapidly-Varying Functions, Smoothly-Varying Functions, Exponentially-Varying Functions, Logarithmically-Varying Functions, Asymptotic Differential Equations, Asymptotic Functional Equations, Asymptotic Variation of Wronskians

1. Introduction

This paper is a direct continuation of [1] [2] and contains some complements to the theory developed therein with the purpose of completing the general theory. This first section, besides a list of notations, contains a summary of the various involved classes of functions and their main characterizations.

- §2 contains a detailed study of the possible types of asymptotic variation of a convergent infinite series $\sum_{n=1}^{\infty} c_n \phi_n(x)$ where $\{\phi_n(x)\}_n$ is an asymptotic sequence at $+\infty$ and each $\{\phi_n(x)\}$ has a definite index of variation. Two quite different situations occur depending on whether

$$\phi_1(x) \gg \cdots \gg \phi_n(x) \gg \cdots,\ x \to +\infty, \quad \text{or} \quad \phi_1(x) \ll \cdots \ll \phi_n(x) \ll \cdots,\ x \to +\infty.$$

The first case is elementary whereas, in treating the second case, we give non-trivial extensions of one known result for power series.

- §3 contains results concerning the type of asymptotic variation of a Wronskian whose arguments are functions with a definite index of asymptotic variation at $+\infty$ so completing the extensive study of the asymptotic behaviors of Wronskians developed in two previous papers. The obtained results are quite natural and are based on the asymptotic study of a Vandermonde determinant with a gap in the exponents, a study which parallels the analogous investigation for standard Vandermondians in the previous papers.
- §4 contains a comparison between the two main standard approaches to the concept of "type of asymptotic variation": via an asymptotic differential equation or an asymptotic functional equation. The theory developed in [1] [2] is necessarily based on asymptotic differential equations in order to define higher-order types of variation for differentiable functions, whereas the more general Karamata theory is based on asymptotic functional equations. We show that for a function with a monotonic derivative the two approaches coincide for each one of the studied classes of functions, a result already known for regular variation.
- §5 contains a discussion about the concept of logarithmic variation starting from suitable asymptotic functional equations and showing, as in the previous section, the equivalence with corresponding asymptotic differential equations. Some of the results may be found in the literature but hidden in a quite-complicated general theory. The studied concept (namely, three related concepts) completes the list of the fundamental types of asymptotic variation in the way that we wished to systematize this theory.
- §6 gives results on the inverse of a function with a definite type of exponential variation. The results require some calculations and are clarified by the concepts of logarithmic variation.
- §7 contains some minor complements to the theory.
- §8 contains the conclusions about the whole theory developed in three papers.
- §9 contains a few bibliographical notes and a list of corrections for [1] [2].

Here is a list of general notations used in [1] [2].

- $\mathbb{N} := \{1, 2, \cdots\}$; $\mathbb{Z} := \{0, \pm 1, \pm 2, \cdots\}$;
- $\mathbb{R} :=$ real line; $\overline{\mathbb{R}} \equiv$ extended real line $:= \mathbb{R} \cup \{\pm\infty\}$;
- $f \in AC^0(I) \equiv AC(I) \Leftrightarrow f$ is absolutely continuous on each compact subinterval of the interval I;

- $f \in AC^k(I) \Leftrightarrow f^{(k)} \in AC(I)$;
- For $f \in AC^k(I)$ we write "$\lim_{x\to x_0} f^{(k+1)}(x)$" meaning that x runs through the points wherein $f^{(k+1)}$ exists as a finite number; $f(+\infty) := \lim_{x\to+\infty} f(x)$.
- The differentiation operators: $Df(x) := f'(x); D^k f(x) := f^{(k)}(x)$.
- The logarithmic derivative: $D_\ell f := f'/f$.
- Hardy's notations:

"$f(x) \ll g(x), x \to x_0$" or, equivalently "$g(x) \gg f(x), x \to x_0$" stands for $f(x) = o(g(x)), x \to x_0$;

"$f(x) \preccurlyeq g(x), x \to x_0$" or, equivalently "$g(x) \succcurlyeq f(x), x \to x_0$" stands for $f(x) = O(g(x)), x \to x_0$.

- The relation of "asymptotic similarity", "$f(x) \asymp g(x), x \to x_0$" means that

$$c_1|g(x)| \le |f(x)| \le c_2|g(x)| \quad \forall x$$

in a deleted neighborhood of x_0 ($c_i = \text{constant} > 0$). (1.1)

- The relation of "asymptotic equivalence":

$$f(x) \sim g(x),\ x \to x_0 \quad \text{stands for} \quad f(x) = g(x)[1+o(1)],\ x \to x_0.$$

- The relation:

$$f(x) = +\infty(g(x)),\ x \to x_0\ (x \in \mathcal{I}) \overset{\text{def}}{\Leftrightarrow} \begin{cases} f(x) = h(x)g(x) \ \ \forall\ x \text{ near } x_0, \\ \lim\limits_{x\to x_0, x\in\mathcal{I}} h(x) = +\infty; \end{cases} \tag{1.2}$$

and a similar definition for notation $f(x) = -\infty(g(x)), x \to x_0\ (x \in \mathcal{I})$. In particular:

$$f(x) = \pm\infty(1),\ x \to x_0\ (x \in \mathcal{I}) \overset{\text{def}}{\Leftrightarrow} \lim_{x\to x_0, x\in\mathcal{I}} f(x) = \pm\infty. \tag{1.3}$$

- Factorial powers:

$$\alpha^{\underline{0}} := 1;\ \alpha^{\underline{1}} := \alpha;\ \alpha^{\underline{k}} := \alpha(\alpha-1)\cdots(\alpha-k+1);\ \ \alpha \in \mathbb{C}, k \in \mathbb{N}; \tag{1.4}$$

where $\alpha^{\underline{k}}$ is termed the "k-th falling ($\equiv$decreasing) factorial power of α". Notice that we have defined $0^{\underline{0}} := 1$.

- Everywhere the symbol "$\log x$" stands for "$\log_e(x)$" := "the natural logarithm" of x.
- Notation for the iterated natural logarithm:

$$\begin{cases} \ell_k(x) := \underbrace{\log(\log(\cdots(\log}_{k} x)\cdots)), k \ge 1, (\text{defined for } x \text{ large enough});\ \ell_0(x) := x; \\ \dfrac{\mathrm{d}}{\mathrm{d}x}(\ell_k(x)) = \left(\prod\limits_{i=0}^{k-1} \ell_i(x)\right)^{-1}, k \ge 1. \end{cases} \tag{1.5}$$

For the reader's convenience we give a list of the special classes of functions characterized in [1] [2] mentioning only the main facts to be used in the present paper.

Classes of functions and their main characterizations.

(I) (Index of asymptotic variation). *If* $f \in AC[T, +\infty)$, f *ultimately* > 0, *its index of asymptotic variation at* $+\infty$ *is defined as the value of the following* *limit* (*assumed to exist*):

$$\lim_{x\to+\infty} xf'(x)/f(x) = \begin{cases} 0 & (\text{slow variation at } +\infty), \\ \alpha\in\mathbb{R}\setminus\{0\} & (\text{regular variation at } +\infty), \\ \pm\infty & (\text{rapid variation at } +\infty). \end{cases} \tag{1.6}$$

(II) (Higher-order regular variation). *A function* $f\in AC^{n-1}[T,+\infty)$, $n\ge1$ *is termed* "*regularly varying at* $+\infty$ (*in the strong sense*) *of order n*" *if each of the functions* $|f|,|f'|,\cdots,|f^{(n-1)}|$ *never vanishes on a neighborhood of* $+\infty$ *and is regularly varying at* $+\infty$ *with its own index of variation. If this is the case we use notation*

$$f\in\{\mathcal{R}_\alpha(+\infty)\text{ of order } n\},\quad \alpha:=\text{"the index of } f\text{"}\in\mathbb{R}. \tag{1.7}$$

If $f\in\{\mathcal{R}_\alpha(+\infty)\text{ of order } n\}$, $n\ge1$, *then relations*

$$\begin{aligned} f^{(k)}(x)/f(x) &= \alpha(\alpha-1)\cdots(\alpha-k+1)x^{-k}+o(x^{-k}) \\ &\equiv \alpha^{\underline{k}}x^{-k}+o(x^{-k}),\ x\to+\infty, 1\le k\le n, \end{aligned} \tag{1.8}$$

hold true whichever $\alpha\in\mathbb{R}$ *may be. The indexes of the derivatives are subject to the restrictions specified in* ([1]; Prop. 2.6, p. 796); *in particular*:

$$f\in\{\mathcal{R}_0(+\infty)\text{ of order } n\},\ n\ge2 \Rightarrow \begin{cases} f'(x)=o(x^{-1}f(x)) \\ f''(x)=\alpha_1x^{-1}f'(x)[1+o(1)] \text{ with } \alpha_1\le-1; \end{cases} \tag{1.9}$$

where α_1 *is the index of* f' *and the index of* $f^{(k)}$ *is* "α_1-k+1" *for* $k\ge2$. *Notice that the last derivative involved in* (1.8), *i.e.* $f^{(n)}$, *may have an arbitrary sign if* $\alpha^{\underline{n}}=0$.

(III) (Smooth variation). Relations in (1.8) characterize higher-order regular variation only for $\alpha\notin\{0,1,\cdots,n-2\}$; as these relations are basic in the applications their validity defines the following concept.

A function $f\in AC^{n-1}[T,+\infty)$, $n\ge1$, $f(x)\ne0$ $\forall x$ *large enough, is termed* "*smoothly varying at* $+\infty$ *of order n and index* α" *if the relations in* (1.8), *referred to* $|f|$, *are satisfied. We denote this class by*: $\{\mathcal{SR}_\alpha(+\infty)$ *of order n*$\}$. *The following inclusions obtain*:

$$\begin{cases} \{\mathcal{R}_\alpha(+\infty)\text{ of order } n\}=\{\mathcal{SR}_\alpha(+\infty)\text{ of order } n\} & \text{if } n=1 \text{ or } \{n\ge2,\alpha\ne0,1,\cdots,n-2\}; \\ \{\mathcal{R}_\alpha(+\infty)\text{ of order } n\}\subsetneqq\{\mathcal{SR}_\alpha(+\infty)\text{ of order } n\} & \text{otherwise}; \end{cases} \tag{1.10}$$

the reason of the last strict inclusion being that some derivatives of a smoothly-varying function may vanish or change sign infinitely often. The following sets of asymptotic relations, for a fixed $\alpha\in\mathbb{R}$, *are equivalent to each other*:

$$x^kf^{(k)}(x)/f(x)=\alpha(\alpha-1)\cdots(\alpha-k+1)+o(1),\ x\to+\infty, 1\le k\le n; \tag{1.11}$$

$$\begin{cases} xf'(x)/f(x)=\alpha+o(1),\ x\to+\infty; \\ (xf'(x)/f(x))^{(k)}=o(x^{-k}),\ x\to+\infty, 1\le k\le n. \end{cases} \tag{1.12}$$

(IV) (Rapid variation of first order). *A function* $f\in AC^1[T,+\infty)$ *is called* "*rapidly varying at* $+\infty$ *of order* 1 (*in the strong restricted sense*)" *if*:

$$\begin{cases} f(x), f'(x) \neq 0 & \forall x \text{ large enough;} \\ f(x)/f'(x) = o(x), & x \to +\infty; \\ \left(f(x)/f'(x)\right)' = o(1), & x \to +\infty; \end{cases} \tag{1.13a}$$

or, equivalently, if:

$$\begin{cases} f(x), f'(x) \neq 0 & \forall x \text{ large enough;} \\ f''(x)/f'(x) \sim f'(x)/f(x), & x \to +\infty; \end{cases} \tag{1.13b}$$

which imply $f''(x) \neq 0$ *for almost all x large enough.*

(Rapid variation of higher order). *A function* $f \in AC^n[T, +\infty)$ *is called* "*rapidly varying at* $+\infty$ *of order* $n \geq 2$ (*in the strong restricted sense*)" *if all the functions* $f, f', \cdots, f^{(n-1)}$ *are rapidly varying at* $+\infty$ *in the above-specified sense and this amounts to say that the following conditions hold true as* $x \to +\infty$:

$$f^{(k)}(x) \neq 0 \ \ \forall\, x \text{ large enough and } 0 \leq k \leq n; \tag{1.14}$$

$$f(x)/f'(x) = o(x);\ f'(x)/f''(x) = o(x); \cdots; f^{(n-1)}(x)/f^{(n)}(x) = o(x); \tag{1.15}$$

$$\left(f(x)/f'(x)\right)' = o(1); \left(f'(x)/f''(x)\right)' = o(1); \cdots; \left(f^{(n-1)}(x)/f^{(n)}(x)\right)' = o(1). \tag{1.16}$$

If f *is rapidly varying at* $+\infty$ *of order* $n \geq 2$ *in the previous sense then all the functions* $f, f', \cdots, f^{(n-1)}$ *belong to the same class, either* $\mathcal{R}_{-\infty}(+\infty)$ *or* $\mathcal{R}_{+\infty}(+\infty)$, *hence we shall use notation* $f \in \{\mathcal{R}_{\pm\infty}(+\infty) \text{ of order } n\}$ *to denote that* f *enjoys the properties in* (1.14)-(1.15)-(1.16) *plus the corresponding value* $\pm\infty$ *of the limit in* (1.6). *For an* $f \in AC^n[T, +\infty)$ *satisfying* (1.14) *we have the characterizations that conditions in* (*1.16*) *hold true, i.e.*

$$f \in \{\mathcal{R}_{+\infty}(+\infty) \text{ of order } n\} \cup \{\mathcal{R}_{-\infty}(+\infty) \text{ of order } n\},$$

if and only if the following equivalent sets of conditions are satisfied:

$$\begin{cases} f'(x)/f(x) \sim f''(x)/f'(x) \sim \cdots \sim f^{(n)}(x)/f^{(n-1)}(x) \sim f^{(n+1)}(x)/f^{(n)}(x), \\ \text{i.e. } D_\ell\left(f^{(k)}(x)\right) \sim D_\ell\left(f(x)\right), x \to +\infty, 1 \leq k \leq n; \end{cases} \tag{1.17}$$

$$f^{(k+2)}(x) \sim \left(f^{(k+1)}(x)\right)^2 / f^{(k)}(x),\ x \to +\infty,\ 0 \leq k \leq n-1; \tag{1.18}$$

It follows that even $f^{(n+1)}(x) \neq 0$ *for almost all x large enough.*

(V) (Types of exponential variation). *If* $f \in AC^{n-1}[T, +\infty)$ *then*:

$$f \in \{\mathcal{E}_0(+\infty) \text{ of order } n\}$$

$$\Leftrightarrow \begin{cases} f^{(k)}(x) \neq 0\ \forall\, x \text{ large enough and } 0 \leq k \leq n-1, \\ f(x) \gg f'(x) \gg \cdots \gg f^{(n-1)}(x) \gg f^{(n)}(x),\ x \to +\infty; \end{cases} \tag{1.19}$$

$$f \in \{\mathcal{E}_c(+\infty) \text{ of order } n\},\ c \in \mathbb{R} \setminus \{0\}$$

$$\Leftrightarrow \begin{cases} f(x) \neq 0\ \forall\, x \text{ large enough,} \\ f^{(k)}(x) \sim c^k f(x),\ x \to +\infty,\ 1 \leq k \leq n; \end{cases} \tag{1.20}$$

$$f \in \{\mathcal{E}_{\pm\infty}(+\infty) \text{ of order } n\}$$
$$\Leftrightarrow \begin{cases} f(x) \neq 0 \ \ \forall\ x \text{ large enough}, \\ f^{(n)}(x) \gg f^{(n-1)}(x) \gg \cdots \gg f'(x) \gg f(x),\ x \to +\infty, \end{cases} \tag{1.21}$$

wherein the correct index "$+\infty$" *or* "$-\infty$" *is determined by the single limit* "$\lim_{x\to+\infty} f'(x)/f(x)$".

For $c = 0$ *there is no sign-restriction on the highest-order derivative* $f^{(n)}$, *whereas for* $c \neq 0$ *also* $f^{(n)}$ *turns out to be ultimately of one strict sign. More precisely, if* $f(x) > 0$ *and* $f \in \{\mathcal{E}_c(+\infty) \text{ of order } n\}$ *then*:

$$\begin{cases} -\infty \le c < 0 \Rightarrow \lim\limits_{x\to+\infty} (-1)^k f^{(k)}(x) = +\infty,\ 0 \le k \le n; \\ 0 < c \le +\infty \Rightarrow \lim\limits_{x\to+\infty} f^{(k)}(x) = +\infty,\ 0 \le k \le n. \end{cases} \tag{1.22}$$

Notice that in our definition of higher-order variation f is allowed to be either >0 or <0, the essential point being that it ultimately assumes only one strict sign.

The reader must remember that this is a semiexpository paper like [1] [2] and, as such, some elementary or known facts are explicitly reported or proved to have an exposition self-contained and easily-read.

2. Types of Asymptotic Variation of Infinite Series

In [1] and [2] there are some results about the index of variation of a linear combination of functions belonging to one of the previously-studied classes; in this section we give some results about the type of asymptotic variation of an infinite series of such functions. We know from ([1]; formula (2.27), p. 784) that

$$\{\phi_i \in \mathcal{R}_{\alpha_i}(+\infty), \alpha_i \in \mathbb{R}, c_i > 0, 1 \le i \le n\} \Rightarrow \sum_{i=1}^{n} c_i \phi_i \in \mathcal{R}_{\max \alpha_i}(+\infty), \tag{2.1}$$

and similar results, with some restrictions, hold true when rapid variation is involved as stated in ([1]; Prop. 2.3-(I), pp.788-789). Hence, when investigating the possible types of asymptotic variation of an infinite linear combination $\sum_{i=1}^{\infty} c_i \phi_i(x), c_i > 0$, there must be an essential difference between the two circumstances

$$\phi_1(x) \gg \cdots \gg \phi_n(x) \gg \cdots,\ x \to +\infty, \tag{2.2}$$

$$\phi_1(x) \ll \cdots \ll \phi_n(x) \ll \cdots,\ x \to +\infty. \tag{2.3}$$

The simplest case is (2.2) and here are two elementary results extending Proposition 2.3-(II) in ([1]; p. 789), with no restriction on the signs of the coefficients c_i. The C^1-regularity assumption simplifies the exposition.

Proposition 2.1. *Let the functions* $\phi_n \in C^1[T, +\infty), n \in \mathbb{N}$, *form the asymptotic scale* (2.2), *let* $\phi_1(x) \neq 0\ \forall x$ *and let* $\{c_n\}_n$ *be a given sequence of arbitrary real numbers with* $c_1 \neq 0$.

(I) *Assume the following further conditions*:

$$|\phi_n(x)| \le A_n |\phi_1(x)| q(x) \ \ \forall\ x \ge T \ \ \forall\ n \ge 2; \ \ \sum_{n=1}^{\infty} |c_n| A_n < +\infty; \tag{2.4}$$

$$\left|\phi_n'(x)\right| \le B_n \left|\phi_1'(x)\right| r(x) \ \ \forall\, x \ge T \ \ \forall\, n \ge 2; \quad \sum_{n=1}^{\infty} |c_n| B_n < +\infty; \tag{2.5}$$

where both q, r *are suitable nonnegative functions such that*

$$q, r \ \text{ bounded on } [T, +\infty) \ \text{ and } = o(1),\ x \to +\infty. \tag{2.6}$$

Then the two series

$$\sum_{n=1}^{\infty} c_n \phi_n(x), \ \sum_{n=1}^{\infty} c_n \phi_n'(x) \tag{2.7}$$

are absolutely and uniformly convergent on each bounded interval of $[T, +\infty)$. *If* $f(x)$ *is the sum of the first series then* $f \in C^1[T, +\infty)$, $f'(x) = \sum_{n=1}^{\infty} c_n \phi_n'(x)$ *and*

$$f'(x)/f(x) \sim \phi_1'(x)/\phi_1(x),\ x \to +\infty. \tag{2.8}$$

(II) *Assume all the conditions in part* (I) *with the exception that* (2.5) *is now replaced by*

$$\left|\phi_n'(x)/\phi_n(x)\right| \le C \left|\phi_1'(x)/\phi_1(x)\right| \ \ \forall\, x \ge T \ \ \forall\, n \ge 2, \tag{2.9}$$

or, more generally, by

$$\left|\phi_n'(x)/\phi_n(x)\right| \le C_n \left|\phi_1'(x)/\phi_1(x)\right| \ \ \forall\, x \ge T \ \ \forall\, n \ge 2, \ \sum_{n=1}^{\infty} |c_n| A_n C_n < +\infty. \tag{2.10}$$

Then all the conclusions in part (I) *still hold true. In the special case*

$$\phi_n \in \mathcal{R}_{\alpha_n}(+\infty),\ \alpha_1 > \cdots > \alpha_n > \cdots,\ (\text{even } \alpha_1 = +\infty), \tag{2.11}$$

relations in (2.2) *are automatically satisfied* ([1]; Prop. 2.3-(III), p. 789), *and the sequence* $\{\phi_n'(x)\}_n$ *is an asymptotic scale as well provided that* "$\alpha_n \ne 0\ \forall\, n$" *because of relations* "$\phi_n'(x) \sim \alpha_n x^{-1} \phi_n(x)$".

Remarks. Conditions in (2.4), (2.5), (2.10) are a kind of uniformity respectively for the infinite families of asymptotic relations: $c_n \phi_n(x) = o(\phi_1(x))$, $c_n \phi_n'(x) = o(\phi_1'(x))$, $c_n \phi_n'(x)/\phi_n(x) = O(\phi_1'(x)/\phi_1(x))$, $n \ge 2$.

These conditions cannot be dispensed with and even in a simple case such as

$$\phi_n \in \mathcal{R}_{\alpha_n}(+\infty);\ \alpha_1 > \alpha_2 \ge \cdots \ge \alpha_n \ge \cdots;\ \alpha_n \in \mathbb{R} \setminus \{0\}, \tag{2.12a}$$

the asymptotic relations

$$\phi_n'(x)/\phi_n(x) \sim (\alpha_n/\alpha_1)\left(\phi_1'(x)/\phi_1(x)\right),\ x \to +\infty, \tag{2.12b}$$

do not in themselves grant (2.10) as shown by the counterexample of

$$\phi_n(x) := x^{1/n} (\log x)^n,\ x \ge 1;\ \phi_n'(x)/\phi_n(x) = \left(\frac{1}{n} + \frac{n}{\log x}\right)\left(\phi_1'(x)/\phi_1(x)\right). \tag{2.13}$$

Proof. For part (I) the estimates

$$\sum_{n=k}^{\infty} \left|c_n \phi_n(x)\right| \le \left(\sum_{n=k}^{\infty} |c_n| A_n\right) \left|\phi_1(x)\right| q(x) = o(\phi_1(x)),\ k \ge 2; \tag{2.14}$$

$$\sum_{n=k}^{\infty} \left|c_n \phi_n'(x)\right| \le \left(\sum_{n=k}^{\infty} |c_n| B_n\right) \left|\phi_1'(x)\right| r(x) = o(\phi_1'(x)),\ k \ge 2; \tag{2.15}$$

imply the assertions concerning the convergence of the two series due to the local boundedness of ϕ_1, ϕ_1', q, r, and moreover:

$$f(x) = c_1\phi_1(x) + \sum_{n=2}^{\infty} c_n\phi_n(x) \sim c_1\phi_1(x);\ f'(x) = c_1\phi_1(x) + \sum_{n=2}^{\infty} c_n\phi_n'(x) \sim c_1\phi_1'(x). \quad (2.16)$$

For part (II) instead of (2.15) we now have:

$$\begin{aligned} \sum_{n=k}^{\infty} \left|c_n\phi_n'(x)\right| &= \sum_{n=k}^{\infty} \left|c_n\phi_n(x)\right| \left|\frac{\phi_n'(x)}{\phi_n(x)}\right| \le \left|\frac{\phi_1'(x)}{\phi_1(x)}\right| \sum_{n=k}^{\infty} C_n \left|c_n\phi_n(x)\right| \\ &\le \left(\sum_{n=k}^{\infty} |c_n| A_n C_n\right) \left|\phi_1'(x)\right| q(x); \end{aligned} \quad (2.17)$$

for $k \ge 2$ and the subsequent conclusions are still valid. □

An elementary example. For any function ϕ such that:

$$\phi \in C^1[T, +\infty);\ \phi(x) = o(1),\ x \to +\infty;\ 0 < \phi(x) \le M \quad \forall\ x \ge T; \quad (2.18)$$

we have the estimates:

$$\begin{aligned} &\phi_n(x) := (\phi(x))^n = (\phi(x))^{n-1}\phi(x) \le M^{n-1}\phi(x),\ n \ge 1; \\ &\left|\phi_n'(x)/\phi_n(x)\right| = n\left|\phi'(x)/\phi(x)\right|,\ n \ge 2. \end{aligned} \quad (2.19)$$

Whatever $M > 0$ and $k \in \mathbb{N}$:

$$\begin{cases} f_k(x) := \sum_{n\ge k} \frac{1}{n!}(\phi(x))^n = \exp(\phi(x)) - 1 - \phi(x) - \frac{1}{2!}(\phi(x))^2 - \cdots - \frac{1}{(k-1)!}(\phi(x))^{k-1} \sim \frac{1}{k!}(\phi(x))^k,\ x \to +\infty; \\ f_k'(x) = \sum_{n\ge k} \frac{1}{(n-1)!}(\phi(x))^{n-1}\phi'(x) \sim \frac{1}{(k-1)!}(\phi(x))^{k-1}\phi'(x),\ x \to +\infty. \end{cases} \quad (2.20)$$

And if $0 < M < 1$ and $k \in \mathbb{N}$:

$$\begin{cases} g_k(x) := \sum_{n\ge k} (\phi(x))^n = \frac{1}{1-\phi(x)} - 1 - \phi(x) - (\phi(x))^2 - \cdots - (\phi(x))^{k-1} \sim (\phi(x))^k,\ x \to +\infty; \\ g_k'(x) = \sum_{n\ge k} n(\phi(x))^{n-1}\phi'(x) \sim k(\phi(x))^{k-1}\phi'(x),\ x \to +\infty. \end{cases} \quad (2.21)$$

In both cases " $f_k'(x)/f_k(x) \sim k\phi_k'(x)/\phi_k(x)$ " and " $g_k'(x)/g_k(x) \sim k\phi_k'(x)/\phi_k(x)$ ", hence:

$$\phi \in \mathcal{R}_\alpha(+\infty),\ -\infty \le \alpha \le 0 \Rightarrow f_k, g_k \in \mathcal{R}_{k\alpha}(+\infty),\ (k \in \mathbb{N}), \quad (2.22)$$

the circumstance " $\alpha > 0$ " being inconsistent with condition " $\phi = o(1)$ ".

A less elementary example. Consider the sequence of "modified iterated logarithms":

$$\begin{cases} \bar{\ell}_1(x) := \log x,\ \bar{\ell}_n(x) := \log\left(1 + \bar{\ell}_{n-1}(x)\right),\ n \ge 2;\ \bar{\ell}_n \in C^\infty[1, +\infty); \\ \frac{\mathrm{d}}{\mathrm{d}x}\bar{\ell}_n(x) = \left[x\left(1 + \bar{\ell}_{n-1}(x)\right)\left(1 + \bar{\ell}_{n-2}(x)\right)\cdots\left(1 + \bar{\ell}_1(x)\right)\right]^{-1},\ n \ge 2; \end{cases} \quad (2.23)$$

which satisfy " $\bar{\ell}_n(x), \frac{\mathrm{d}}{\mathrm{d}x}\bar{\ell}_n(x) > 0$ for $x > 1$ " and:

$$\begin{aligned} &\bar{\ell}_1(x) \gg \bar{\ell}_2(x) \gg \cdots \gg \bar{\ell}_n(x) \gg \cdots; \\ &\frac{\mathrm{d}}{\mathrm{d}x}\bar{\ell}_1(x) \gg \frac{\mathrm{d}}{\mathrm{d}x}\bar{\ell}_2(x) \gg \cdots \gg \frac{\mathrm{d}}{\mathrm{d}x}\bar{\ell}_n(x) \gg \cdots,\ x \to +\infty. \end{aligned} \quad (2.24)$$

To get useful estimates for our aim we must take $x \ge x_0 > 1$; for instance, using the elementary inequality "$\log(1+x) < x$ for $x > 0$" we get the following estimates for $x \ge e$:

$$\begin{cases} \overline{\ell}_2(x) \equiv \ell_2(x) + \log\left(1+(\log x)^{-1}\right) < \ell_2(x) + (\log x)^{-1}, \ x \ge e; \\ \overline{\ell}_3(x) \le \overline{\ell}_2(x) < \ell_2(x) + (\log x)^{-1}, \ x \ge e; \\ \qquad\qquad \vdots \\ \overline{\ell}_n(x) < \ell_2(x) + (\log x)^{-1} \le \ell_2(x) + 1, \ x \ge e, \ n \ge 2; \end{cases} \tag{2.25}$$

whence:

$$\begin{cases} 0 < \overline{\ell}_n(x)\big/\overline{\ell}_1(x) \le \left[\ell_2(x)+1\right](\log x)^{-1} \equiv q(x), \ (A_n \equiv 1), \ x \ge e, \ n \ge 2; \\ 0 < \dfrac{\mathrm{d}}{\mathrm{d}x}\overline{\ell}_n(x)\Big/\dfrac{\mathrm{d}}{\mathrm{d}x}\overline{\ell}_1(x) = \left[\left(1+\overline{\ell}_{n-1}(x)\right)\left(1+\overline{\ell}_{n-2}(x)\right)\cdots\left(1+\overline{\ell}_1(x)\right)\right]^{-1} \\ \le \left(1+\overline{\ell}_1(x)\right)^{-1} \equiv r(x), \ (B_n \equiv 1), \ x \ge e, \ n \ge 2. \end{cases} \tag{2.26}$$

It follows that for any sequence $\{c_n\}_n$ such that $\sum_{n\ge1}|c_n| < +\infty$, both series

$$f(x) := \sum_{n=1}^{\infty} c_n \overline{\ell}_n(x), \quad f'(x) := \sum_{n=1}^{\infty} c_n \frac{\mathrm{d}}{\mathrm{d}x}\overline{\ell}_n(x) \tag{2.27}$$

are absolutely convergent on $[e,+\infty)$ and uniformly convergent on each bounded interval and $f'(x)/f(x) \sim \dfrac{\mathrm{d}}{\mathrm{d}x}\overline{\ell}_1(x)\Big/\overline{\ell}_1(x), \ x \to +\infty$, *i.e.* f is slowly varying.

Let us now examine the case (2.3), the classical case being that of a power series with an infinite radius of convergence; here coefficients of nonconstant signs may generate entire functions with no definite type of asymptotic variation at $+\infty$ such as the trigonometric functions, hence in this case we must restrict our study to positive coefficients. A problem solved in American Mathematical Monthly, [3], states that if the function $f(x) := \sum_{n=0}^{\infty} c_n x^n$, with $c_n \ge 0$, is defined for all x and $k \in \mathbb{N}$ and is not a polynomial then $f \in \mathcal{R}_{+\infty}(+\infty)$, a result that can be extended to infinite series of regularly-varying functions.

Proposition 2.2. Assumptions:

$$\phi_n \in C^1[T,+\infty) \text{ and } \phi_n \in \mathcal{R}_{\alpha_n}(+\infty), \ \alpha_n \in \mathbb{R}\setminus\{0\}, \ \forall\, n \in \mathbb{N}; \tag{2.28}$$

$$\phi_n(x) > 0, \ \phi_n'(x) \ne 0, \ c_n > 0 \quad \forall\, x \in [T,+\infty) \ \forall\, n \in \mathbb{N}; \tag{2.29}$$

$$f(x) := \sum_{n=1}^{\infty} c_n \phi_n(x) \text{ and } f'(x) = \sum_{n=1}^{\infty} c_n \phi_n'(x) \text{ uniformly convergent on each compact interval;} \tag{2.30}$$

$$x\phi_n'(x) = \alpha_n \phi_n(x)\left[1 + A_n(x)\right] \text{ with } \begin{cases} |A_n(x)| \le A(x) = o(1), \ x \to +\infty \\ \alpha_n \ne 0 \end{cases}, \forall\, n \in \mathbb{N}; \tag{2.31}$$

$$\sum_{n=1}^{\infty} c_n |\alpha_n| \phi_n(x) \text{ convergent on each compact interval.} \tag{2.32}$$

We also assume (2.3) *which we express by saying that* $\{\phi_n(x)\}_n$ *forms an "inverted asymptotic scale at* $+\infty$*"; this implies "*$\{\alpha_n\}_n$ *nondecreasing" and we put* $\alpha := \lim_n \alpha_n$; *hence* $\alpha_n \le \alpha \ \forall n$. *We separate the cases: "*$-\infty < \alpha < 0$;

$\alpha = 0$; $0 < \alpha \le +\infty$". *In the first two cases monotonicity and restriction* $\alpha_n \ne 0$ *imply* "$\alpha_n < 0\ \forall n$"; *whereas in the third case we may suppose, without loss of generality, that* "$\alpha_n > 0\ \forall n$" *due to the fact that for the sum of a finite number of terms we have* $\sum_{n=1}^{p} \phi_n(x) \in \mathcal{R}_p(+\infty)$ *and we then split the given series* "$\sum_{n=1}^{\infty} = \sum_{n=1}^{p} + \sum_{n>p}$" *and apply the results below. These agreements on the signs of* α_n *imply that*:

$$\left|\sum_{n=1}^{\infty} c_n x\phi_n'(x)\right| = \sum_{n=1}^{\infty} c_n x\left|\phi_n'(x)\right|; \quad \left|\sum_{n=1}^{\infty} c_n \alpha_n \phi_n(x)\right| = \sum_{n=1}^{\infty} c_n \left|\alpha_n\right| \phi_n(x). \tag{2.33}$$

<u>Thesis</u>:

$$xf'(x) = \left(\sum_{n=1}^{\infty} c_n \alpha_n \phi_n(x)\right) \cdot \left[1 + r(x)\right], \text{ with } \left|r(x)\right| \le A(x); \tag{2.34}$$

$$f \in \mathcal{R}_\alpha(+\infty) \quad \text{whichever the value of } \alpha : -\infty < \alpha \le +\infty. \tag{2.35}$$

Remark. Of course, instead of (2.30) it is enough to assume that $\sum c_n \phi_n'(x)$ is uniformly convergent on each compact interval and that $\sum c_n \phi_n(x)$ converges for at least one value of *x*; but in the present context it may sometimes be more convenient to use the convergence of $\sum c_n \phi_n(x)$ in order to prove the convergence of $\sum c_n \phi_n'(x)$ thanks to relations in (2.31).

Proof. From (2.31)-(2.32) we get (2.34) as:

$$\begin{cases} \sum_{n=1}^{\infty} c_n x\phi_n'(x) = \sum_{n=1}^{\infty} c_n \alpha_n \phi_n(x)\left[1 + A_n(x)\right] = \sum_{n=1}^{\infty} c_n \alpha_n \phi_n(x) + \sum_{n=1}^{\infty} c_n \alpha_n A_n \phi_n(x); \\ \sum_{n=1}^{\infty} c_n \left|\alpha_n A_n\right| \phi_n(x) \le A(x) \sum_{n=1}^{\infty} c_n \left|\alpha_n\right| \phi_n(x) = o\left(\sum_{n=1}^{\infty} c_n \alpha_n \phi_n(x)\right). \end{cases} \tag{2.36}$$

Case "$0 < \alpha \le +\infty$", *which implies* "$\phi_n'(x) > 0\ \forall x\ \forall n$". *First step*:

$$\begin{aligned} &\overline{\lim_{x\to+\infty}} \frac{f(x)}{xf'(x)} = \overline{\lim_{x\to+\infty}} \frac{\sum_{n\ge1} c_n \phi_n(x)}{\sum_{n\ge1} c_n x\phi_n'(x)} = \overline{\lim_{x\to+\infty}} \frac{\sum_{n\ge1} c_n \phi_n(x)}{\sum_{n\ge1} c_n \alpha_n \phi_n(x)\left[1 + A_n(x)\right]} \ge \cdots \\ &\left(\text{as } 0 < 1 + A_n(x) \le 1 + A(x) \text{ for } x \text{ large enough}\right) \\ &\cdots \ge \overline{\lim_{x\to+\infty}} \frac{\sum_{n\ge1} c_n \phi_n(x)}{\left[1 + A(x)\right] \sum_{n\ge1} c_n \alpha_n \phi_n(x)} \ge \cdots \left(\text{as } 0 < \alpha_n \le \alpha\right) \cdots \\ &\cdots \ge \overline{\lim_{x\to+\infty}} \frac{1}{\left[1 + A(x)\right]\alpha} = \frac{1}{\alpha} \text{ if } \alpha < +\infty. \end{aligned} \tag{2.37}$$

Second step. Using a different device we get for each fixed *k*:

$$\begin{aligned} &\overline{\lim_{x\to+\infty}} \frac{f(x)}{xf'(x)} \le \overline{\lim_{x\to+\infty}} \frac{\sum_{n=1}^{k-1} c_n \phi_n(x)}{\sum_{n=1}^{\infty} c_n x\phi_n'(x)} + \overline{\lim_{x\to+\infty}} \frac{\sum_{n=k}^{\infty} c_n \phi_n(x)}{\sum_{n=1}^{\infty} c_n x\phi_n'(x)} \le \cdots \\ &\left(\text{by the positivity of } \phi_n'\right) \cdots \le \overline{\lim_{x\to+\infty}} \frac{\sum_{n=1}^{k-1} c_n \phi_n(x)}{c_k x\phi_k'(x)} + \overline{\lim_{x\to+\infty}} \frac{\sum_{n=k}^{\infty} c_n \phi_n(x)}{\sum_{n=k}^{\infty} c_n x\phi_n'(x)}. \end{aligned} \tag{2.38}$$

For the last two ratios we have:

$$\frac{\sum_{n=1}^{k-1} c_n \phi_n(x)}{c_k x \phi_k'(x)} = \cdots \text{ by (2.3) and (2.31) } \cdots = \frac{o(\phi_k(x))}{c_k x \phi_k'(x)} = o(1),\ x \to +\infty; \quad (2.39)$$

$$R(x) := \frac{\sum_{n=k}^{\infty} c_n \phi_n(x)}{\sum_{n=k}^{\infty} c_n x \phi_n'(x)} = \frac{\sum_{n=k}^{\infty} c_n \phi_n(x)}{\sum_{n=k}^{\infty} c_n \alpha_n \phi_n(x)} \cdot [1+o(1)] \le \frac{1}{\alpha_k} \cdot [1+o(1)],\ x \to +\infty, \quad (2.40)$$

whence

$$\overline{\lim_{x\to+\infty}} R(x) \le \frac{1}{\alpha_k} \text{ for each fixed } k; \quad (2.41)$$

and (2.38) yields:

$$\overline{\lim_{x\to+\infty}} \frac{f(x)}{xf'(x)} \le \frac{1}{\alpha_k} \text{ for each fixed } k. \quad (2.42)$$

Taking the limit as $k \to \infty$:

$$\overline{\lim_{x\to+\infty}} \frac{f(x)}{xf'(x)} \le \begin{cases} 1/\alpha & \text{if } 0 < \alpha < +\infty, \\ 0 & \text{if } \alpha = +\infty, \end{cases} \quad (2.43)$$

whence:

$$\lim_{x\to+\infty} \frac{f(x)}{xf'(x)} = 1/\alpha \text{ if } 0 < \alpha < +\infty, \text{ by the inequality in (2.37);} \quad (2.44a)$$

$$\lim_{x\to+\infty} \frac{f(x)}{xf'(x)} = 0 \text{ if } \alpha = +\infty, \text{ as the ratio } f(x)/xf'(x) \text{ is ultimately } >0; \quad (2.44b)$$

and this last limit implies the limit "$+\infty$" for the inverted ratio.

Case "$-\infty < \alpha \le 0$", *which, by* (2.29) *and* (2.31), *implies* "$\phi_n'(x) < 0\ \forall x\ \forall n$". Recall: $\alpha_n < 0$ and nondecreasing. First step:

$$\frac{xf'(x)}{f(x)} \overset{(2.34)}{=} \frac{[1+r(x)] \cdot \sum_{n=1}^{\infty} c_n \alpha_n \phi_n(x)}{\sum_{n=1}^{\infty} c_n \phi_n(x)} \le \alpha \cdot [1+r(x)], \quad (2.45)$$

for x so large that "$1+r(x) > 0$"; whence

$$\underline{\lim_{x\to+\infty}} \frac{xf'(x)}{f(x)} \le \alpha. \quad (2.46)$$

Second step. For each fixed k:

$$\begin{aligned} \underline{\lim_{x\to+\infty}} \frac{xf'(x)}{f(x)} &= \underline{\lim_{x\to+\infty}} \frac{[1+r(x)] \cdot \sum_{n=1}^{\infty} c_n \alpha_n \phi_n(x)}{\sum_{n=1}^{\infty} c_n \phi_n(x)} \\ &\ge \underline{\lim_{x\to+\infty}} \frac{[1+r(x)] \cdot \sum_{n=1}^{k-1} c_n \alpha_n \phi_n(x)}{\sum_{n=1}^{\infty} c_n \phi_n(x)} + \underline{\lim_{x\to+\infty}} \frac{[1+r(x)] \cdot \sum_{n=k}^{\infty} c_n \alpha_n \phi_n(x)}{\sum_{n=1}^{\infty} c_n \phi_n(x)}. \end{aligned} \quad (2.47)$$

For the first ratio on the right we have "$|\alpha_n| \le |\alpha_{n-1}| \;\forall n$" and:

$$\left| \frac{\sum_{n=1}^{k-1} c_n \alpha_n \phi_n(x)}{\sum_{n=1}^{\infty} c_n \phi_n(x)} \right| \le \frac{|\alpha_1| \sum_{n=1}^{k-1} c_n \phi_n(x)}{\sum_{n=1}^{\infty} c_n \phi_n(x)} \le \frac{|\alpha_1| \sum_{n=1}^{k-1} c_n \phi_n(x)}{c_k \phi_k(x)} \overset{(2.3)}{=} o(1),\; x \to +\infty; \quad (2.48a)$$

whereas for the second ratio, as it stands without the absolute value, we have:

$$\frac{[1+r(x)] \cdot \sum_{n=k}^{\infty} c_n \alpha_n \phi_n(x)}{\sum_{n=1}^{\infty} c_n \phi_n(x)} \ge \frac{[1+r(x)] \cdot \alpha_n \sum_{n=k}^{\infty} c_n \phi_n(x)}{\sum_{n=1}^{\infty} c_n \phi_n(x)} \ge \alpha_k \quad \text{for } x \text{ large enough.} \quad (2.48b)$$

From (2.47) we get:

$$\varliminf_{x \to +\infty} \frac{x f'(x)}{f(x)} \ge \alpha_k \quad \text{for each fixed } k; \tag{2.49}$$

and taking the limit as $k \to \infty$:

$$\varliminf_{x \to +\infty} \frac{x f'(x)}{f(x)} \ge \alpha, \tag{2.50}$$

which, together with (2.46) yields "$\lim_{x \to +\infty} x f'(x)/f(x) = \alpha$". □

Examples. Let $\{\alpha_n\}_n$ be a strictly increasing sequence of real numbers and $c_n > 0 \;\forall n$.

1) For $\phi_n(x) := x^{\alpha_n}, x \ge 1$, condition in (2.31) is trivially satisfied with $A_n(x) \equiv 0$.

First case. If $\lim_n \alpha_n = \alpha \in \mathbb{R}$ then:

$$\sum_{n\ge1} c_n x^{\alpha_n} \le x^{\alpha} \sum_{n\ge1} c_n,\; x \ge 1;\; \sum_{n\ge1} c_n |\alpha_n| x^{\alpha_n - 1} \le \left(\sup|\alpha_n|\right) x^{\alpha-1} \sum_{n\ge1} c_n,\; x \ge 1; \tag{2.51}$$

and conditions in (2.30), (2.32) reduce to "$\sum_{n\ge1} c_n < +\infty$" which implies (2.35). The identities "$x\phi_n^{(k+1)}(x) \equiv (\alpha_n - k)\phi_n^{(k)}(x) \;\forall k \ge 0$" and similar estimates for the formally differentiated series of higher order yield:

$$\begin{cases} \sum_{n\ge1} c_n < +\infty, \\ f(x) := \sum_{n\ge1} c_n x^{\alpha_n} \end{cases} \Rightarrow \begin{cases} f \in C^{\infty}[1, +\infty), \\ f^{(k)} \in \mathcal{R}_{\alpha-k}(+\infty) \;\forall\, k \in \mathbb{N} \cup \{0\}. \end{cases} \tag{2.52}$$

Consistently with the statement of Proposition 2.2 the circumstance "$\alpha_n - k = 0$" for some value of k and at most one value of n, say $n = p$, is treated by splitting the series

$$\sum_{n\ge1} (\alpha_n - k)\phi_n^{(k)}(x) = \left(\sum_{n\le p} + \sum_{n>p}\right)(\alpha_n - k)\phi_n^{(k)}(x).$$

Second case. If $\lim_n \alpha_n = +\infty$ then:

$$f(x) := \sum_{n\ge1} c_n x^{\alpha_n} < +\infty \;\; \forall\, x \ge T > 0 \Rightarrow \begin{cases} f \in C^{\infty}[T, +\infty), \\ f^{(k)} \in \mathcal{R}_{+\infty}(+\infty) \;\forall\, k \in \mathbb{N}. \end{cases} \tag{2.53}$$

In fact, assuming $\alpha_n > 0 \;\forall n$ and with "$[\alpha_n]$ = integer part of α_n", we have

"$[\alpha_n] \le \alpha_n < [\alpha_n]+1$" and:

$$c_n \alpha_n x^{\alpha_n} = x c_n \alpha_n x^{\alpha_n - 1} \le x c_n \left([\alpha_n]+1\right) x^{[\alpha_n]};$$

so that:

$$\sum_{n\ge1} c_n \alpha_n x^{\alpha_n} \le x^2 \sum_{n\ge1} c_n [\alpha_n] x^{[\alpha_n]-1} + x \sum_{n\ge1} c_n x^{[\alpha_n]}, \tag{2.54}$$

where the second series on the right is a power series majorized by the convergent series $\sum_{n\ge1} c_n x^{\alpha_n}$ and the first series on the right is the formal derivative of a convergent power series, hence itself convergent uniformly on each compact subinterval of $]0,+\infty)$. Analogous situations for the higher-order derivatives and (2.53) follows.

2) For $\phi_n(x) := x^{\alpha_n} (\log x)^{\beta_n}$, $x \ge T > 1$, we have

$$x\phi_n'(x) = \alpha_n \phi_n(x)\left[1 + \frac{\beta_n}{\alpha_n}(\log x)^{-1}\right], \tag{2.55}$$

being $\alpha_n \ne 0$ ultimately. Hence condition in (2.31) is satisfied if $\beta_n = O(\alpha_n)$, $n \to \infty$ without any additional condition on β_n.

The cases "$\alpha_n = \text{constant} \in \overline{\mathbb{R}}$" deserve a separate brief discussion as they can be treated quite simply. The following results do not require any growth-order chain between the ϕ_n's.

Proposition 2.3. (The case $\alpha_n = \text{constant} \in \mathbb{R}$). *Assume that*:

$$L_n \in C^1[T,+\infty) \cap \mathcal{R}_0(+\infty);\ c_n > 0\ \forall n \in \mathbb{N}; \tag{2.56}$$

$$\sum_{n=1}^{\infty} c_n L_n(x) \text{ uniformly and absolutely convergent on each compact interval;} \tag{2.57}$$

$$x L_n'(x) = r_n(x) L_n(x) \text{ with } |r_n(x)| \le r(x) \begin{cases} \le A \quad \forall x \ge T, \\ = o(1),\ x \to +\infty\ \forall n \in \mathbb{N}. \end{cases} \tag{2.58}$$

Then the function $f(x) := x^\alpha L_n(x)$, $\alpha \in \mathbb{R}$, *satisfies* "$f \in C^1[T,+\infty) \cap \mathcal{R}_\alpha(+\infty)$". *Without either the restriction on the signs of* c_n *or the conditions in* (*2.58*) *no definite conclusion on the type of variation of* f *can be drawn.*

Proof. Notice that (2.56) implies $L_n(x) > 0$ for x large enough and that we may always assume $T > 0$. For $\alpha = 0$ we have:

$$\sum_{n\ge1} c_n |L_n'(x)| \le x^{-1} r(x) \cdot \sum_{n\ge1} c_n |L_n(x)| \begin{cases} \le \dfrac{A}{T} \cdot \sum\limits_{n\ge1} c_n |L_n(x)|,\ x \ge T; \\ = o\left(x^{-1} f(x)\right),\ x \to +\infty; \end{cases} \tag{2.59}$$

whence it follows that

$$\sum_{n=1}^{\infty} c_n L_n'(x) \text{ is uniformly and absolutely convergent on each compact interval of } [T,+\infty).$$

Hence $\sum_{n=1}^{\infty} c_n L_n'(x) = f'(x)\ \forall x \ge T$ and $f'(x) = o\left(x^{-1} f(x)\right)$. For $\alpha \ne 0$ just apply the previous result to "$x^{-\alpha} f(x)$" so obtaining "$\left(x^{-\alpha} f(x)\right)' = o\left(x^{-\alpha-1} f(x)\right)$", whence " $f'(x) \sim \alpha x^{-1} f(x)$ " follows. □

Example. Let $\{a_n\}_n$ be any strictly decreasing sequence convergent to $a \ge 0$ and define

$$\phi_n(x) := (\log x)^{-a_n}, \; x \ge e; \; |x\phi_n'(x)| = a_n (\log x)^{-1} \phi_n(x) \le a_1 (\log x)^{-1} \phi_n(x). \quad (2.60)$$

If $\sum_{n\ge1} c_n$ is any convergent series of constant positive terms then the function $f(x) := \sum_{n\ge1} c_n \phi_n(x)$ belongs to the class: $f \in C^1[e, +\infty) \cap \mathcal{R}_0(+\infty)$.

Counterexamples. Let L be any function such that:

$$L \in C^1[T, +\infty) \cap \mathcal{R}_0(+\infty); \; L(+\infty) = +\infty; \quad (2.61)$$

define $\phi_n(x) := (L(x))^n$ and let $\Phi(x) := \sum_{n\ge0} c_n x^n, c_n \ge 0$ be an entire non-polynomial function. Here condition in (2.58) cannot be satisfied as $x\phi_n'(x)/\phi(x) = n(xL'(x)/L(x))$, except in case $L'(x) \equiv 0$, and the function $f(x) := \sum_{n\ge0} c_n (L(x))^n = \Phi(L(x)), x \ge T$, may have an arbitrarily large growth-order at $+\infty$ or may be slowly varying. To visualize, just consider the exponential power series and various elementary choices for L:

$$\begin{cases} \sum_{n\ge0} (c\ell_2(x))^n / n! = (\log x)^c \in \mathcal{R}_0(+\infty), \; c \ne 0; \; x \in [e, +\infty); \\ \sum_{n\ge0} \left(c(\log x)^\delta\right)^n / n! = \exp\left(c(\log x)^\delta\right) \in \mathcal{R}_0(+\infty), \; 0 < \delta < 1, \; c \ne 0; \; x \in [1, +\infty); \\ \sum_{n\ge0} (c\log x)^n / n! = \exp(c\log x) \equiv x^c \in \mathcal{R}_c(+\infty), \; c \ne 0; \; x \in [1, +\infty); \\ \sum_{n\ge0} \left(\pm(\log x)^\delta\right)^n / n! = \exp\left(\pm(\log x)^\delta\right) \in \mathcal{R}_{\pm\infty}(+\infty), \; \delta > 1; \; x \in [1, +\infty); \\ \sum_{n\ge0} \left[\pm\exp(\log x)^\delta\right]^n / n! = \exp\left[\pm\exp(\log x)^\delta\right] \in \mathcal{R}_{\pm\infty}(+\infty), \; 0 < \delta < 1; \; x \in [1, +\infty). \end{cases} \quad (2.62)$$

Proposition 2.4. (The case $\alpha_n = \pm\infty$). Assumptions: (2.28) *with* $\alpha_n = +\infty \; \forall n$ *or* $\alpha_n = -\infty \; \forall n$, *and* $c_n > 0 \; \forall n$; *the uniform convergence of both series* $\sum_{n=1}^{\infty} c_n \phi_n(x), \sum_{n=1}^{\infty} c_n \phi_n'(x)$ *on each compact interval. If* $\alpha_n = +\infty$ *and*

$$\forall M > 0 \; \exists T_M, \textit{ independent of } n,$$
$$\textit{such that } x\phi_n'(x) > M\phi_n(x) \; \forall x > T_M \; \forall n \quad (2.63)$$

then $f \in \mathcal{R}_{+\infty}(+\infty)$. *If* $\alpha_n = -\infty$ *and*

$$\forall M < 0 \; \exists T_M, \textit{ independent of } n,$$
$$\textit{such that } x\phi_n'(x) < M\phi_n(x) \; \forall x > T_M \; \forall n \quad (2.64)$$

then $f \in \mathcal{R}_{-\infty}(+\infty)$.

Proof. Both proofs are trivial using the stated assumptions of "uniformity with respect to n" of the relations " $\lim_{x\to+\infty} x\phi_n'(x)/\phi_n(x) = \pm\infty$ ". For $\alpha_n = +\infty$:

$$M > 0 \Rightarrow \sum_{n=1}^{\infty} c_n x\phi_n'(x) \Big/ \sum_{n=1}^{\infty} c_n \phi_n(x) \ge M \quad \forall x \ge T_M,$$

and analogously for $\alpha_n = -\infty$. □

For series, with positive coefficients, of functions having hypo-exponential, or exponential, or hyper-exponential variation at $+\infty$ corresponding results hold true suppressing the factor x on the left of the asymptotic relation in (2.31). All the proofs are exactly the same and, in particular, we have the result:

$$\begin{cases} \phi_n \in \mathcal{E}_{\alpha_n}(+\infty), -\infty < \alpha_n \le +\infty, \\ \phi_n'(x) = \phi_n(x)\left[\alpha_n + A_n(x)\right], \ |A_n(x)| \le A(x) = o(1), \ x \to +\infty, \\ \alpha_n \nearrow \alpha, \end{cases} \tag{2.65}$$
$$\Rightarrow f \in \mathcal{E}_\alpha(+\infty),$$

provided that the remaining assumptions in Proposition 2.2 are satisfied.

Three more examples. The following examples are not included in the previous results but elucidate techniques based on the type of asymptotic variation.

1) Consider the sequence:

$$\begin{cases} \phi_n(x) := \exp\left[(\log x)^{1-1/n}\right], \ x \ge e, \ n \ge 2; \quad \phi_n \in \mathcal{R}_0(+\infty); \\ \phi_{n-1}(x) \le \phi_n(x) \ \ \forall x \ge \mathrm{e}; \quad \phi_{n-1}(x) \ll \phi_n(x), \ x \to +\infty, \ \forall\, n \ge 2; \\ \lim_n \phi_n(x) = x \quad \text{uniformly on each compact subset of } [e, +\infty); \\ \phi_n'(x) = \dfrac{n-1}{n} x^{-1} (\log x)^{-1/n} \phi_n(x) \equiv r_n(x)\phi_n(x) > 0, \ x \ge e. \end{cases} \tag{2.66}$$

This is a sequence of slowly varying functions which is locally uniformly convergent to a regularly-varying function of index 1. The factor $r_n(x)$ satisfies:

$$0 < r_n(x) \le x^{-1}, \text{ hence } r_n(x) = O\left(x^{-1}\right) \text{ uniformly with respect to } n; \tag{2.67}$$

but, though "$r_n(x) = o\left(x^{-1}\right), x \to +\infty$" for each fixed *n*, this last relation is not uniform with respect to *n*. Suppose now that for a sequence of positive numbers c_n the series $\sum_{n\ge 2} c_n \phi_n(x)$ is locally uniformly convergent; then

$$0 < \sum_{n\ge 2} c_n \phi_n'(x) \le \sum_{n \ge 2} c_n \phi_n(x);$$

hence the series of the derivatives is locally uniformly convergent as well and, by (2.65), $f \in \mathcal{E}_o(+\infty)$. But direct calculations yield more precise information:

$$f'(x) \begin{cases} = x^{-1} \cdot \sum\limits_{n\ge 2} \dfrac{n-1}{n} c_n (\log x)^{-1/n} \phi_n(x) \le x^{-1} \sum\limits_{n\ge 2} c_n \phi_n(x) = x^{-1} f(x), \\ \ge x^{-1} (\log x)^{-1} \sum\limits_{n \ge 2} \dfrac{n-1}{n} c_n \phi_n(x) \ge \dfrac{1}{2} x^{-1} (\log x)^{-1} \sum\limits_{n\ge 2} c_n \phi_n(x) = \dfrac{1}{2} x^{-1} (\log x)^{-1} f(x), \end{cases} \tag{2.68}$$

valid for $x \ge \mathrm{e}$; whence, integrating the ratio f'/f:

$$\log\left(f(x)/f(e)\right) \begin{cases} \le \log(x/e) \\ \ge \dfrac{1}{2} \ell_2(x) \end{cases}, \ x \ge e, \tag{2.69a}$$

$$f(x) \begin{cases} \le \left(f(e)/e\right) x = \dfrac{1}{e}\left(\sum\limits_{n\ge 2} c_n \phi_n(e)\right) x = \left(\sum\limits_{n\ge 2} c_n\right) x \\ \ge e\left(\sum\limits_{n\ge 2} c_n\right) (\log x)^{1/2} \end{cases}, \ x \ge e. \tag{2.69b}$$

The relations in the first line of (2.68) imply the two limits

$$\underline{\lim}_{x\to+\infty} x f'(x)/f(x) \ge 0, \quad \overline{\lim}_{x\to+\infty} x f'(x)/f(x) \le 1;$$

hence, if f happens to be regularly varying of some index α, then $0 \le \alpha \le 1$.

2) Consider the sequence:

$$\begin{cases} \phi_n(x) := \exp\left[(\log x)^n\right],\ x \ge e,\ n \ge 2; \quad \phi_n \in \mathcal{R}_{+\infty}(+\infty); \\ \phi_{n-1}(x) \le \phi_n(x)\ \ \forall x \ge e; \quad \phi_{n-1}(x) \ll \phi_n(x),\ x \to +\infty,\ \forall n \ge 2; \\ \phi_n'(x) = nx^{-1}(\log x)^{n-1}\phi_n(x),\ x \ge e. \end{cases} \tag{2.70}$$

If the two conditions in (2.30) hold true then:

$$\begin{aligned} f'(x) &= x^{-1}\sum_{n\ge2} nc_n(\log x)^{n-1}\phi_n(x) \ge x^{-1}\cdot(\log x)\cdot\sum_{n\ge2}c_n\phi_n(x) \\ &= x^{-1}\cdot(\log x)\cdot f(x);\ (c_n > 0); \end{aligned}$$

whence: $xf'(x)/f(x) \to +\infty$ *i.e.* $f \in \mathcal{R}_{+\infty}(+\infty)$. The inequality "$f'(x)/f(x) \ge x^{-1}\log x,\ x \ge e$" yields the global estimates:

$$\begin{cases} \log\left(f(x)/f(e)\right) \ge \int_e^x t^{-1}\log t\,\mathrm{d}t = \frac{1}{2}(\log x)^2 - \frac{1}{2},\ x \ge e; \\ f(x) \ge f(e)e^{-1/2}\exp\left[(\log x)^2/2\right] = e^{1/2}\exp\left[(\log x)^2/2\right],\ x \ge e; \end{cases} \tag{2.71}$$

and obviously:

$$\begin{aligned} &f(x) \ge c_k\phi_k(x)\ \ \forall x, k \text{ and} \\ &f(x) = +\infty\left(\exp(\log x)^k\right),\ x \to +\infty, \text{ for each fixed } k \ge 2. \end{aligned} \tag{2.72}$$

3) For a bounded sequence of real numbers of arbitrary signs, $|c_n| \le M$, consider the following two series:

$$f(x) := \sum_{n\ge1}\frac{c_n}{1+(nx)^\alpha}, \quad f'(x) := -\alpha x^{\alpha-1}\sum_{n\ge1}\frac{c_n n^\alpha}{\left(1+(nx)^\alpha\right)^2},\ \alpha > 1, \tag{2.73}$$

which are absolutely and uniformly convergent on the whole interval $[1,+\infty)$ because:

$$\begin{cases} \sum_{n\ge1}\frac{c_n}{1+(nx)^\alpha} = x^{-\alpha}\sum_{n\ge1}\frac{c_n}{n^\alpha\left(1+(nx)^{-\alpha}\right)} \le x^{-\alpha}\sum_{n\ge1}\frac{c_n}{n^\alpha}; \\ \sum_{n\ge1}\frac{c_n n^\alpha}{\left(1+(nx)^\alpha\right)^2} = x^{-2\alpha}\sum_{n\ge1}\frac{c_n}{n^{2\alpha}\left(1+(nx)^{-\alpha}\right)^2} \le x^{-2\alpha}\sum_{n\ge1}\frac{c_n}{n^\alpha}. \end{cases} \tag{2.74}$$

We also have the estimates:

$$\begin{cases} \frac{1}{2}x^{-\alpha}\sum_{n\ge1}\frac{c_n}{n^\alpha} \le f(x) \le x^{-\alpha}\sum_{n\ge1}\frac{c_n}{n^\alpha},\ x \ge 1; \\ \frac{\alpha}{4}x^{-\alpha-1}\sum_{n\ge1}\frac{c_n}{n^{2\alpha}} \le \left|f'(x)\right| \le \alpha x^{-\alpha-1}\sum_{n\ge1}\frac{c_n}{n^{2\alpha}} \le \alpha x^{-\alpha-1}\sum_{n\ge1}\frac{c_n}{n^\alpha},\ x \ge 1; \end{cases} \tag{2.75}$$

hence f is positive, strictly decreasing and:

$$-2\alpha \le xf'(x)/f(x) \le -\alpha/4, \tag{2.76}$$

which implies:

$$\lambda^{-2\alpha} \le \frac{f(\lambda x)}{f(x)} \le \lambda^{-\alpha/4} \quad \forall \text{ fixed } \lambda > 1, \tag{2.77}$$

a weaker concept of regular variation thoroughly studied in ([4]; Ch. 2).

3. Type of Asymptotic Variation of a Wronskian

In two papers [5] [6] the author described a number of techniques to obtain the asymptotic behaviors of Wronskians whose entries are regularly- or rapidly-varying functions of higher order; here we point out some cases wherein one can specify the type of asymptotic variation of the involved Wronskians under the natural additional assumption on the nth derivatives. Proofs are based on the evaluation of a type of Vandermonde determinants with a gap in the exponents, determinants which are still a part of the classical theory of determinants.

Lemma 3.1. (I) *If* $(c_1,\cdots,c_n)$ *is an ordered n-tuple of complex numbers with* $n \ge 2$, *its Vandermondian with a one-unit gap in the highest exponent is defined as the number*:

$$\tilde{V}(c_1,c_2) := \begin{vmatrix} 1 & 1 \\ c_1^2 & c_2^2 \end{vmatrix} = \left(c_2^2 - c_1^2\right) = V(c_1,c_2)\cdot(c_1+c_2); \tag{3.1}$$

$$\tilde{V}(c_1,\cdots,c_n) := \begin{vmatrix} 1 & 1 & \cdots & 1 \\ c_1 & c_2 & \cdots & c_n \\ c_1^2 & c_2^2 & \cdots & c_n^2 \\ \vdots & \vdots & & \vdots \\ c_1^{n-2} & c_2^{n-2} & \cdots & c_n^{n-2} \\ c_1^n & c_2^n & \cdots & c_n^n \end{vmatrix} = V(c_1,\cdots,c_n)\cdot(c_1+\cdots+c_n),\ n \ge 3, \tag{3.2}$$

where $V(c_1,\cdots,c_n)$ *denotes the standard Vandermondian.*

(II) *The following formula holds true*:

$$\begin{vmatrix} 1 & \cdots & 1 \\ c_1 & \cdots & c_n \\ (c_1)^{\underline{2}} & \cdots & (c_n)^{\underline{2}} \\ \vdots & \cdots & \vdots \\ (c_1)^{\underline{n-2}} & \cdots & (c_n)^{\underline{n-2}} \\ (c_1)^{\underline{n}} & \cdots & (c_n)^{\underline{n}} \end{vmatrix} = V(c_1,\cdots,c_n)\cdot\left(c_1+\cdots+c_n - n(n-1)/2\right),\ n \ge 2. \tag{3.3}$$

Proof. Part I is mentioned as an exercise by Mirsky ([7]; Problem 30 on Chapter I, p. 38): it can be proved by polynomial algebra adapting the argument in ([7]; p. 18). Part II is checked at once for $n = 2$ and can be proved for $n \ge 3$ by repeating, as a first step, the elementary procedure used in reducing the determinant $\det\left[1, c_i, (c_i)^{\underline{2}}, \cdots, (c_i)^{\underline{n}}\right]_{i=1}^{n}$ to the Vandermondian: this is sketched in ([5]; Lemma 4 part 5, p. 10) and then used twice, in ([5]; proof of Th. 9, p. 19) and ([6]; proof of Th. 3, p.19). In the present case the procedure, when iterated until the exponent $n-2$, yields:

$$\begin{vmatrix} 1 \\ c_i \\ (c_i)^{\underline{2}} \\ \vdots \\ (c_i)^{\underline{n-2}} \\ (c_i)^{\underline{n}} \end{vmatrix}_{i=1,\cdots,n} = \begin{vmatrix} 1 \\ c_i \\ c_i^2 \\ \vdots \\ c_i^{n-2} \\ (c_i)^{\underline{n}} \end{vmatrix}_{i=1,\cdots,n}, \tag{3.4}$$

where, as easily checked, the polynomial in $c_i, (c_i)^{\underline{n}}$, has an expression of the form

$$(c_i)^{\underline{n}} \equiv c_i^n - \frac{n(n-1)}{2} c_i^{n-1} + a_{n-2} c_i^{n-2} + \cdots + a_1 c_1,$$

with coefficients a_k independent of the index i so that, by subtracting the appropriate linear combination of the preceding rows, we get that the determinant on the right in (3.4) equals

$$\begin{vmatrix} 1 \\ c_i \\ c_i^2 \\ \vdots \\ c_i^{n-2} \\ c_i^n - \frac{n(n-1)}{2} c_i^{n-1} \end{vmatrix}_{i=1,\cdots,n} = \begin{vmatrix} 1 \\ c_i \\ c_i^2 \\ \vdots \\ c_i^{n-2} \\ c_i^n \end{vmatrix}_{i=1,\cdots,n} - \frac{n(n-1)}{2} \begin{vmatrix} 1 \\ c_i \\ c_i^2 \\ \vdots \\ c_i^{n-2} \\ c_i^{n-1} \end{vmatrix}_{i=1,\cdots,n} \tag{3.5}$$

$$= \tilde{V}(c_1,\cdots,c_n) - \frac{n(n-1)}{2} V(c_1,\cdots,c_n),$$

and (3.3) follows. □

The following is a complement to Theorem 9 in ([5]; p. 18).

Theorem 3.2. (Wronskians of smoothly-varying functions). *Let* $\phi_i \in C^n[T,+\infty), \phi_i(x) \neq 0$ *for x large enough,* $1 \le i \le n, n \ge 2$, *and*

$$\phi_i^{(k)}(x)/\phi_i(x) = x^{-k}\left[(a_i)^{\underline{k}} + o(1)\right],\ x \to +\infty,\ \forall\, i,k \in \{0,1,\cdots,n\};\ a_i \neq a_j\ \forall i \neq j; \tag{3.6}$$

which means that each ϕ_i *is smoothly varying of order n and index* a_i *according to Definition* 3.2 *in* ([1]; p. 803). *Then*:

$$\begin{cases} W(x) := W(\phi_1(x),\cdots,\phi_n(x)) \sim V(a_1,\cdots,a_n)\cdot\left(\prod_{i=1}^n \phi_i(x)\right)\cdot x^{-n(n-1)/2}, \\ W'(x)/W(x) = x^{-1}\left[a_1+\cdots+a_n - n(n-1)/2 + o(1)\right], \end{cases} x \to +\infty, \tag{3.7}$$

i.e. W is regularly varying of order n and index "$a_1+\cdots+a_n - n(n-1)/2$" *in the strong sense of Definition* 2.1 *in* ([1]; p. 781). *In particular W is slowly varying whenever* $a_1+\cdots+a_n = n(n-1)/2$ *such as the elementary Wronskian*

"$W(x^{a_1},\cdots,x^{a_n}) = V(a_1,\cdots,a_n) = \text{constant}$": ([5]; formula (68), p. 10).

Proof. The first relation in (3.7) is proved in ([5]; Theorem 9, p. 18) whereas for the behavior of W' we have:

$$W'(x) \equiv \begin{vmatrix} \phi_i(x) \\ \phi_i'(x) \\ \vdots \\ \phi_i^{(n-2)}(x) \\ \phi_i^{(n)}(x) \end{vmatrix}_{i=1,\cdots,n} = \left(\prod_{i=1}^{n}\phi_i(x)\right)\cdot \begin{vmatrix} 1 \\ x^{-1}\left[a_i + o(1)\right] \\ x^{-2}\left[(a_i)^{\underline{2}} + o(1)\right] \\ \vdots \\ x^{n-2}\left[(a_i)^{\underline{n-2}} + o(1)\right] \\ x^{-n}\left[(a_i)^{\underline{n}} + o(1)\right] \end{vmatrix}_{i=1,\cdots,n} \tag{3.8}$$

$$= \left(\prod_{i=1}^{n}\phi_i(x)\right)\cdot x^{-\frac{n(n-1)}{2}-1}\cdot \begin{vmatrix} 1 \\ \left[a_i + o(1)\right] \\ \left[(a_i)^{\underline{2}} + o(1)\right] \\ \vdots \\ \left[(a_i)^{\underline{n-2}} + o(1)\right] \\ \left[(a_i)^{\underline{n}} + o(1)\right] \end{vmatrix}_{i=1,\cdots,n}.$$

Applying the procedure in (3.4)-(3.5) it is seen that the last determinant equals "$\tilde{V}(c_1,\cdots,c_n)+o(1)$", and by Lemma 3.1 we get:

$$W'(x) = \left(\prod_{i=1}^{n}\phi_i(x)\right)\cdot x^{-\frac{n(n-1)}{2}-1} \cdot\left[V(a_1,\cdots,a_n)\cdot\left(a_1+\cdots+a_n-n(n-1)/2\right)+o(1)\right],\ x\to+\infty; \tag{3.9}$$

whence the second relation in (3.7) follows. □

To obtain a result in the case of rapid variation we need a correct statement of the analogue of Theorem 6 in ([5]; pp. 11-12) for the determinant $\tilde{V}$. Rereading the proof of this theorem we get the following claims.

Lemma 3.3. *Let* $g_1,\cdots,g_n$ *and* $f_1,\cdots,f_n$ *be functions defined on a deleted neighborhood of* $x_0\in\overline{\mathbb{R}}$ *and such that*:

$$g_i(x) = f_i(x)+o\left(f_i(x)\right),\ x\to x_0,\ 1\le i\le n,\ n\ge 2. \tag{3.10}$$

The following relations hold true. (I) *The general estimate*

$$\begin{aligned} &\tilde{V}\left(g_1(x),\cdots,g_n(x)\right) \\ &\equiv \tilde{V}\left(f_1(x)+o\left(f_1(x)\right),\cdots,f_n(x)+o\left(f_n(x)\right)\right) \\ &= \tilde{V}\left(f_1(x),\cdots,f_n(x)\right)+o\left(\sum_{(p_1,\cdots,p_n)\in\mathcal{P}'}\prod_{i=1}^{n}\left|f_i(x)\right|^{p_i}\right),\ x\to x_0, \end{aligned} \tag{3.11}$$

where $\mathcal{P}'$ *denotes the set of all permutations of the n-tuple* $(0,1,\cdots,n-2,n)$. *Formula* (3.11) *must be read with the agreement that* "$\left|f_i(x)\right|^0\equiv 1$" *regardless of the possible zeros of* $f_i(x)$. *For* $n=2$ *it reduces to*:

$$\begin{aligned} &\tilde{V}\left(f_1(x)+o\left(f_1(x)\right),\ f_2(x)+o\left(f_2(x)\right)\right) \\ &= \tilde{V}\left(f_1(x),\ f_2(x)\right)+o\left(\left|f_1(x)\right|^2+\left|f_2(x)\right|^2\right),\ x\to x_0. \end{aligned} \tag{3.12}$$

(II) *If we assume the following relationships between the* f_i*'s*:

$$f_1(x) \succeq f_2(x) \succeq \cdots \succeq f_n(x),\ x \to x_0,\ \forall i, \tag{3.13a}$$

[*where* "$f \succeq g$" *means* "$g = O(f)$"], *in practice*:

$$\text{either} \quad f_i(x) \gg f_{i+1}(x) \quad \text{or} \quad f_i(x) \asymp f_{i+1}(x),\ x \to x_0,\ \forall i, \tag{3.13b}$$

then relation (3.11) *takes the simpler form*:

$$\begin{aligned} &\tilde{V}\left(f_1(x)+o\left(f_1(x)\right),\cdots,f_n(x)+o\left(f_n(x)\right)\right) \\ &= \tilde{V}\left(f_1(x),\cdots,f_n(x)\right)+o\left(f_1(x)\cdot\prod_{i=1}^{n-1}\left|f_i(x)\right|^{n-i}\right),\ x \to x_0, \end{aligned} \tag{3.14}$$

noticing, in the right-hand side, the lack of f_n *which is one of the functions* f_i *with the lowest growth-order. The two most meaningful cases are highlighted in the following statements.*

(III) *If* $(f_1,\cdots,f_n)$ *is an asymptotic scale, i.e.* $f_1(x) \gg \cdots \gg f_n(x),\ x \to x_0$, *then* (3.14) *becomes*

$$\begin{aligned} &\tilde{V}\left(f_1(x)+o\left(f_1(x)\right),\cdots,f_n(x)+o\left(f_n(x)\right)\right) \\ &= (-1)^{n(n-1)/2}\left(f_1(x)\cdot\prod_{i=1}^{n-1}\left(f_i(x)\right)^{n-i}\right)+o\left(f_1(x)\cdot\prod_{i=1}^{n-1}\left|f_i(x)\right|^{n-i}\right),\ x \to x_0. \end{aligned} \tag{3.15}$$

(IV) *If all the* f_i*'s have the same growth-order in the sense that*

$$f_i(x) = c_i f(x)+o\left(f(x)\right),\ x \to x_0,\ 1 \le i \le n, \tag{3.16}$$

for a fixed f *and arbitrary pairwise-distinct constants* c_i, *then*:

$$\begin{aligned} &\tilde{V}\left(c_1 f(x)+o\left(f(x)\right),\cdots,c_n f(x)+o\left(f(x)\right)\right) \\ &= V(c_1,\cdots,c_n)\cdot(c_1+\cdots+c_n)\cdot\left(f(x)\right)^{\frac{n(n-1)}{2}+1}+o\left(\left|f(x)\right|^{\frac{n(n-1)}{2}+1}\right),\ x \to x_0. \end{aligned} \tag{3.17}$$

In the special case $f(x) \equiv 1$ *we get*:

$$\begin{aligned} \tilde{V}\left(c_1+o(1),\cdots,c_n+o(1)\right) &\equiv \begin{vmatrix} 1 & \cdots & 1 \\ c_1+o(1) & \cdots & c_n+o(1) \\ c_1^2+o(1) & \cdots & c_n^2+o(1) \\ \vdots & & \vdots \\ c_1^{n-2}+o(1) & \cdots & c_n^{n-2}+o(1) \\ c_1^{n-1}+o(1) & \cdots & c_n^{n-1}+o(1) \end{vmatrix} \\ &= V(c_1,\cdots,c_n)\cdot(c_1+\cdots+c_n)+o(1),\ x \to x_0, \end{aligned} \tag{3.18}$$

a relation already used in the proof of Theorem 3.2.

Hints for the proof. When rereading Theorem 6 in ([5]; pp. 11-14) the reader will notice that the agreement "$\left|f_i(x)\right|^0 \equiv 1$" is not explicitly stated in the statement but it is clearly specified in the proof ([5]; p. 13, line 8 from below). At this point of the proof in [5] the estimate in (3.11) is proved; to proceed, the reader will replace the quantity in formula (95) in ([5]; p. 13) with:

$$\left|f_1(x)\right|^n \cdot \left|f_2(x)\right|^{n-2} \cdots \left|f_{n-1}(x)\right|^1 \cdot \left|f_n(x)\right|^0, \tag{3.19}$$

which stands for one of the terms with maximal growth-order appearing in the sum inside the "o"-term in (3.11), an assertion justified by noticing that an interchange of exponents in two factors in (3.19), leaving unchanged the other factors, does not increase the growth-order:

$$\begin{cases} \text{for the product } |f_i|^{n-i}\cdot|f_j|^{n-j}, \text{ replaced by } |f_i|^{n-j}\cdot|f_j|^{n-i}, 2\le i<j, \text{ we have:} \\ |f_i|^{n-j}\cdot|f_j^{n-i}| = |f_i|^{n-i}\cdot|f_j^{n-j}|\cdot|f_i|^{i-j}\cdot|f_j^{j-i}| = |f_i|^{n-i}\cdot|f_j^{n-j}|\cdot\left[|f_i|^{i-j}\cdot O\left(|f_i^{j-i}|\right)\right] \\ = \left(|f_i|^{n-i}\cdot|f_j^{n-j}|\right)\cdot O(1), (i<j); \end{cases} \quad (3.20)$$

$$\begin{cases} \text{for the product } |f_1|^{n}\cdot|f_j|^{n-j}, \text{ replaced by } |f_1|^{n-j}\cdot|f_j|^{n}, j>1, \text{ we have:} \\ |f_1|^{n-j}\cdot|f_j|^{n} = |f_1|^{n}\cdot|f_j^{n-j}|\cdot|f_1|^{-j}\cdot|f_j^{j}| = |f_1|^{n}\cdot|f_j|^{n-j}\cdot O(1), (j>1). \end{cases} \quad (3.21)$$

This argument is valid under condition (3.13a) and is independent of the possible zeros of the f_i's unlike the original device used in ([5]; formula (96), p. 13) as pointed out in ([6]; §2, Comments on Theorem 6, part (C), p. 39). The subsequent reasonings in the proof of Theorem 6 in [5] remain unchanged with the only obvious change of the expression of $V(f_1(x),\cdots,f_n(x))$ with that of $\tilde{V}(f_1(x),\cdots,f_n(x))$ in formulas (97)-(98) in ([5]; p. 14). □

The following is a complement to Theorem 10 in ([5]; pp. 19-20).

Theorem 3.4. (Wronskians of rapidly-varying functions). *Consider n functions* $\phi_i\in C^n[T,+\infty), i=1,\cdots,n$ *with both* $\phi_i(x)$ *and* $\phi_i'(x)\ne 0$ *for x large enough*, $1\le i\le n, n\ge 2$; *and let them satisfy the following relations*:

$$\phi_i^{(k)}(x)/\phi_i(x)\sim\left(\phi_i'(x)/\phi_i(x)\right)^k, x\to+\infty; 1\le i,k\le n, \quad (3.22)$$

which, in particular, are satisfied by functions which are rapidly varying at $+\infty$ *of order* $n-1$ *in the strong restricted sense of* ([5]; Def. 4.1, p. 807, and Prop. 4.1, p. 808).

(I) *If*

$$\phi_1'(x)/\phi_1(x)\gg\cdots\gg\phi_n'(x)/\phi_n(x), x\to+\infty, \quad (3.23)$$

then, as $x\to+\infty$:

$$\begin{cases} W(x):=W(\phi_1(x),\cdots,\phi_n(x))\sim(-1)^{-n(n-1)/2}\cdot\left(\prod_{i=1}^{n}\phi_i(x)\right)\cdot\left(\prod_{i=1}^{n-1}\left(\dfrac{\phi_i'(x)}{\phi_i(x)}\right)^{n-i}\right), \\ W'(x)/W(x)\sim\phi_1'(x)/\phi_1(x), \end{cases} \quad (3.24)$$

which means that W has the same type of asymptotic variation as ϕ_1 *according to the concept first formulated by Hardy* [8] *and, in particular, W has the same type of exponential variation as* ϕ_1 *according to* ([2]; Def. 8.1, p. 832).

(II) *If*

$$\phi_1'(x)/\phi_1(x)=c_i\phi(x)+o(\phi(x)), x\to+\infty, i=1,\cdots,n, \quad (3.25)$$

for some fixed function ϕ *and pairwise-distinct constants* c_i *then, as* $x\to+\infty$:

$$\begin{cases} W(x):=W(\phi_1(x),\cdots,\phi_n(x))\sim V(c_1,\cdots,c_n)\cdot\left(\prod_{i=1}^{n}\phi_i(x)\right)\cdot(\phi(x))^{n(n-1)/2}, \\ W'(x)/W(x)=(c_1+\cdots+c_n)\phi(x)+o(\phi(x)). \end{cases} \quad (3.26)$$

If, moreover, all the c_i*'s and their sum* $(c_1+\cdots+c_n)$ *are non-zero then*:

1) if " $\lim_{x\to+\infty}\phi(x)$ = *either 0 or* $\pm\infty$ ", W *has the same type of hypo-exponential or hyper-exponential variation as all the* ϕ_i*'s*;

2) if " $\lim_{x\to+\infty}\phi(x)\in\mathbb{R}\setminus\{0\}$ ", *relations in* (3.25) *may be written* (*possibly changing the constants*) *as*:

$$\phi_i'(x)/\phi_i(x)=c_i+o(1),\ x\to+\infty,\ i=1,\cdots,n,\ \text{i.e. } \phi_i\in\mathcal{E}_{c_i}(+\infty), \tag{3.27}$$

and $W\in\mathcal{E}_{c_1+\cdots+c_n}(+\infty)$. *If the* c_i*'s are pairwise-distinct non-zero numbers and* $c_1+\cdots+c_n=0$ *then* W *is hypoexponentially varying such as the Wronskian* " $W(e^{a_1x},\cdots,e^{a_nx})=V(a_1,\cdots,a_n)=\text{constant}$ ": ([5]; formula (68), p. 10, case $g(x)\equiv e^x$).

Proof. The asymptotic relations for W are given in ([5]; pp. 19-20); those for W' and $n\ge 3$ —the case $n=2$ being trivial—follow from Lemma 3.3 and the simple calculations in ([5]; proof of Th. 10, p. 21) in the two cases. In fact, under conditions in (3.23):

$$W'(x)\equiv\begin{vmatrix}\phi_i(x)\\ \phi_i'(x)\\ \vdots\\ \phi_i^{(n-2)}(x)\\ \phi_i^{(n)}(x)\end{vmatrix}_{i=1,\cdots,n}=\left(\prod_{i=1}^{n}\phi_i(x)\right)\cdot\begin{vmatrix}1\\ \phi_i'(x)/\phi_i(x)\\ \left(\phi_i'(x)/\phi_i(x)\right)^2[1+o(1)]\\ \vdots\\ \left(\phi_i'(x)/\phi_i(x)\right)^{n-2}[1+o(1)]\\ \left(\phi_i'(x)/\phi_i(x)\right)^{n}[1+o(1)]\end{vmatrix}_{i=1,\cdots,n} \tag{3.28}$$

$$\overset{(3.15)}{\sim}(-1)^{-n(n-1)/2}\cdot\left(\prod_{i=1}^{n}\phi_i(x)\right)\cdot\left(\prod_{i=1}^{n-1}\left(\frac{\phi_i'(x)}{\phi_i(x)}\right)^{n-i}\right)\cdot\frac{\phi_1'(x)}{\phi_1(x)},\ x\to+\infty;$$

and under conditions in (3.25):

$$W'(x)\sim V(c_1,\cdots,c_n)\cdot(c_1+\cdots+c_n)\cdot(\phi(x))^{\frac{n(n-1)}{2}+1},\ x\to+\infty. \tag{3.29}$$

□

4. Asymptotic Differential Equations versus Asymptotic Functional Equations

The theory developed in [1] [2] concerns functions differentiable a certain number of times and it starts from various types of asymptotic relations; for instance, the relation

$$xf'(x)/f(x)=\alpha+o(1),\ x\to+\infty, \tag{4.1a}$$

defines the basic concept of regular variation of index α in the strong sense, a concept (but not the locution) dating back to Hardy [8]. The above relation may be termed an "asymptotic differential equation". But, as mentioned in ([1]; p. 782), a larger class of functions with substantially the same fundamental properties was first introduced by Karamata, and then extensively studied by other mathematicians, starting from the "asymptotic functional equation"

$$f(\lambda x) \sim \lambda^{\alpha} f(x),\ x \to +\infty,\ \text{for each fixed } \lambda > 0,$$
$$\text{with } f \text{ measurable and positive.} \tag{4.1b}$$

It is known that for a function f with a monotonic derivative conditions (4.1a) and (4.1b) are equivalent and we show in this section that the same is true for the pair of equations (differential and functional) pertinent to each of the other types of variation: rapid, hypo-exponential, exponential or hyper-exponential. The first result of this type has been proved by Lamperti ([9]; pp. 382-383) for everywhere-differentiable functions using the Lagrange mean-value formula: $f(x) - f(y) = f'(c)(x-y)$. The integral version of the mean-value formula provides proofs for absolutely continuous functions when suitably reading the asymptotic relations for the derivative. For the sake of completeness, we make explicit some elementary remarks about condition "f' monotonic" in three possible interpretations.

- If "f' exists everywhere on an interval I and is monotonic thereon" then an elementary argument shows that f is absolutely continuous on each compact subinterval of I ([10]; Problem A, p. 13), and obviously is either concave or convex on *I*.
- *If* "f' exists on I save possibly a countable set N and is monotonic on $I \setminus N$" then the above conclusion about absolute continuity follows as a corollary of a non-trivial result in the Lebesgue theory ([11]; p. 299); and the concavity-convexity character follows from the classical characterization ([10]; Ch. 12, Th. A, pp. 9-10) or ([11]; Ch. V (18.43), p. 300), via an integral representation:

$$f(x) = f(c) + \int_c^x \phi(t)\,\mathrm{d}t;\ x, c \in \mathring{I},\ \text{with } \phi \text{ monotonic on } I, \tag{4.2}$$

and where the role of ϕ may be played indifferently by the left or the right derivative of f, both existing everywhere and coinciding except possibly on a countable set: ϕ non-decreasing for convexity and non-increasing for concavity.

- If "f' *exists on I save possibly a set* $\tilde{N}$ *of Lebesgue measure zero and is monotonic on* $I \setminus \tilde{N}$ *and if the absolute continuity of* f *is explicitly assumed*" then for any three numbers in I, $x < z < y$, and, say, f' non-decreasing we have the inequalities:

$$\frac{f(z) - f(x)}{z - x} = \frac{1}{z - x}\int_x^z f' \le \sup_{[x,z]\setminus\tilde{N}} f' \le \inf_{[x,y]\setminus\tilde{N}} f' \le \frac{1}{y - z}\int_z^y f' = \frac{f(y) - f(z)}{y - z}, \tag{4.3}$$

whence, by standard calculations, we get the inequality:

$$\frac{f(z) - f(x)}{z - x} \le \frac{f(y) - f(x)}{y - x},\quad x < z < y. \tag{4.4}$$

In fact:

$$\frac{f(z) - f(x)}{z - x} - \frac{f(y) - f(x)}{y - x} \le 0$$
$$\Leftrightarrow (y - x)\big(f(z) - f(x)\big) - (z - x)\big(f(y) - f(x)\big) \le 0$$
$$\text{writing } y - x = y - z + z - x$$
$$\Leftrightarrow (y - z)\big(f(z) - f(x)\big) + (z - x)\big(f(z) - f(x) - f(y) + f(x)\big) \le 0$$
$$\Leftrightarrow (y - z)\big(f(z) - f(x)\big) + (z - x)\big(f(z) - f(y)\big) \le 0,$$

which is true by (4.3). This shows that the difference quotient with a fixed end-point is non-decreasing, hence the left and right derivatives exists as finite numbers at each interior point of I and are non-decreasing. Analogously for f' non-increasing. These remarks show that:

In the setting of absolute continuity, condition "f' monotonic" even in its weakest meaning unambiguously refers to a function whose left and right derivatives exist as finite numbers at each interior point, coincide except possibly on a countable set and are monotonic with the appropriate type of monotonicity.

Moreover, an asymptotic relation involving f' such as, for instance (4.1a) may be read in any of the following three ways:

$$\begin{cases} xf'_+(x)/f(x) = \alpha + o(1),\ x \to +\infty,\ x \in [T,+\infty); \\ xf'_-(x)/f(x) = \alpha + o(1),\ x \to +\infty,\ x \in [T,+\infty); \\ xf'(x)/f(x) = \alpha + o(1),\ x \to +\infty,\ x \in [T,+\infty) \setminus \{\text{a suitable countable set}\}; \end{cases} \tag{4.5}$$

whereas, strictly speaking, the shortened notation in (4.1a) refers to the case of an everywhere-differentiable function. The following two theorems show that the differential and the functional approaches to the theory of asymptotic variation coincide for functions which are ultimately either concave or convex which is certainly the case of functions whose first derivatives are regularly varying of index $\neq 0$ or rapidly varying. In the proofs use is made of the integral representation in (4.2).

Theorem 4.1. (Regular and rapid variation). *For a positive function f either concave or convex on an interval $[T,+\infty)$ the following equivalences hold true*:

$$\begin{aligned} &\left\{ f(\lambda x) \sim \lambda^\alpha f(x),\ x \to +\infty,\ \text{for each fixed } \lambda > 0 \text{ and a constant } \alpha \in \mathbb{R} \right\} \\ &\Leftrightarrow xf'(x)/f(x) = \alpha + o(1),\ x \to +\infty; \end{aligned} \tag{4.6}$$

$$f(\lambda x) = \begin{cases} o(f(x)) & \text{if } 0 < \lambda < 1, \\ +\infty(f(x)) & \text{if } \lambda > 1, \end{cases} \Leftrightarrow xf'(x)/f(x) = +\infty(1),\ x \to +\infty; \tag{4.7}$$

$$f(\lambda x) = \begin{cases} +\infty(f(x)) & \text{if } 0 < \lambda < 1, \\ o(f(x)) & \text{if } \lambda > 1, \end{cases} \Leftrightarrow xf'(x)/f(x) = -\infty(1),\ x \to +\infty. \tag{4.8}$$

Theorem 4.2. (Types of exponential variation). *For a positive function f either concave or convex on an interval $[T,+\infty)$ the following equivalences hold true*:

$$\begin{aligned} &\left\{ f(x+\lambda) \sim f(x),\ x \to +\infty,\ \text{for each fixed } \lambda \in \mathbb{R} \right\} \\ &\Leftrightarrow f'(x) = o(f(x)),\ x \to +\infty; \end{aligned} \tag{4.9}$$

$$\begin{aligned} &\left\{ f(x+\lambda) \sim e^{c\lambda} f(x),\ x \to +\infty,\ \text{for each fixed } \lambda \in \mathbb{R} \text{ and } c \neq 0 \right\} \\ &\Leftrightarrow f'(x) = f(x) \cdot [c + o(1)],\ x \to +\infty; \end{aligned} \tag{4.10}$$

$$f(x+\lambda) = \begin{cases} o(f(x)) & \text{if } \lambda < 0, \\ +\infty(f(x)) & \text{if } \lambda > 0, \end{cases} \Leftrightarrow f'(x)/f(x) = +\infty(1),\ x \to +\infty; \tag{4.11}$$

$$f(x+\lambda)=\begin{cases}+\infty\left(f(x)\right) & \text{if } \lambda<0,\\ o\left(f(x)\right) & \text{if } \lambda>0,\end{cases} \Leftrightarrow f'(x)/f(x)=-\infty(1),\ x\to+\infty. \quad (4.12)$$

Proof of Theorem 4.1. All the inferences from the right to the left in both theorems elementarily follow from the integral representations and are to be found in ([1]; §5) and in ([2]; §8). Here we have to prove the converses. All the proofs are based on estimating the integral $\int_x^y f' \equiv f(y)-f(x)$ in terms of f' where f' indifferently stands for the right or left derivative of f which both exist everywhere. Proof of (4.6); for f' non-decreasing we have:

$$f'(x)\le\frac{1}{(\lambda-1)x}\int_x^{\lambda x} f' \equiv \frac{f(\lambda x)-f(x)}{(\lambda-1)x}\le f'(\lambda x),\ \forall\,\lambda>1, \quad (4.13)$$

whence:

$$\frac{xf'(x)}{f(x)}\le\frac{1}{\lambda-1}\left(\frac{f(\lambda x)}{f(x)}-1\right)\le x\frac{f'(\lambda x)}{f(x)}\equiv\lambda x\frac{f'(\lambda x)}{f(\lambda x)}\cdot\lambda^{-1}\frac{f(\lambda x)}{f(x)},\ \forall\,\lambda>1; \quad (4.14)$$

and letting $x\to+\infty$:

$$\overline{\lim}_{x\to+\infty}\frac{xf'(x)}{f(x)}\le\frac{\lambda^{\alpha}-1}{\lambda-1}\le\lambda^{\alpha-1}\underline{\lim}_{x\to+\infty}\frac{xf'(x)}{f(x)},\ \forall\,\lambda>1; \quad (4.15)$$

and we get our claim letting $\lambda\to1^+$. For f' non-increasing use the same argument reversing the inequalities in (4.13). For the proof of (4.7) we have:

$$\begin{aligned}
&f' \text{ non-decreasing} \Rightarrow f'(\lambda x)\ge\frac{f(\lambda x)-f(x)}{(\lambda-1)x},\ \forall\,\lambda>1,\\
&\Rightarrow\frac{\lambda xf'(\lambda x)}{f(\lambda x)}\ge\frac{\lambda}{\lambda-1}\left(1-\frac{f(x)}{f(\lambda x)}\right),\ \forall\,\lambda>1, \qquad (4.16)\\
&\overset{\text{as } x\to+\infty}{\Rightarrow}\ \underline{\lim}_{y\to+\infty}\frac{yf'(y)}{f(y)}\ge\frac{\lambda}{\lambda-1},\ \forall\,\lambda>1;
\end{aligned}$$

and then let $\lambda\to1^+$. If f' would be non-increasing (see however Remark 2 about growth-estimates after the proof of Theorem 4.2) then we might more simply write:

$$f'(x/2)\ge\frac{f(x)-f(x/2)}{x/2}\quad\text{whence}\quad\frac{(x/2)f'(x/2)}{f(x/2)}\ge\left(\frac{f(x)}{f(x/2)}-1\right)=+\infty(1). \quad (4.17)$$

The result in (4.8) is brought back to (4.7) putting $g(x):=1/f(x)$. □

Proof of Theorem 4.2. Notice that both the functional equations and the differential equations listed in Theorem 4.2 may be brought back to the corresponding ones in Theorem 4.1 by a change of variable; in fact, putting $g(x):=f\left(e^x\right)$, where $f(x)\ne0$ for x large enough, we have:

$$\begin{cases}
f(x)=g(\log x);\quad f(\lambda x)=g(\log x+\log\lambda),\ \lambda>0;\\
\lim\limits_{x\to+\infty}\left[f(\lambda x)/f(x)\right]=\lim\limits_{x\to+\infty}\left[g(\log x+\log\lambda)/g(\log x)\right]\\
\qquad=\lim\limits_{y\to+\infty}\left[g(y+\log\lambda)/g(y)\right]\equiv\phi(\lambda),\ \lambda>0;\\
\text{hence}: \lim\limits_{y\to+\infty}\left[g(y+\lambda)/g(y)\right]=\phi\left(e^{\lambda}\right),\ \lambda\in\mathbb{R};
\end{cases} \quad (4.18a)$$

and the following are easily checked:

$$\lim_{x\to+\infty}\frac{f(\lambda x)}{f(x)}=\begin{cases}1 & \text{for } \lambda>0,\\ \lambda^{\alpha} & \text{for } \lambda>0,\\ 0 & \text{for } 0<\lambda<1,\\ +\infty & \text{for } \lambda>1,\end{cases} \Leftrightarrow \lim_{x\to+\infty}\frac{g(x+\lambda)}{g(x)}=\begin{cases}1 & \text{for } \lambda\in\mathbb{R},\\ e^{\alpha\lambda} & \text{for } \lambda\in\mathbb{R},\\ 0 & \text{for } \lambda<0,\\ +\infty & \text{for } \lambda>0;\end{cases} \quad (4.18b)$$

$$\lim_{x\to+\infty}\frac{xf'(x)}{f(x)}=\ell\in\overline{\mathbb{R}} \Leftrightarrow \lim_{x\to+\infty}\frac{g'(x)}{g(x)}=\ell. \quad (4.18c)$$

But a correspondence between the monotonicity of f' and g' is not granted; hence it is better to give direct proofs of Theorem 4.2 following the same patterns as above. We write them down only for "f' non-decreasing". For the proof of (4.9) notice that the claim does not depend on the sign of f; hence we may suppose "$f'\ge 0$" changing, if necessary, the sign of f. For "f' non-decreasing" we have:

$$0\le f'(x)\le f(x+1)-f(x)=f(x)+o(f(x))-f(x)=o(f(x)). \quad (4.19)$$

For the proof of (4.10) and "f' non-decreasing" we have:

$$f'(x)\le\frac{f(x+\lambda)-f(x)}{\lambda}\le f'(x+\lambda),\ \lambda>0; \quad (4.20)$$

whence:

$$\frac{f'(x)}{f(x)}\le\frac{1}{\lambda}\left(\frac{f(x+\lambda)}{f(x)}-1\right)\le\frac{f'(x+\lambda)}{f(x)}\equiv\frac{f'(x+\lambda)}{f(x+\lambda)}\frac{f(x+\lambda)}{f(x)},\ \lambda>0; \quad (4.21)$$

and as $x\to+\infty$:

$$\begin{aligned}\overline{\lim_{x\to+\infty}}\, f'(x)/f(x)&\le\frac{1}{\lambda}\left(e^{c\lambda}-1\right)\le e^{c\lambda}\underline{\lim_{x\to+\infty}}\, f'(x+\lambda)/f(x+\lambda)\\ &=e^{c\lambda}\underline{\lim_{x\to+\infty}}\, f'(x)/f(x),\ \lambda>0;\end{aligned} \quad (4.22)$$

and for $\lambda\to0^+$:

$$\overline{\lim_{x\to+\infty}}\, f'(x)/f(x)\le c\le\underline{\lim_{x\to+\infty}}\, f'(x)/f(x). \quad (4.23)$$

And for the proof of (4.11) we write:

$$f'(x+\lambda)\ge\frac{f(x+\lambda)-f(x)}{\lambda},\ \text{hence}\ \frac{f'(x+\lambda)}{f(x+\lambda)}\ge\frac{1}{\lambda}\left(1-\frac{f(x)}{f(x+\lambda)}\right),\ \forall\lambda>0, \quad (4.24)$$

and the assertion follows by taking first the "lim inf" as $x\to+\infty$ and then the limit as $\lambda\to0^+$. □

Remarks about growth-estimates. 1) One of the basic properties of the functions in the studied classes is the estimates, though rough, of their growth-orders. For the subclasses defined via asymptotic differential equations the estimates are elementarily inferred from the pertinent integral representations whereas for the general classes it is true that they are inferred from suitable integral representations as well, but such representations are not elementary facts but consequences of the nontrivial core of the theory, namely the "uniform convergence theorems".

For slow (hence for regular) variation the estimates are explicitly mentioned in ([4]; Prop.1.3.6, p.16):

$$\text{“} f \text{ satisfying the functional equation in (4.1b)”} \Rightarrow x^{\alpha-\epsilon} \ll f(x) \ll x^{\alpha+\epsilon},\ x \to +\infty,\ \forall\, \epsilon > 0. \tag{4.25}$$

For rapid variation a “representation theorem” ([4]; Prop. 2.4.4, p. 85, and Th. 2.4.5, p. 86), can be obtained only for a certain subclass; in particular, if f is monotonic on an interval $[T,+\infty)$ the two functional equations in (4.7) imply that f is non-decreasing and admits of the integral representation:

$$f(x) = \exp\left\{z(x) + \eta(x) + \int_T^x t^{-1}\xi(t)\,\mathrm{d}t\right\},\ x \ge T, \tag{4.26}$$

where the measurable functions z,η,ξ are such that: z is non-decreasing, $\eta(x) \to 0$ and $\xi(x) \to +\infty$ as $x \to +\infty$ (This is the case of Theorem 4.1 wherein the monotonicity of f' implies the ultimate monotonicity of f). From (4.26) the following estimate is trivially inferred:

$$x^{-\alpha} f(x) \ge A\exp\left(\int_T^x t^{-1}\left[\xi(t) - \alpha\right]\mathrm{d}t\right),\ (A = \text{constant} > 0), \tag{4.27a}$$

whence

$$f(x) \gg x^{\alpha},\ x \to +\infty,\ \forall\, \alpha > 0. \tag{4.27b}$$

By putting $g(x) := f(e^x)$ one gets the corresponding growth-estimates for exponential variations listed in ([2]; §8).

2) In the case of (4.7) the contingency “f' non-increasing” is excluded otherwise we would have:

$$\lim_{x\to+\infty} f(x)/x = \lim_{x\to+\infty} f'(x) = \begin{cases} a \in \mathbb{R} \\ -\infty \end{cases}, \tag{4.28}$$

contradicting either the growth-estimate in (4.27b) or the positivity of f.

3) In passing we point out that the above-mentioned “uniform convergence theorems” imply that a function satisfying one of the asymptotic functional equation mentioned in Theorems 4.1-4.2 satisfies a quite stronger functional equation. A list appears in ([1]; §5) inferred from the simple integral representations valid when the corresponding asymptotic differential equations hold true: the pertinent calculations are elementary, though not trivial, implicitly using the uniform convergence with respect to the parameter. In the more general context we have:

$$\begin{cases} f(\lambda x) \sim \lambda^{\alpha} f(x), \ \text{for each } \lambda > 0, \\ f \text{ measurable, without any monotonicity restriction,} \end{cases} \tag{4.29}$$
$$\Rightarrow f(\lambda x + o(x)) \sim \lambda^{\alpha} f(x),\ x \to +\infty,\ \alpha \in \mathbb{R};$$

a result inferred from the factorization $f(x) = x^{\alpha}\ell(x)$, with ℓ slowly varying, and from ([4]; Th. 1.2.1, p. 6). And we also have:

$$\begin{cases} f(\lambda x) = \begin{cases} o(f(x)) & \text{if } 0 < \lambda < 1, \\ +\infty(f(x)) & \text{if } \lambda > 1, \end{cases} \\ f \text{ measurable, without any monotonicity restriction,} \end{cases}$$

$$\Rightarrow f\left(\lambda x+o(x)\right)=\begin{cases} o\left(f(x)\right) & \text{if } 0<\lambda<1, \\ +\infty\left(f(x)\right) & \text{if } \lambda>1, \end{cases}, \ x\to+\infty; \tag{4.30}$$

inferred from ([4]; Cor. 2.4.2, p.85).

The reader may write down the corresponding versions of Theorems 4.1-4.2 for the functional equations listed in ([1]; Prop. 5.2, pp. 814-815) under the monotonicity of f'.

5. Concepts Related to Logarithmic Variation

The concept of regular variation of order $\alpha \neq 0$ generalizes the asymptotic behavior of a power whereas the three concepts of exponential variation generalize the asymptotic behaviors of the exponential of a power. These generalizations are quite natural whereas the concept of slow variation is not the appropriate generalization of the behavior of the logarithm: it shares some asymptotic properties of the logarithm but it encompasses functions with orders of growth either greater than the order of each positive power of the logarithm or less than the order of each negative power of the logarithm such as the functions: $\exp\left[\pm(\log x)^{\delta}\right], 0<\delta<1$. Looking at the asymptotic functional equation of a slowly-varying function ℓ,

$$\ell(\lambda x)\sim \ell(x), \ x\to+\infty, \tag{5.1a}$$

we see no link with the parameter λ, whereas the arithmetic functional equation characterizing the logarithm,

$$\ell(\lambda x)=\ell(x)+\ell(\lambda) \ \ \forall\, \lambda, x>0, \tag{5.1b}$$

if interpreted asympotically, gives an expression for the remainder. One of the possible asymptotic counterparts of (5.1b), as $x\to+\infty$, is the asymptotic functional equation:

$$f(\lambda x)=f(x)+\phi(\lambda)+o(1), \ x\to+\infty, \ \forall \text{ fixed } \lambda>0. \tag{5.2}$$

It is known, ([4]; Lemma 3.2.1-case $g(x)\equiv 1$, p. 140) that if (5.2) is satisfied by a measurable function f defined on a neighborhood of $+\infty$ then "$\phi(\lambda)=c\log\lambda \ \forall\lambda>0$" for some real constant *c*. Together with (5.2) we shall describe two other classes of functions highlighting a possible concept of sublogarithmic variation which fits, e.g., to iterated logarithms.

Definition 5.1. *Let* f *be a measurable function defined on a neighborhood of* $+\infty$.

(I) *We say that* f *is quasi-logarithmically varying at* $+\infty$ *if:*

$$f(\lambda x)=f(x)+O(1), \ x\to+\infty, \ \forall \text{ fixed } \lambda>0. \tag{5.3}$$

(II) *We say that* f *is logarithmically varying at* $+\infty$ *if:*

$$f(\lambda x)=f(x)+c\log\lambda+o(1), \ x\to+\infty, \ \forall \text{ fixed } \lambda>0 \text{ and some constant } c\in\mathbb{R}\setminus\{0\}. \tag{5.4}$$

(III) *We say that* f *is hypo* [$\equiv$ *sub*] *-logarithmically varying at* $+\infty$ *if:*

$$f(\lambda x)=f(x)+o(1), \ x\to+\infty, \ \forall \text{ fixed } \lambda>0. \tag{5.5}$$

Equations (5.3) and (5.4) have trivial solutions, namely all bounded functions and all functions convergent as $x \to +\infty$ respectively satisfy (5.3) and (5.4); and obviously the behaviors of such functions have nothing to partake of the intuitive meanings associated with "logarithmic variation". For this reason someone might prefer to use the locutions of "logarithmically varying" and "hypo-logarithmically varying" exclusively for functions which, besides satisfying the appropriate above-specified equation, enjoy additional properties such as: strict positivity, or monotonicity, or divergence to $+\infty$ as $x \to +\infty$, or the last two properties. But this is a matter of agreement. Now we collect the essential properties of the three classes using the shortened locution "f *differentiable*" to mean that:

either "f *is everywhere differentiable on some interval* $[T, +\infty)$"

or "f *is absolutely continuous on* $[T, +\infty)$",

with the agreement that an asymptotic relation involving f', say "$f'(x) \sim \phi(x),\ x \to +\infty$" for an absolutely continuous f is to be meant as "$x \to +\infty,\ x \in [T, +\infty) \setminus N$" where N is a suitable set of measure zero.

Three admissible meanings of the locution "f' *monotonic*" have been highlighted at the outset of the preceding section and the proofs involving this property may be done using the integral mean-value theorem which applies to each of the three cases.

Theorem 5.1. (I) *If* f *is a measurable function defined on a neighborhood of* $+\infty$ *satisfying Equation* (5.3) *then the following two properties hold true*:

1) *The asympyotic relation* (5.3) *holds true uniformly with respect to the parameter* λ *on each compact subset of* $]0, +\infty)$ *and this implies that* f *satisfies the more general asymptotic functional equation*:

$$f(xI(x)) = f(x) + O(1),\ x \to +\infty,\ \forall \text{ function } I \text{ such that } 0 < c_1 \le I(x) \le c_2 < +\infty; \quad (5.6)$$

and in particular:

$$\begin{cases} f(\lambda x + o(x)) = f(x) + O(1),\ x \to +\infty, \\ \text{uniformly for } \lambda \text{ varying on a compact subset of }]0, +\infty). \end{cases} \quad (5.7)$$

2) f *satisfies the estimate*:

$$f(x) = O(\log x),\ x \to +\infty. \quad (5.8)$$

(II) *If* f *is a differentiable function then condition* $f'(x) = O(x^{-1}),\ x \to +\infty$ *implies that* f *satisfies Equation* (5.3).

(III) *If* f' *is monotonic then* (5.3) *is satisfied iff*

$$f'(x) = O(x^{-1}),\ x \to +\infty. \quad (5.9)$$

Theorem 5.2. (I) *If* f *is a measurable function defined on a neighborhood of* $+\infty$ *satisfying Equation* (5.4) *then it obviously satisfies* (5.3), *hence* (5.7); *moreover*:

$$\begin{cases} f(\lambda x + o(x)) = f(x) + c \log \lambda + o(1),\ x \to +\infty, \\ \text{uniformly for } \lambda \text{ varying on a compact subset of }]0, +\infty); \end{cases} \quad (5.10)$$

and the growth-order of f *is*:

$$f(x) \sim c\log x,\ x \to +\infty. \tag{5.11}$$

(II) *If* f *is a differentiable function then condition* $f'(x) \sim cx^{-1},\ x \to +\infty$ *implies that* f *satisfies Equation* (5.4).

(III) *If* f' *is monotonic then* (5.4) *is satisfied if and only if*

$$f'(x) \sim cx^{-1},\ x \to +\infty. \tag{5.12}$$

Theorem 5.3. *If* f *is a measurable function defined on a neighborhood of* $+\infty$ *satisfying Equation* (5.5) *then a word-for-word restatement of Theorem* 5.1 *holds true with the symbol* "O" *replaced everywhere by* "o".

Proof of the theorems. The fundamental results about the uniformity of equations (5.3)-(5.4)-(5.5) with respect to λ may be found in the monograph ([4]; Corollaries 3.1.8a and 3.1.8c, case $g \equiv 1$, p. 133) for equations (5.3) and (5.5); whereas (5.4) is a special case of (5.3). From such results the following useful representations are derived ([4]; Theorem 3.6.1, p. 152):

$$f(x) = C + T_0 + \eta(x) + \int_{T_0}^{x} \xi(t)t^{-1}\mathrm{d}t,\ x \ge T_0, \tag{5.13}$$

where C, T_0 are suitable constants and the measurable functions η, ξ are such that:

$$\eta(x), \xi(x) = \begin{cases} O(1) & \text{if } f \text{ satisfies } (5.3), \\ o(1) & \text{if } f \text{ satisfies } (5.5); \end{cases} \tag{5.14}$$

$$\begin{cases} \eta(x) = O(1), \\ \xi(x) = c + o(1), \end{cases} \quad \text{if } f \text{ satisfies } (5.4). \tag{5.15}$$

The estimates in (5.14) are explicitly stated in ([4]; Th. 3.6.1, p. 152) whereas those in (5.15) are checked at once by inspecting a formula in ([4]; formula (3.6.2), p. 152). From (5.13) the relations in (5.8) and (5.11) and the corresponding one in Theorem 5.3 are easily inferred. So the statements in part (I) of each theorem are proved, and the statements in parts (II) trivially follow from the integral representation:

$$f(\lambda x) - f(x) = \int_x^{\lambda x} f'(t)\mathrm{d}t. \tag{5.16}$$

For parts (III) in the theorems we must prove the "only if" inferences noticing that, changing f in - f if necessary, we may always suppose $f' \ge 0$. Relation (5.9) follows from:

$$\begin{cases} f' \text{ non-decreasing} \Rightarrow \left[f(2x) - f(x)\right] = \int_x^{2x} f' \ge xf'(x); \\ f' \text{ non-increasing} \Rightarrow \left[f(x/2) - f(x)\right] = \int_{x/2}^{x} f' \ge \dfrac{x}{2} f'(x). \end{cases} \tag{5.17}$$

The very same calculations also give the estimate with "o" for Theorem 5.3. Relation in (5.12) requires less immediate calculations. For f' non-decreasing and each $\lambda > 1$ we have:

$$f'(x) \le \frac{1}{(\lambda - 1)x} \int_x^{\lambda x} f' \equiv \frac{f(\lambda x) - f(x)}{(\lambda - 1)x} \le f'(\lambda x), \tag{5.18}$$

whence:

$$xf'(x) \le \frac{c\log\lambda + o(1)}{\lambda - 1} \le \lambda^{-1}(\lambda x) f'(\lambda x); \tag{5.19}$$

and letting $x \to +\infty$:

$$\overline{\lim}_{x\to+\infty} xf'(x) \le \frac{c\log\lambda}{\lambda-1} \le \underline{\lim}_{x\to+\infty}\lambda^{-1}(\lambda x) f'(\lambda x) = \lambda^{-1}\underline{\lim}_{x\to+\infty} xf'(x),\ \forall\ \lambda > 1. \tag{5.20}$$

As $\lambda \to 1^+$ we get relation (5.12). For f' non-increasing and $\lambda > 1$ the inequalities in (5.19) are reversed and an analogous reasoning may be done. We also get the estimate with "o" for Theorem 5.3. □

And now what can be said about the concept of higher-order logarithmic variation? Higher-order regular variation is defined by imposing on each derivative of order not less than 2 an asymptotic behavior consistent with that of the first derivative according to a preliminary result: ([1]; Prop. 2.6, p. 796, and Def. 3.1, p. 798); the foregoing Theorems 5.1-5.3 justify the following:

Definition 5.2. *If* f *is n-times differentiable on an interval* $[T,+\infty)$, $n \ge 1$, *in the sense that* $f^{(n-1)}$ *is either absolutely continuous or everywhere differentiable, then one may use the following locutions*:

$$\begin{cases} f \text{ is quasi-logarithmically varying at } +\infty \text{ of order } n \\ \Leftrightarrow f^{(k)}(x) = O(x^{-k}),\ x\to+\infty, 1\le k\le n; \\ f \text{ is logarithmically varying at } +\infty \text{ of order } n \\ \Leftrightarrow f^{(k)}(x) \sim c\cdot D^k(\log x),\ x\to+\infty; c\ne 0, 0\le k\le n; \left(D^0h(x) := h(x)\right); \\ f \text{ is hypo-logarithmically varying at } +\infty \text{ of order } n \\ \Leftrightarrow f^{(k)}(x) = o(x^{-k}),\ x\to+\infty, 1\le k\le n. \end{cases} \tag{5.21}$$

Remarks. 1) Some people would like to add to the above definition a condition such as "strict positivity or monotonicity or divergence" for the sole function f (and not for its derivatives!) to adhere more consistently to the intuitive notion of logarithmic variation; but this is a matter of agreement as noticed above.

2) The asymptotic estimate (5.9) obviously implies (5.8) but the converse fails regardless of any monotonicity restriction; in fact for, say, a C^1 function we have:

$$\begin{cases} f(x) \equiv \log x\cdot g(x) \\ g(x) = O(1) \end{cases}$$
$$\Rightarrow f'(x) \equiv x^{-1}g(x) + \log x\cdot g'(x) = O(x^{-1}) \tag{5.22a}$$
$$\Leftrightarrow g'(x) = O\left((x\log x)^{-1}\right).$$

And in the case "$g(x) = c + o(1), c\in\mathbb{R}$, the characteristic condition for f' to satisfy the corresponding asymptotic relation "$f'(x) = cx^{-1} + o(x^{-1})$" is "$g'(x) = o\left((x\log x)^{-1}\right)$". These remarks show that for the validity of the inference "(5.8) $\Rightarrow$ (5.9)" the right additional condition (beside differentiability) is the following restriction on the order of growth of the derivative:

$$\left(f(x)/\log x\right)' = O\left((x\log x)^{-1}\right). \tag{5.22b}$$

From (5.22a) and Theorem 5.1 it follows that the solutions of (5.3) which are ultimately concave or convex satisfy the three estimates: of

$$f(x) = O(\log x);\ f'(x) = O\left(x^{-1}\right);\left(f(x)/\log x\right)' = O\left((x\log x)^{-1}\right),\ x \to +\infty. \tag{5.23}$$

Some examples. 1) *The standard ones*. The following functions are hypo-logarithmically varying at $+\infty$ of a non-trivial type:

$$\begin{cases} (\log x)^{\delta},\ 0<\delta<1; \quad \prod_{2\le k\le n}\left(\ell_k(x)\right)^{\delta_k},\ \delta_k \in \mathbb{R}; \\ (\log x)^{\delta} \cdot \prod_{2\le k\le n}\left(\ell_k(x)\right)^{\delta_k},\ 0<\delta<1,\ \delta_k \in \mathbb{R} \end{cases} \tag{5.24}$$

The first of the foregoing function shows the obvious fact that no one of the three classes in Definition 5.1 is closed under multiplication.

The following examples concern various types of compositions.

2) *Logarithm of a regularly-varying function*:

$$\begin{cases} f \in \mathcal{R}_{\alpha}(+\infty),\ \alpha \in \mathbb{R} \Rightarrow \log \circ f \text{ logarithmically varying at } +\infty, \text{ as} \\ \log\left(f(\lambda x)\right) = \log\left(\lambda^{\alpha} f(x)\left[1+o(1)\right]\right) = \log\left(f(x)\right) + \alpha\log\lambda + o(1),\ x \to +\infty. \end{cases} \tag{5.25}$$

3) *A regularly-varying function of the logarithm*. The general result is:

$$\begin{cases} g \text{ quasi-logarithmically varying at } +\infty, \\ g(+\infty) = +\infty, \\ f \text{ asymptotically sublinear at } +\infty, \end{cases} \tag{5.26}$$
$$\Rightarrow f \circ g \text{ hypo-logarithmically varying at } +\infty.$$

In fact the assumptions mean that g satisfies "$g(\lambda x) = g(x) + O(1)$" and f satisfies "$f\left(x+O(1)\right) = f(x) + o(1)$" ([1]; Prop. 5.2-(I), p. 814), hence:

$$f\left(g(\lambda x)\right) = f\left(g(x) + O(1)\right) = f\left(g(x)\right) + o(1),\ x \to +\infty. \tag{5.27}$$

The most meaningful contingency for f in (5.27) is:

$$\begin{cases} f \text{ differentiable with } f'(x) = o(1),\ x \to +\infty; \\ \text{in particular}: f \in \mathcal{R}_{\delta}(+\infty),\ \delta < 1, \text{ so generalizing the first example in } (5.24). \end{cases} \tag{5.28}$$

The case "$f \in \mathcal{R}_1(+\infty)$" needs restrictions:

$$\begin{cases} g \text{ hypo-logarithmically varying at } +\infty, \\ g(+\infty) = +\infty, \\ f \text{ asymptotically uniformly continuous at } +\infty, \end{cases} \tag{5.29}$$
$$\Rightarrow f \circ g \text{ hypo-logarithmically varying at } +\infty;$$

whose proof is as above, noticing that now "$g(\lambda x) = g(x) + o(1)$" and f satisfies, by definition, "$f\left(x+o(1)\right) = f(x) + o(1)$" ([1]; Prop. 5.2-(I), p. 814); and a sufficient condition for such an f is "$f'(x) = O(1)$".

4) *A slowly-varying function of the logarithm*. Here is a result useful in inverting an hyper-exponential function:

Proposition 5.4. (I) *If* $\ell \in \{\mathcal{R}_0(+\infty) \text{ of order } n\}$ *and* $\ell^{(k)} \in \mathcal{R}_{-k}(+\infty), 1 \le k \le n-1$, *then the function* $\mathcal{L} := \ell \circ \log$ *has the following properties*:

$$\begin{cases} \mathcal{L} \in \{\mathcal{R}_0(+\infty) \text{ of order } n\}; \ \mathcal{L}^{(k)} \in \mathcal{R}_{-k}(+\infty), 1 \le k \le n-1; \\ \mathcal{L}^{(k)}(x) \sim (-1)^{\underline{k-1}} x^{-k} \cdot \ell'(\log x) \sim (-1)^{\underline{k-1}} x^{1-k} \cdot \mathcal{L}'(x), \ x \to +\infty, 1 \le k \le n; \\ \mathcal{L}^{(k)}(x) = o\left(x^{-k} (\log x)^{-1} \cdot \ell(\log x)\right) = o\left(x^{-k} (\log x)^{-1+\epsilon}\right) \quad \forall \epsilon > 0 \\ \qquad = o\left(x^{-k}\right), \ x \to +\infty, 1 \le k \le n; \end{cases} \tag{5.30}$$

hence $\mathcal{L}$ *is hypo-logarithmically varying at* $+\infty$ *of order n according to our Definition* 5.2.

(II) *If* $\ell \in \{\mathcal{SR}_0(+\infty) \text{ of order } n\}$, *then*:

$$\mathcal{L}^{(k)}(x) = o\left(x^{-k} (\log x)^{-1} \cdot \mathcal{L}(x)\right) = o\left(x^{-k}\right), \ x \to +\infty, 1 \le k \le n; \tag{5.31}$$

hence $\mathcal{L} \in \{\mathcal{SR}_0(+\infty) \text{ of order } n\}$ *and it is hypo-logarithmically varying at* $+\infty$ *of order n*.

Proof. Part (I). The types of asymptotic variation of $\mathcal{L}$ and $\mathcal{L}^{(k)}$ follow from ([2]; Prop. 7.5-(III), p. 825) and, anyway, they follow from the detailed calculations below to prove the remaining relations in (5.30). The assumptions on ℓ stand for the set of relations:

$$\ell'(x) = o\left(x^{-1}\ell(x)\right); \ \ell^{(k+1)}(x) \sim -kx^{-1} \cdot \ell^{(k)}(x), \ x \to +\infty, 1 \le k \le n-1; \tag{5.32}$$

which imply:

$$\ell(x) \ll x^{\varepsilon}, \ x \to +\infty, \forall \epsilon > 0; \quad \ell^{(k+1)}(x) \sim (-1)^{\underline{k}} x^{-k} \cdot \ell'(x), \ x \to +\infty; 1 \le k \le n-1; \tag{5.33}$$

and from these we get:

$$\begin{cases} \mathcal{L}(x) \equiv \ell(\log x); \ \mathcal{L}'(x) = x^{-1} \cdot \ell'(\log x) = o\left(x^{-1} (\log x)^{-1} \cdot \ell(\log x)\right) = o\left(x^{-1}\right); \\ \mathcal{L}''(x) = -x^{-2} \cdot \ell'(\log x) + x^{-2} \cdot \ell''(\log x) \sim -x^{-2} \cdot \ell'(\log x) \ \text{ as } \ell''(\log x) \ll \ell'(\log x). \end{cases} \tag{5.34}$$

For any higher derivative we use Faà Di Bruno's formula ([2]; formulas (6.1)-(6.2), p.818):

$$\mathcal{L}^{(k)}(x) = \sum_{0 \le i_j \le k}^{i_1 + 2i_2 + \cdots + ki_k = k} a_{i_1, \cdots, i_k} \cdot \ell^{(i_1 + \cdots + i_k)}(\log x) \cdot (D \log x)^{i_1} \cdot \left(D^2 \log x\right)^{i_2} \cdots \left(D^k \log x\right)^{i_k}, \tag{5.35}$$

where the summation is taken over all possible ordered k-tuples of non-negative integers i_j such that

$$i_1 + 2i_2 + \cdots + ki_k = k \ \left(\text{hence } 1 \le i_1 + i_2 + \cdots + i_k \le k\right); \tag{5.36}$$

and $a_{i_1, \cdots, i_k}$ are suitable coefficients with $a_{0, \cdots, 0, 1} = 1$. Now we have by (5.33):

$$\begin{cases} \ell^{(i_1 + \cdots + i_k)}(\log x) \sim (-1)^{\underline{m_k}} \cdot (\log x)^{1 - i_1 - i_2 - \cdots - i_k} \cdot \ell'(\log x), \ m_k := i_1 + \cdots + i_k - 1; \\ (D \log x)^{i_1} \cdot \left(D^2 \log x\right)^{i_2} \cdots \left(D^k \log x\right)^{i_k} \sim x^{-i_1} \cdot \left((-1)^{\underline{1}} x^{-2}\right)^{i_2} \cdots \left((-1)^{\underline{k-1}} x^{-k}\right)^{i_k} \equiv c_{i_1, \cdots, i_k} x^{-k}, \end{cases} \tag{5.37}$$

where $c_{i_1,\cdots,i_k}$ are suitable non-zero constants. Hence the term with the greatest growth-order in the sum is obtained when "$i_1+\cdots+i_k=1$", *i.e.* when "$(i_1,\cdots,i_k)=(0,\cdots,0,1)$, so that "$\ell^{(i_1+\cdots+i_k)}(\log x)\sim\ell'(\log x)$", and we get the principal part of $\mathcal{L}^{(k)}$. The last estimates in (5.30) follow from the estimates for $\mathcal{L}'$ in (5.34) and from "$\ell(\log x)\ll(\log x)^{\epsilon}$". For part (II) the claim about the smooth variation of $\mathcal{L}$ follows from ([2]; Prop. 7.1, formula (7.3), p. 820) whereas the estimates in (5.31) follow from (5.35) where each term in the sum is

$$o\left((\log x)^{-i_1-i_2-\cdots-i_k}\cdot\ell(\log x)\cdot x^{-k}\right)=o\left(x^{-k}\cdot(\log x)^{-1}\cdot\mathcal{L}(x)\right). \qquad \square$$

In conclusion, the discussion in this section gives a theoretical description of the possible concepts related to a "logarithmic variation" at $+\infty$ but these seem to have a limited import for practical applications, though linked with the general theory of asymptotic variation. An application is presented in the next section.

6. Inverse of a Function with a Type of Exponential Variation

This is termed "Open Problem 4" in ([2]; p. 866); a complete treatment requires some calculations and it is appropriately understood in the context of logarithmic variation. Let f have a definite type of exponential variation namely, either:

$$f\in\mathcal{E}_c(+\infty) \text{ with } -\infty\le c\le+\infty,\ c\ne 0; \tag{6.1}$$

or

$$f\in\mathcal{E}_0(+\infty)\cap\mathcal{R}_{+\infty}(+\infty). \tag{6.2}$$

The restriction in (6.2) is unavoidable as the class $\mathcal{E}_0(+\infty)$ contains functions with different types of asymptotic variation whose inverses (if they exist) may have quite different behaviors and no definite general result may be given. Assumptions in (6.1) or (6.2) imply that f is ultimately positive and f' has ultimately one strict sign, hence f has an inverse f^{-1} on some neighborhood of $+\infty$; f^{-1} is defined on a right neighborhood of zero if $c<0$, or on a neighborhood of $+\infty$ if $c\ge 0$. By the change of variable $x=f^{-1}(y)$, as in ([1]; formula (2.38), p. 786), we get:

$$\lim_{\substack{y\to 0^+\\ [y\to+\infty]}}\frac{\log y}{f^{-1}(y)}\overset{H}{=}\lim_{\substack{y\to 0^+\\ [y\to+\infty]}}\frac{f'\left(f^{-1}(y)\right)}{y}=\lim_{x\to+\infty}\frac{f'(x)}{f(x)}=c, \tag{6.3}$$

whence:

$$f^{-1}(y)\begin{cases}\sim c^{-1}\log y, \begin{cases} y\to+\infty & \text{if } 0<c<+\infty,\\ y\to 0^+ & \text{if } -\infty<c<0;\end{cases}\\ =o(\log y), \begin{cases} y\to+\infty & \text{if } c=+\infty,\\ y\to 0^+ & \text{if } c=-\infty;\end{cases}\\ =+\infty(\log y),\ y\to+\infty,\ \text{if } c=0;\end{cases} \tag{6.4}$$

and analogously we get that in each case:

$$\lim_{\substack{y\to 0^+\\ [y\to+\infty]}}\frac{y\cdot D\left(f^{-1}(y)\right)}{f^{-1}(y)}=\lim_{x\to+\infty}\frac{f(x)}{xf'(x)}=0; \tag{6.5}$$

hence f^{-1} is slowly varying at 0^+ or at $+\infty$. Moreover, the limit in (6.3) also yields:

$$f'\left(f^{-1}(y)\right)\begin{cases}=o(y), & \text{if } c=0,\\ \sim cy, & \text{if } c\in\mathbb{R}\setminus\{0\},\\ =\pm\infty(y) & \text{if } c=\pm\infty.\end{cases} \tag{6.6}$$

1) For $c=0$ the sole assumption "$f\in\{\mathcal{R}_{+\infty}(+\infty)\text{ of order } n\}, n\geq 2$" implies, by a result in ([2]; Prop. 7.7-(II), p. 830) that:

$$f^{-1}\in\{\mathcal{R}_0(+\infty)\ \text{ of order } n\};\quad D^k f^{-1}\in\mathcal{R}_{-k}(+\infty), 1\leq k\leq n-1, \tag{6.7}$$

which stand for the set of relations in (5.32) with ℓ' replaced by f^{-1}. These relations imply:

$$\left(D^k f^{-1}\right)(y)=o\left(y^{-1}\cdot\left(D^{k-1}f^{-1}\right)(y)\right),\ y\to+\infty, 1\leq k\leq n; \tag{6.8a}$$

which, together with the estimate "$f^{-1}(y)\ll y^{\epsilon}, \forall\,\epsilon>0$", yield:

$$\left(D^k f^{-1}\right)(y)\ll y^{-k+\epsilon},\ y\to+\infty,\ \forall\,\epsilon>0; 1\leq k\leq n; \tag{6.8b}$$

and better estimates do not obtain in general as shown by the inverse of the function $\exp\left(x^{\alpha}\right), \alpha>1$.

2) If "$c\in\mathbb{R}\setminus\{0\}$" and "$f\in\{\mathcal{E}_c(+\infty)\text{ of order } n\}$" we get from ([2]; formula (8.69), p. 840):

$$f^{(i)}\left(f^{-1}(y)\right)\sim c^i f\left(f^{-1}(y)\right)\equiv c^i y, i\geq 1; \tag{6.9a}$$

and from the formula for the higher derivatives of an inverse function ([2]; formulas (6.4)-(6.5), p. 819), we get the estimates:

$$\begin{aligned} D^k\left(f^{-1}(y)\right)&=(cy)^{1-2k}\left[1+o(1)\right]\cdot\sum_{i_1,\cdots,i_k}^{\cdots}\left\{a_{i_1,\cdots,i_k}\cdot c^{2k-2}y^{k-1}\left[1+o(1)\right]\right\}\\ &=c^{-1}y^{-k}\cdot\left(\sum_{i_1,\cdots,i_k}^{\cdots}a_{i_1,\cdots,i_k}\right)\cdot\left[1+o(1)\right], 1\leq k\leq n,\end{aligned} \tag{6.9b}$$

provided that the last sum is not zero. In fact, choosing $f(x)=e^x$ in ([2]; formula (6.4), p. 819) it is checked at once that:

$$y^{-k}\sum_{i_1,\cdots,i_k}^{\cdots}a_{i_1,\cdots,i_k}=D^k\log y, k\geq 1; \tag{6.9c}$$

hence the sum equals the numerical non-zero coefficient appearing in the explicit expression of $D^k\log y$. In conclusion, we have the relations:

$$f^{-1}(y)\sim c^{-1}\cdot\log y;\ \left(D^k f^{-1}\right)(y)\sim c^{-1}\cdot D^k\log y,\ y\to+\infty; 1\leq k\leq n; \tag{6.10}$$

which mean that f is logarithmically varying at $+\infty$ of order n according to Definition 5.2.

3) For $c=+\infty$ we give the two most meaningful results concerning the cases "$\log f\in\mathcal{R}_\alpha(+\infty), \alpha>1$" and "$\log f\in\mathcal{R}_{+\infty}(+\infty)$".

Proposition 6.1. *Let*

$$f(x):=\exp\left(R_\alpha(x)\right),\ \text{ with } R_\alpha\in\{\mathcal{R}_\alpha(+\infty)\ \text{ of order } n\}, \alpha>1;\ R_\alpha>0; \tag{6.11}$$

which, by ([2]; Prop. 9.4-(II), p. 847) *implies* "$f\in\{\mathcal{E}_{+\infty}(+\infty)\ \text{ of order } n\}$". *Then, besides the relations in* (6.7) *we have the additional properties*:

$$\begin{cases} f^{-1}(y) = (\log y)^{1/\alpha} \cdot \ell(\log y) \ \text{ with a suitable constant } c > 0 \text{ and a function } \ell \text{ such that:} \\ \ell \in \{\mathcal{SR}_0(+\infty) \text{ of order } n\}, \ i.e.\ \ \ell^{(k)}(x) = o\left(x^{-k}\ell(x)\right), \ x \to +\infty, 1 \le k \le n; \end{cases} \tag{6.12a}$$

$$\left(D^k f^{-1}\right)(y) \begin{cases} \sim (-1)^{k-1} \dfrac{(k-1)!}{\alpha} y^{-k} \cdot (\log y)^{-1} \cdot f^{-1}(y) \\ = o\left(y^{-k}\right) \end{cases}, \ y \to +\infty; 1 \le k \le n; \tag{6.12b}$$

hence $f^{-1} \in \{\mathcal{R}_0(+\infty) \text{ of order } n\}$ *and it is hypo-logarithmically varying at* $+\infty$ *of order n.*

Proof. The inverse of the function R_α, denoted by $R_{1/\alpha}$, has the following properties:

$$\begin{cases} R_{1/\alpha} \in \{\mathcal{R}_{1/\alpha}(+\infty) \text{ of order } n\}, \quad \text{by ([2]; Prop. 7.7-(I), p.830);} \\ D^k R_{1/\alpha} \in \mathcal{R}_{(1/\alpha)-k}(+\infty), 1 \le k \le n-1, \ \text{ by ([1]; formula (3.3), p.798);} \\ R_{1/\alpha}(x) \equiv x^{1/\alpha} \cdot \ell(x) \ \text{ with a suitable } \ell \in \mathcal{R}_0(+\infty). \end{cases} \tag{6.13}$$

We cannot in general assert that $\ell \in \{\mathcal{R}_0(+\infty) \text{ of order } n\}$; for instance it may be a constant as in the case $R_\alpha(x) = x^\alpha$. However it is certainly differentiable n times by its definition "$\ell(x) := x^{-1/\alpha} R_{1/\alpha}(x)$", and the asymptotic relations for $D^k R_{1/\alpha}$ ([1]; formula (3.5), p. 799), yield the following ones for $\ell^{(k)}$:

$$\begin{aligned} \ell^{(k)}(x) &= \sum_{i=0}^{k} \binom{k}{i} D^i x^{-1/\alpha} \cdot D^{k-i} R_{1/\alpha}(x) \\ &= \sum_{i=0}^{k} \left\{ \binom{k}{i} \left(\frac{-1}{\alpha}\right)^{\underline{i}} x^{-(1/\alpha)-i} \cdot \left(\frac{1}{\alpha}\right)^{\underline{k-i}} x^{i-k} R_{1/\alpha}(x)\left[1+o(1)\right] \right\} \\ &= x^{-(1/\alpha)-k} \cdot R_{1/\alpha}(x) \cdot \sum_{i=0}^{k} \left\{ \binom{k}{i} \left(\frac{-1}{\alpha}\right)^{\underline{i}} \cdot \left(\frac{1}{\alpha}\right)^{\underline{k-i}} \left[1+o(1)\right] \right\} \\ &= x^{-(1/\alpha)-k} \cdot R_{1/\alpha}(x) \cdot o(1) = o\left(x^{-k}\ell(x)\right), \ x \to +\infty, 1 \le k \le n, \end{aligned} \tag{6.14}$$

because the following sum is zero:

$$\sum_{i=0}^{k} \left\{ \binom{k}{i} \left(\frac{-1}{\alpha}\right)^{\underline{i}} \cdot \left(\frac{1}{\alpha}\right)^{\underline{k-i}} \right\} = D^k \left(x^{-1/\alpha} x^{1/\alpha}\right)\Big|_{x=1} = 0, \ k \ge 1. \tag{6.15}$$

Now, inverting the equation defining f one gets "$f^{-1}(y) = R_{1/\alpha}(\log y)$" whence the factorization for f^{-1} in (6.12a) with the above-studied slowly-varying ℓ. For the higher derivatives of f^{-1} one may use the relations:

$$\begin{cases} D^i (\log y)^{1/\alpha} = o\left(y^{-i} (\log y)^{1/\alpha}\right), \text{because } (\log y)^{1/\alpha} \in \{\mathcal{R}_0(+\infty) \text{ of any order } n\}; \\ D^i \ell(\log y) = o\left(y^{-i} (\log y)^{-1} \cdot \ell(\log y)\right) \quad \text{by (5.31);} \end{cases} \tag{6.16}$$

so getting:

$$\begin{aligned} D^k f^{-1}(y) &= \sum_{i=0}^{k} \binom{k}{i} \cdot D^i (\log y)^{1/\alpha} \cdot D^{k-i} \ell(\log y) \\ &= \sum_{i=0}^{k} o\left(y^{-i} (\log y)^{1/\alpha}\right) \cdot o\left(y^{i-k} (\log y)^{-1} \cdot \ell(\log y)\right) \\ &= o\left(y^{-k} (\log y)^{(1/\alpha)-1} \cdot \ell(\log y)\right) = o\left(y^{-k}\right). \end{aligned} \tag{6.17}$$

But one may also use an explicit expression for the derivatives given in ([1]; formula (3.30), p. 802), namely:

$$D^k\phi(\log x)=(-1)^{k-1}(k-1)!x^{-k}\cdot\phi'(\log x)+x^{-k}\sum_{i=2}^{k}c_i\cdot\phi^{(i)}(\log x),\ k\ge 1, \quad (6.18)$$

where " $\phi^{(i)}(\log x)=(\mathrm{d}^i/\mathrm{d}t^i)\phi(t)\big|_{t=\log x}$ "; and this implies:

$$\begin{aligned} D^i(\log y)^{1/\alpha} &=(-1)^{i-1}\frac{(i-1)!}{\alpha}y^{-i}\cdot(\log y)^{(1/\alpha)-1}+y^{-i}\sum_{j=2}^{i}c_j\cdot(1/\alpha)^j(\log y)^{(1/\alpha)-j} \\ &\sim(-1)^{i-1}\frac{(i-1)!}{\alpha}y^{-i}\cdot(\log y)^{(1/\alpha)-1},\ y\to+\infty,\ i\ge 1. \end{aligned} \quad (6.19)$$

Now in (6.17) we take both the first and the last term out of the sum in the first line and use the relations in (6.19) and the same estimates for $D^{k-i}\ell(\log y)$ as above, so getting:

$$\begin{aligned} D^k f^{-1}(y) &= D^k(\log y)^{1/\alpha}\cdot\ell(\log y)+(\log y)^{1/\alpha}\cdot D^k\ell(\log y) \\ &\quad+\sum_{i=1}^{k-1}\binom{k}{i}\cdot D^i(\log y)^{1/\alpha}\cdot D^{k-i}\ell(\log y) \\ &\sim D^k(\log y)^{1/\alpha}\cdot\ell(\log y),\ y\to+\infty,\le k\le n, \end{aligned} \quad (6.20)$$

because we have:

$$(\log y)^{1/\alpha}\cdot D^k\ell(\log y)=o\left(y^{-k}\cdot(\log y)^{(1/\alpha)-1}\ell(\log y)\right),$$

and each term in the sum is " $o\left(y^{-k}\cdot(\log y)^{(1/\alpha)-2}\cdot\ell(\log y)\right)$ ". Relations in (6.19)-(6.20) yield (6.12b) and the proof is complete. □

Proposition 6.2. *If* $R\in\{\mathcal{R}_{+\infty}(+\infty)\text{ of order }n\}, R>0$, $\tilde{R}$ *is its inverse and* $f(x):=\exp(R(x))$, *then*:

$$\begin{cases} f^{-1}\in\{\mathcal{R}_0(+\infty)\text{ of order }n\};\ D^k f^{-1}\in\mathcal{R}_{-k}(+\infty), 1\le k\le n-1; \\ (D^k f^{-1})(y)\sim(-1)^{\underline{k-1}}\ y^{-k}\cdot\tilde{R}'(\log x)\sim(-1)^{\underline{k-1}}\ x^{1-k}\cdot(Df^{-1})(y),\ y\to+\infty, 1\le k\le n; \\ (D^k f^{-1})(y)=o\left(y^{-k}(\log y)^{-1}\tilde{R}(\log x)\right)=o\left(y^{-k}\right),\ y\to+\infty, 1\le k\le n; \end{cases} \quad (6.21)$$

hence f^{-1} *is hypo-logarithmically varying at* $+\infty$ *of order n.*

Proof. We have $f^{-1}(y)=R^{-1}(\log y)$ hence the properties in the first line in (6.21) follow from ([2]; Prop. 7.7-(II), p. 830); the remaining properties follow from (5.30). □

7. Various Minor Complements

7.1. Counterexamples on Formal Differentiation: First Group

The counterexamples listed in Remarks 2 and 3 after the proof of Proposition 2.3 ([1]; p. 791), may be rearranged and completed to show the total independence of the following two statements:

$$<< f,\ g \text{ strongly or weakly comparable of order } 1>>, \quad (7.1)$$

$$<< D_\ell f, D_\ell g \text{ strongly or weakly comparable}>>, \quad (7.2)$$

where "strong [resp. weak] comparison" means the validity of a "o"-relation [resp. "O"-relation] linking f and g; and "order 1" means that the same type of comparison may be established between the derivatives f', g'. All asymptotic relations in the following examples refer to $x \to +\infty$.

Counterexample 1:

$$\begin{cases} f(x) := x^{\alpha}, \alpha \neq 0; \ g(x) := \exp\left(-x^{\delta}\right), \delta > 0; \\ g(x) \ll f(x); \ g'(x) \ll f'(x); \ D_\ell g(x) \gg D_\ell f(x); \\ f, g \text{ strongly comparable of order 1;} \\ D_\ell f, D_\ell g \text{ strongly comparable but with inverted order.} \end{cases} \tag{7.3}$$

For this counterexample any pair of functions f, g may be used where: f is of the type in I-(2.7) and g of the type in I-(2.8).

Counterexample 2:

$$\begin{cases} f(x) := x + \sin\left(x^2\right); \ g(x) := x^3; \\ f(x) \ll g(x); \ f'(x) \ll g'(x); \ D_\ell f(x) = 2\cos\left(x^2\right) + o(1); D_\ell g(x) = 3x^{-1}; \\ f, g \text{ strongly comparable of order 1; } D_\ell f(x) = O(1); \\ D_\ell f, D_\ell g \text{ not weakly comparable.} \end{cases} \tag{7.4}$$

Counterexample 3:

$$\begin{cases} f(x) := \exp\left[x + x^{\alpha}\sin\left(x^{\beta}\right)\right]; \ g(x) := \exp\left[x + x^{\alpha}\cos\left(x^{\beta}\right)\right], 0 < \alpha < 1, 0 < \beta < 1 - \alpha; \\ \underline{\lim}_{x\to+\infty}\left(f(x)/g(x)\right) = \underline{\lim}_{x\to+\infty}\left(f'(x)/g'(x)\right) = 0; \\ \overline{\lim}_{x\to+\infty}\left(f(x)/g(x)\right) = \overline{\lim}_{x\to+\infty}\left(f'(x)/g'(x)\right) = +\infty; \\ D_\ell f(x) \sim D_\ell g(x) \sim 1; \\ f, g \text{ not weakly comparable; } f', g' \text{ not weakly comparable.} \end{cases} \tag{7.5}$$

Counterexample 4:

$$\begin{cases} f(x) := e^{-x}; \ g(x) := (1/2)x^2 + x\sin x + \cos x; \\ f(x) \ll g(x); \ D_\ell g(x) \sim 2(1+\cos x)x^{-1} \ll D_\ell f(x) \equiv -1; \\ D_\ell f, D_\ell g \text{ strongly comparable;} \\ f, g \text{ strongly comparable but with inverted order;} \\ f', g' \text{ not weakly comparable.} \end{cases} \tag{7.6}$$

7.2. Counterexamples on Formal Differentiation: Second Group

The following may be considered remarks on Prop. 2.5 in ([1]; pp. 794-796) and concern the formal differentiation of an asymptotic relation between two functions having the same type of asymptotic variation. To be definite consider the inference:

$$\begin{gathered} \left\{ f(x) \sim g(x), x \to +\infty, \text{either } f, g \in \mathcal{R}_{\alpha}(+\infty), \alpha \in \overline{\mathbb{R}}, \text{or } f, g \in \mathcal{E}_c(+\infty), c \in \overline{\mathbb{R}} \right\} \\ \overset{?}{\Rightarrow} f'(x) \sim g'(x), x \to +\infty. \end{gathered} \tag{7.7}$$

By definition "$f(x), g(x) > 0$ ultimately" and if we add the assumption

" $f'(x), g'(x) \neq 0$ ultimately" we may write:

$$\frac{f'}{g'} \equiv \frac{f'}{f} \cdot \frac{g}{g'} \cdot \frac{f}{g} \sim \frac{f'}{f} \cdot \frac{g}{g'}, \quad \text{whence } f'/g' \sim 1 \Leftrightarrow f'/f \sim g'/g; \tag{7.8}$$

hence the thesis in (7.7) is automatically granted for $f, g \in \mathcal{R}_\alpha(+\infty), \alpha \in \mathbb{R} \setminus \{0\}$ and for $f, g \in \mathcal{E}_c(+\infty), c \in \mathbb{R} \setminus \{0\}$, whereas the following counterexamples show that it is false in the remaining cases.

Counterexample 1:

$$\begin{cases} g(x) := \log x \cdot \left[1 + a\int_x^{+\infty} (t \log t)^{-1} \sin t dt\right] \sim \log x \equiv f(x),\, x \to +\infty;\, a \neq 0; \\ g'(x) = x^{-1}\left[1 - a \sin x + o(1)\right] \begin{cases} = O(x^{-1}), \\ \not\sim x^{-1}; \end{cases} \\ \lim\limits_{x \to +\infty} xg'(x)/g(x) = 0; \end{cases} \tag{7.9}$$

which is a counterexample for " $f, g \in \mathcal{R}_0(+\infty)$ " and for " $f, g \in \mathcal{E}_0(+\infty)$ ".

Counterexample 2:

$$\begin{cases} g(x) := ae^{x^\alpha} + \sin\left(e^{x^\alpha}\right) \sim ae^{x^\alpha} \equiv f(x),\, x \to +\infty;\, a \neq 0,\, \alpha > 0; \\ g'(x) = \alpha x^{\alpha-1} e^{x^\alpha}\left[a + \cos\left(e^{x^\alpha}\right)\right] \begin{cases} = O(f'(x)), \\ \not\sim f'(x); \end{cases} \\ g'(x)/g(x) = a\alpha x^{\alpha-1}\left[1 + a^{-1} \cdot \cos\left(e^{x^\alpha}\right)\right] \cdot \left[1 + o(1)\right] \\ \qquad = \begin{cases} +\infty(1) & \text{if } \alpha > 1,\, a > 1, \\ o(1) & \text{if } 0 < \alpha < 1,\, a \neq 0; \end{cases} \end{cases} \tag{7.10}$$

which is a counterexample for " $f, g \in \mathcal{R}_{+\infty}(+\infty)$ " and for " $f, g \in \mathcal{E}_{+\infty}(+\infty)$ " with the choice " $a, \alpha > 1$ ". It is also a counterexample for " $f, g \in \mathcal{E}_0(+\infty)$ " with the choice " $0 < \alpha < 1; a \neq 0$ ".

Counterexample 3:

$$\begin{cases} g(x) := e^{-x^\alpha}\left[a + x^{-\beta} \sin\left(x^{\alpha+\beta}\right)\right] \sim ae^{-x^\alpha} \equiv f(x),\, x \to +\infty;\, a, \alpha, \beta > 0; \\ g'(x) = e^{-x^\alpha}\left[-a\alpha x^{\alpha-1} - \alpha x^{\alpha-\beta-1} \sin\left(x^{\alpha+\beta}\right) - \beta x^{-\beta-1} \sin\left(x^{\alpha+\beta}\right) + (\alpha+\beta) x^{\alpha-1} \cos\left(x^{\alpha+\beta}\right)\right] \\ \qquad = -a\alpha x^{\alpha-1} e^{-x^\alpha}\left[1 - \dfrac{\alpha+\beta}{a\alpha} \cos\left(x^{\alpha+\beta}\right) + o(1)\right] \not\sim f'(x); \\ g'(x)/g(x) = -\alpha x^{\alpha-1}\left[1 - \dfrac{\alpha+\beta}{a\alpha} \cos\left(x^{\alpha+\beta}\right) + o(1)\right]\left[1 + o(1)\right] \\ \qquad = -\infty(1) \text{ if } a, \alpha > 1;\, \beta < (a-1)\alpha; \end{cases} \tag{7.11}$$

which is a counterexample for " $f, g \in \mathcal{R}_{-\infty}(+\infty)$ ". It is also a counterexample for " $f, g \in \mathcal{E}_0(+\infty)$ " with the choice " $0 < \alpha < 1;\, a, \beta > 0$ ".

7.3. A Correction in the Statement of Proposition 4.1 in ([1]; p. 808)

The last few lines in the statement of this proposition were erroneously inserted from another file. Starting from the words "*Relations in* (4.10) *imply...*" to the end of the statement the text must be replaced by the following:

Relations in (4.10) *do not imply the validity of the asymptotic scale*:

$$f^{(n+1)}(x) \gg f^{(n)}(x) \gg \cdots \gg f'(x) \gg f(x),\ x \to +\infty, \tag{4.12}$$

which characterizes a subclass of rapidly-varying functions. However, a different way of writing relations in (4.6) *yields the asymptotic scale*:

$$f^{(n)}(x) \gg x^{-1} f^{(n-1)}(x) \gg x^{-2} f^{(n-2)}(x) \gg \cdots \gg x^{-(n-1)} f'(x) \gg x^{-n} f(x),\ x \to +\infty. \tag{4.13}$$

The mentioned subclass is that of hyper-exponentially varying functions of order n ([2]; Def. 8.2, formula (8.70), p. 840).

7.4. Clarifications Concerning Three Examples in [1]

- The function in formula (2.16), p. 783, "$f(x) := x^{\alpha}(2 + \sin\phi(x))$" belongs to the class $\mathcal{R}_{\alpha}(+\infty)$ iff the condition

$$\cos\phi(x) \cdot \phi'(x) = o(x^{-1}),\ x \to +\infty \tag{7.12}$$

is satisfied, because there is no a-priori-specified condition on the asymptotic behavior of ϕ. With the added condition "$\phi(x) = o(1)$" then (7.12) is obviously equivalent to condition "$\phi'(x) = o(x^{-1})$" stated in ([1]; first line of p. 784).

- Remark 2, p. 791. In the three asymptotic relations appearing in the third line in this Remark the roles of the two functions f, g must be interchanged so that the relations read:

$$g \ll f,\ g' \ll f' \quad \text{but} \quad g/g' \gg f/f'.$$

We have the same situation a few lines below where in the third line in formula (2.70) f, g must be interchanged as well, so that this line must be replaced by:

$$\limsup_{x\to+\infty}\left(f(x)/g(x)\right) = \limsup_{x\to+\infty}\left(f'(x)/g'(x)\right) = +\infty;$$

and this is done for consistency with the preceding line though, in this particular instance, the very same relations involving "liminf, limsup" obviously hold true for the inverted ratios.

- In the last function listed in formula (2.105), p. 796, the first term $(1/2)e^{-x}$ is to be replaced by Ae^{-x} with a constant $0 < A < 1/4$. In fact for the function

$$f(x) := Ae^{-x} + e^{-2x}\sin\left(e^{x/2}\right)$$

we have the following relations:

$$\begin{cases} f'(x) \sim -Ae^{-x}; \quad f''(x) = e^{-x}\left[A - \dfrac{1}{4}\sin\left(e^{x/2}\right) + o(1)\right]; \\ xf''(x)/f'(x) = -x\left[1 - \dfrac{1}{4A}\sin\left(e^{x/2}\right) + o(1)\right]; \end{cases}$$

whence $xf''(x)/f'(x)$ is "oscillatory and unbounded for $0 < A < 1/4$".

7.5. A Clarification Concerning Definition 8.1 in ([2]; p. 832)

In this definition, in the second line of formula (8.1) there is the condition "f'

strictly one-sided" referred to the case of hyper-exponential variation, a redundant condition as it automatically follows from the assumption on the one-signedness of f. This is obviously the case also for exponential variation of index $c \neq 0$, whereas for hypo-exponential variation, *i.e.* $f'(x) \ll f(x)$, this restriction is not required.

7.6. Completing the Proof of Proposition 8.4 in ([2]; p. 839)

We give the proof of the last claim in this proposition which is missing in [2]. The assumptions are:

$$\begin{cases} f \in \mathcal{E}_{c_0}(+\infty), |f'| \in \mathcal{E}_{c_1}(+\infty), -\infty \leq c_1 < 0, \quad \text{whence}: \\ \lim_{x\to+\infty} f'(x)/f(x) = c_0, \ \lim_{x\to+\infty} f''(x)/f'(x) = c_1; \end{cases} \tag{7.13}$$

and the assertion is:

$$c_0 = 0 \Leftrightarrow f(+\infty) \in \mathbb{R}\setminus\{0\}.$$

In fact, a simple consequence of $c_1 < 0$ is that "$f'(+\infty) = 0$" and "$\int^{+\infty} f'$ converges" ([2]; formulas (8.27) and (8.47)); hence $f(+\infty)$ exists as a finite number. Now, if $c_0 = 0$ then it cannot be $f(+\infty) = 0$, otherwise L'Hospital's rule applied to the first limit in (7.13) would give a contradiction with the second limit. Viceversa, if $f(+\infty) \neq 0$ then $f'(x)/f(x) \sim f'(x)/f(+\infty) = o(1)$, as $x \to +\infty$.

We point out three misprinted references in the original proof: in the first line following formula (8.63) the reference to formula (8.60) must be changed in a reference to formula (8.63); in the third line following formula (8.64) the reference to formula (8.63) must be changed in a reference to formula (8.64); and in the fifth line following formula (8.64) the reference to formula (8.13) must be changed in a reference to formula (8.12). Also notice that the function f_1 in formula (8.65), p. 839, satisfies:

$$f_1(x) := 1 + \exp(-cx^\gamma), c > 0; \ f_1 \in \mathcal{R}_0(+\infty) \cap \{\mathcal{SR}_0(+\infty) \text{ of any order } n\} \ \forall \gamma \in \mathbb{R}.$$

8. Conclusions

In this final section we give a bird's eye view of the matter treated in [1] [2] and the present paper altogether, the intention having been that of systematizing the theory of what we termed "higher-order types of asymptotic variation". The classical Karamata and de Haan theories of regular and rapid variation, whose detailed exposition is (in [4]; Chapters 1, 2, 3), are based on various types of asymptotic functional equations satisfied by measurable or Baire functions; they are quite demanding and have their meaningful applications to probability, number theory and other fields. But for applications to the asymptotic study of ordinary differential equations or the analytic theory of asymptotic expansions in the real domain one has to deal with functions differentiable a certain number of times and whose derivatives show the same types of behavior so that one is led to formulating concepts of higher-order types of variation. For once-differentiable (namely, absolutely continuous) functions the theory of regular or rapid variation of order 1 (in a stronger sense than Karamata's) is quite elementary directly inferred from the value of the limit "$\lim_{x\to+\infty} xf'(x)/f(x) \equiv \alpha$", assumed to exist as an extended real number; but for types of variation of order ≥ 2 the pertinent theory must be based on appropriate definitions and preliminary results. Here is a list of the various types of asymptotic variation we defined and studied in our work. The locution "strong sense" refers to the involvement of certain derivatives and the regularity of the functions must grant that the highest-order involved derivative is the derivative of an absolutely continuous function.

- Slow variation of order $n \geq 1$ (in a strong sense involving derivatives up to order n);
- Regular variation of index $\alpha \in \mathbb{R}$ and order $n \geq 1$ (in a strong sense involving derivatives up to order n);
- Smooth variation of order $n \geq 1$ (involving derivatives up to order n);
- Rapid variation of order $n \geq 1$ (in a strong restricted sense involving derivatives up to order $n+1$);
- Three classes related to the concept of exponential variation:
- hypo-exponential variation,
- exponential variation,
- hyper-exponential variation;
- Three special subclasses of slow variation related to the concept of logarithmic variation:
- quasi-logarithmic variation,
- logarithmic variation,
- hypo-logarithmic variation.

Let us say something about each class starting from "regular variation", slow variation being the special case of index zero. A basic lemma by the author ([1]; Prop. 2.6, p. 796), establishes precise relationships between the indexes of variation of a function and its derivatives so that, when defining higher-order regular variation, one knows the possible links between the indexes of the involved derivatives, and this is absolutely necessary for applying these concepts. The main feature of regularly-varying functions of order n is the n asymptotic relations satisfied by the ratios $f^{(k)}(x)/f(x)$:

$$\begin{aligned} f^{(k)}(x)/f(x) &= \alpha(\alpha-1)\cdots(\alpha-k+1)x^{-k} + o\left(x^{-k}\right) \\ &\equiv \alpha^{\underline{k}} x^{-k} + o\left(x^{-k}\right),\ x \to +\infty, 1 \leq k \leq n, \end{aligned} \tag{8.1}$$

where α is the index of regular variation of f. These relations turn out to characterize such a class whenever α does not assume the exceptional values "$0,1,\cdots,n-2$": ([1]; Prop. 3.1, p.799). The relations valid in the exceptional cases are simple "o"-estimates and lead in a natural way to the concept of higher-order smooth variation. The concept of smooth variation used in the literature until now refers to functions satisfying (8.1) for each $k \in \mathbb{N}$ (smooth variation of any order n, in our terminology) and an exposition is in ([4]; §1.8, pp.

44-49) where the authors mention as a simple check the equivalence between the relations in (8.1), assumed valid for each $k \in \mathbb{N}$, and the relations satisfied by the associated function:

$$h(x) := \log(f(x)); \lim_{x \to +\infty} h'(x) = \alpha; \lim_{x \to +\infty} h^{(k)}(x) = 0, x \to +\infty, k \geq 2. \quad (8.2)$$

As a matter of fact this equivalence is anything but trivial and the original ingenious proof by Balkema, Geluk and de Haan is revisited in ([1]; pp. 802-803) to highlight the technical ideas. Besides the equivalence of (8.1) and (8.2) for $1 \leq k \leq n$, we give two other simple but useful characterizations: ([1]; Prop. 3.2 and Def. 3.2, pp. 801-804).

Next, the concept of rapid variation of order n needs some critical a-priori remarks leading to a restricted class which is the right class for most applications. For this class, which involves derivatives up to order $n+1$, we give five different characterizations expressing asymptotic relations satisfied by either the logarithmic derivatives $D_\ell\left(f^{(k)}(x)\right), 0 \leq k \leq n$, or their reciprocals: ([1]; Def. 4.1 and Prop. 4.1, pp. 807-808).

The three concepts related to exponential variation are studied in some detail in ([2]; §§ 8-9-10). Though these concepts may in principle be brought back to the concepts of slow, regular and rapid variation by the simple change of variable $g(x) := f(e^x)$, they are quite important for applications and deserve a separate explicit treatment not to be found in the literature in a systematized way. After the exposition of the properties for order 1 the pertinent concepts of higher-order exponential variation are defined quite simply provided a preliminary lemma clarifies the relationships between the indexes of exponential variation of a function and its derivative: ([2]; pp. 838-840).

Let us now come back to the concept of slow variation ($\equiv$ regular variation of index 0). In many a respect slowly-varying functions behave differently from regularly-varying ones of index $\alpha \neq 0$ and it is improper to speak of higher-order slow variation because a derivative of a slowly-varying function, if it exists and is regularly varying, has a non-zero index of variation. Also, the concept of slow variation, though including all iterated logarithms, does not exactly define what is intuitively attached to "logarithmic variation" when the variable tends to $+\infty$. So, for the sake of completeness, we added in §5 of the present paper a discussion about a concept of logarithmic variation which does not seem to be explicitly stated in the literature though it is to be implicitly found in the more advanced theories in [4]. In fact we had to use some non-trivial results from this monograph to prove some estimates. Unlike the to-be-avoided concept of higher-order slow variation, the definitions of the three types of logarithmic variation allow respective types of higher order.

Fundamental features of the various classes of functions are certain asymptotic functional equations and some of them represent in fact the starting points for the more general definitions of the classes in Karamata and de Haan theories. Many such functional equations are collected in ([1]; §5), ([2]; §8) and §5 of the present paper; they motivate the discussion in §4 of the present paper comparing the two approaches to the concept of "type of asymptotic variation": via an asymptotic differential equation or an asymptotic functional equation. The main contribution consists in proving that for a function which is either ultimately concave or ultimately convex the two approaches coincide for each one of the studied classes including those related to logarithmic variation.

Now some words about the types of asymptotic variation for functions obtained by means of algebraic operations: linear combination, multiplication, composition and inversion. While the pertinent results are quite simple if all the involved functions are smoothly varying, an hard job was required in ([2]; §7) to obtain complete results in the cases of regular or rapid variation due to the exceptional values of the indexes which cause anomalies; pathological counterexamples are given. An equally hard job was done to obtain results on the higher-order types of compositions involving exponential and regular variations in ([2]; §9); three cases needed separate treatments: order 1, order 2, order $n \geq 3$. The study of the inverse of a function with a type of exponential variation, left open in ([2]; Open Problem 4, p. 866), received adequate study in §6 of the present paper clarified by the concepts related to logarithmic variation.

The remaining material in our three papers concerns elementary applications of the theory such as: relations between "integral of a product" and "product of integrals" ([2]; §10.1); sums of exponentially-varying terms ([2]; §10.2); certain types of asymptotic expansions ([2]; §11); types of variation of infinite series and Wronskians ([this paper]; §§2, 3).

We hope that our exposition (in these three semi-expository papers) of the general theory of "higher-order types of asymptotic variation" will reveal exhaustive and apt to applications. Until now the author has applied this theory to obtain non-elementary results about the asymptotic behaviors of Wronskians [5] [6]. A minor fact remain unsolved and this is "Open Problems 1 and 3" in ([2]; p. 866) and an open problem stated after relation (2.110) in ([1]; p. 796). We think that the three problems are interrelated and if one finds out a counterexample for one of them, then suitable adaptations will yield counterexamples for the other questions. This puzzled the author and a lot of scribbled-out sheets of paper have been wasted.

Conflicts of Interest

The author declares no conflicts of interest regarding the publication of this paper.

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Bibliographical Notes and Corrections to Parts I and II

- For the matter treated in §2 we have no reference in the literature apart from the elementary case in [3].
- Chapter 3 in the monograph [4] is devoted to a deep study of various asymptotic relations satisfied by the ratio $\left[f(\lambda x)-f(x)\right]/g(x)$ where g is a fixed "auxiliary" function; in §5 we reported on the special case $g\equiv 1$ with some complements pertinent to differentiable functions in order to complete our theory of "higher-order types of asymptotic variation". The exposition in §5 clarifies and expands the notion cursorily touched on in ([1]; Prop. 5.2-(III), p. 815).
- Here is a list of corrections for a number of typos in the papers [1] [2]: page-number univocally identifies the pertinent paper because the two papers have a consecutive page-numbering.

page	**line**	**original string of words or symbols**	**to be replaced by**
783	2nd line after formula (2.12)	*two numbers* $c\neq 0, T>0$	*two numbers* $c, T>0$
784	6th line from above	quite elementary	elementarily
784	1st line in formula (2.22)	$+\sin\left((\log x)^{\delta}\right)$	$c+\sin\left((\log x)^{\delta}\right)$
788	inside formula (2.48)	f ultimately strictly increasing,	$f'>0$ ultimately,
788	1st line after formula (2.49)	are proved as in (2.38).	are proved as in (2.38) noticing that condition " $f'>0$ is required.
789	3rd line in formula (2.52)	**in the conclusion these two relations may be added**	$f=o(g);\ f'=o(g');$
789	2nd line from below	$\forall\ c_i\in\mathbb{R}$.	$\forall\ c_i\in\mathbb{R}$ and $c_1\neq 0$.
790	1st line from above	Besides (2.56)	Besides (2.60)
792	5th line from below	with relations in (2.104) below.	with the rough estimates in (2.81) – (2.82).
793	inside formula (2.87)	**the last integral on the right is obviously to be replaced by**	$\int_T^x f(t)\left(f(t)/f'(t)\right)'\mathrm{d}t$
793	1st line after formula (2.87)	the third condition	the fourth condition
794	9th line from above	Proposition 4.2	Proposition 4.1, formula (4.25),
798	last line in formula (3.4)	$\alpha_k=\beta-(k-k_0+1)$	$\alpha_k=\beta-(k-k_0-1)$
804	2nd line from below	$f\in\{\mathcal{SR}_\alpha\dots\},\ \alpha\neq 0,\ f'\in\dots$	$f\in\{\mathcal{SR}_\alpha\dots\},\ \alpha\neq 0,\ \implies\ f'\in\dots$
815	3rd line from below	$f'(x)/f(x)\geq M$	$f'(x)/f(x)\geq Mx^{-1}$
820	last line in formula (7.3)	$\forall\ \alpha\in\mathbb{R},\beta>0$	$\forall\ \beta>0$
826	2nd line after formula (7.47)	<u>save the last</u>	<u>save possibly the last</u>
829	3rd line after formula (7.65)	Proposition 4.2	Corollary 4.2