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Transference Principles for the Series of Semigroups with a Theorem of Peller

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Abstract

A general approach to transference principles for discrete and continuous sequence of operators (semi) groups is described. This allows one to recover the classical transference results of Calderon, Coifman and Weiss and of Berkson, Gilleppie and Muhly and the more recent one of the author. The method is applied to derive a new transference principle for (discrete and continuous) the sequence of operators semigroups that need not be grouped. As an application, functional calculus estimates for bounded sequence of operators with at most polynomially growing powers are derived, leading to a new proof of classical results by Peller from 1982. The method allows for a generalization of his results away from Hilbert spaces to $L^{(1+\varepsilon)}$-spaces and—involving the concept of γ-boundedness—to general spaces. Analogous results for strongly-continuous one-parameter (semi) groups are presented as well by Markus Haase [1]. Finally, an application is given to singular integrals for one-parameter semigroups.

Keywords

Transference, Operator Semigroup, Functional Calculus, Analytic Besov, Peller, γ-Boundedness, γ-Radonifying, γ-Summing, Power-Bounded Operator

1. Introduction

The purpose of this article is twofold. The short part devotes to a generalization of this classical transference principle of Calderon, Coifman and Weiss. The major part gives applications of this new abstract result to discrete and continuous operator (semi) groups, in particular shall recover and generalize important results of Peller. In the classical transference principle(s) the objects under investigation are derived from the sequence of operators of the form.

$$\sum_j T^j_{\mu^j} = \int_G \sum_j T^j(1+\varepsilon)\mu^j(d(1+\varepsilon)) \tag{1.1}$$

where G is a locally compact group and $T^j = (T^j(1+\varepsilon))_{(1+\varepsilon)\in G} : G \to \mathcal{L}(x)$ is a strongly bounded continuous representation of G on a Banach space X. The integral (1.1) has to be understood in the strong sense, *i.e.*,

$$\sum_j T^j_{\mu^j} x = \int_G \sum_j T^j(1+\varepsilon) x\mu^j(d(1+\varepsilon)) \quad (x \in X)$$

And μ^j are scalar measure that renders the meaningful expression. Since such sequence of operators occurs in a variety of situations, the applications of transference principles are manifold, and the literature on this topic is vast.

Originally, Calder'n [2] considered representations on $L^{\varepsilon+1}$ induced by a G-flow of measure-preserving transformations of the underlying measure space. H is considerations that were motivated by ergodic theory and his aim was to obtain maximal inequalities. Subsequently, Coifman and Weiss [3] [4] shifted the focus to norm estimates and were able to drop Calder'n's assumption of an underlying measure-preserving G-flow towards general G-representations on $L^{\varepsilon+1}$-spaces. Berkson, Gillespie and Muhly [5] were able to generalize the method towards general Banach spaces X. However, the representations considered in these works were still (uniformly) bounded. In the continuous one-parameter case (*i.e.*, $G=\mathbb{R}$) Blower [6] showed that the original proof method could fruitfully be applied also to non-bounded representations. However, his result was in a sense "local" and did not take into account the growth rate of the group $(T^j(1+\varepsilon))_{(1+\varepsilon)\in\mathbb{R}}$ at infinity. In [7] we discovered Blower's result and in [8] we could refine it towards a "global" transference result for strongly continuous one-parameter groups.

Markus Haase [1] showed a developing method of generating transference results and showed that the known transference principles, (the classical Berkson-Gillespie-Muhly result and the central results of [8]) are special instances of it. The method has three important new features. Firstly, it allows to pass from groups to semigroups. More precisely, consider closed sub-semigroups S of a locally compact group G together with a strongly continuous representation $T^j : S \to \mathcal{L}(X)$ on a Banach space, and try to estimate the norms of sequence of operators of the form

$$\sum_j T^j_{\mu^j} = \int_{(1+\varepsilon)} \sum_j T^j(1+\varepsilon)\mu^j(d(1+\varepsilon)) \tag{1.2}$$

by means of the transference method. The second feature is the role of weights in the transference procedure, somehow hidden in the classical version. Thirdly, the account brings to light the formal structure of the transference argument. in the first step one establishes a factorization of the sequence of operators (1.2) over a convolution (*i.e.*, Fourier multiplier) operator on a space of X-valued functions on G, then, in a second step, one uses this factorization to estimate the series norms, and finally, one may vary the parameters to optimize the obtained inequalities, So one can briefly subsume our method under the scheme.

factorize-estimate-optimize

where use one particular way of constructing the initial factorization. One reason for the power of the method lies in choosing different weight in the factorization, allowing for the optimization in the last step. The second reason lies in the purely formal nature of the factorization: this allows to re-interpret the same factorization involving different function spaces.

Devoted to applications of the transference method. These applications deal exclusively with the cases $S=\mathbb{Z},\mathbb{Z}_+$, and $S=\mathbb{R},\mathbb{R}_+$ which we for short call the discrete and the continuous case, respectively. However, let us point out that the general transference method works even for sub-semigroups of non-abelian groups.

To clarify what kind of applications we have in mind, let us look at the discrete case first. Here the semigroup consists of the powers $\left(\left(T^j\right)^n\right)_{n\in\mathbb{N}_0}$ of one single bounded sequence of operators T^j, and the derived sequence of operators (1.2) take the form

$$\sum_{n\geq 0}(\beta+\varepsilon)_n\sum_j\left(T^j\right)^n$$

for some (complex) scalar sequence $\beta+\varepsilon=\left((\beta+\varepsilon)_n\right)_{n\geq 0}$. In order to avoid convergence questions, suppose that $(\beta+\varepsilon)$ is a finite sequence, hence

$$\widehat{(\beta+\varepsilon)}(z)=\sum_{n\geq 0}(\beta+\varepsilon)_n(z)^n$$

is a complex polynomial. One usually writes

$$\widehat{(\beta+\varepsilon)}\sum_j\left(T^j\right)=\sum_{n\geq 0}(\beta+\varepsilon)_n\sum_j\left(T^j\right)^n$$

and is interested in continuity properties of the functional calculus

$$\mathbb{C}[z]\to\mathcal{L}(X),f_j\leftrightarrow f_j\left(T^j\right).$$

That is, one looks for a function algebra norm $\|\cdot\|_{(A_j)}$ on $\mathbb{C}[z]$ that allows an estimates of the form

$$\sum_j\left\|f_j\left(T^j\right)\right\|\lesssim\left\|\sum_j f_{j\mathcal{A}}\right\|\quad\left(f_j\in\mathbb{C}[z]\right)\tag{1.3}$$

(The symbol $\lesssim$ is short for $<\varepsilon$ for some unspecified constant $\varepsilon>-1$, see also the Terminology-paragraph at the end of the introduction). A rather trivial instance of (1.3) is based on the estimates

$$\sum_j\left\|f_j\left(T^j\right)\right\|=\sum_j\left\|\sum_{n\geq 0}(\beta+\varepsilon)_n\left(T^j\right)^n\right\|\leq\sum_{n\geq 0}\left|(\beta+\varepsilon)_n\right|\left\|\sum_j\left(T^j\right)^n\right\|$$

Defining the positive sequence $(1+\varepsilon)=((1+\varepsilon)_n)_n$ by $(1+\varepsilon)_n := \sum_j \left\|(T^j)^n\right\|$, hence have

$$\sum_j \left\|f_j(T^j)\right\| \le \left\|\sum_j f_j\right\|_{(1+\varepsilon)} = \sum_{n\ge 0}\left|(\beta+\varepsilon)_n\right|(1+\varepsilon)_n \tag{1.4}$$

and by the submultiplicativity $(1+\varepsilon)_{n+m} \le (1+\varepsilon)_n(1+\varepsilon)_m$ one sees that $\|\cdot\|_{(1+\varepsilon)}$ is a functional gebra (semi)norm on $\mathbb{C}[z]$.

The "functional calculus" given by (1.4) is tailored to the sequence of operators T^j and uses no other information than the growth of the power of T^j. The central question is: under which conditions can one obtain better estimates for $\sum_j \left\|f_j(T^j)\right\|$ *i.e.*, in terms of weaker function norms? The conditions have in mind may involve T^j (or better: the semigroup $\left((T^j)^n\right)_{n\ge 0}$) or the underlying Banach space. To recall a famous example: von Neumann's inequality [9] states that if $X=H$ is a Hilbert space and $\sum_j \left\|T^j\right\| \le 1$ (*i.e.*, T^j is a contraction), then

$$\sum_j \left\|f_j(T^j)\right\| \le \left\|\sum_j f_j\right\|_\infty \quad \text{for every } f_j \in \mathbb{C}[z] \tag{1.5}$$

where $\sum_j \left\|f_j\right\|_\infty$ are the series norms of f_j in the Banach algebra $\mathcal{A}=H^\infty(\mathbb{D})$ of bounded analytic functions on the open unit disc $\mathbb{D}$.

Von Neumann's result is optimal in the trivial sense that the estimate (1.5) of course implies that T_j are contraction, but also in the sense that one cannot improve the estimate without further conditions: If $H=L^2(\mathbb{D})$ and $(T^j(h))(z)=zh(z)$ is multiplication with the complex coordinate, then $\sum_j \left\|f_j(T^j)\right\| = \sum_j \left\|f_j\right\|_\infty$ for any $f_j \in C[z]$. A natural question then is to ask which sequence of operators satisfy the slightly weaker estimates

$$\sum_j \left\|f_j(T^j)\right\| \le \left\|\sum_j f_j\right\|_\infty \quad (f_j \in \mathbb{C}[z])$$

(called "polynomial boundedness of T^j"). On a general Banach space this may fail even for a contraction: simply take $X=\ell^1(\mathbb{Z})$ and T^j the shift sequence of operators, given by $(T^jX)_n = X_{n+1}$, $n\in\mathbb{Z}$, $x\in\ell^1(\mathbb{Z})$. On the other hand, Lebow [10] has shown that even on a Hilbert space polynomial boundedness of sequence of operators T^j may fail if it is only assumed to be power-bounded, *i.e.*, if one has merely $\sup_{n\in\mathbb{N}}\left\|\sum_j (T^j)^n\right\| < \infty$ instead of $\left\|\sum_j (T^j)^n\right\| \le 1$. The class of power-bounded sequence of operators on Hilbert spaces is notoriously enigmatic, and it can be considered one of the most important problems in sequence of operators theory to find good functional calculus estimates for this class.

Let us shortly comment on the continuous case. Here one is given a strongly continuous (in short: C_0) semigroup $\left(T^j(1+\varepsilon)\right)_{\varepsilon\ge -1?}$ of the sequence of operators on a Banach space *X*, and one considers integrals of the form

$$\int_{\mathbb{R}_+} \sum_j T^j(1+\varepsilon)\mu^j(d(1+\varepsilon)) \tag{1.6}$$

where assume for simplicity that the support of the measure μ^j are bounded. Shall use only basic results from semigroup theory, and refer to [11] [12] for further information. The sequence of generators of the semigroup $\left(T^j(1+\varepsilon)\right)_{\varepsilon>-1}$ is in general unbounded, closed and densely defined the sequence of operators $-A_j$ satisfying

$$\sum_j \left((1+\varepsilon)+A_j\right)^{-1} = \int_0^\infty e^{-(1+\varepsilon)(1+\varepsilon)} \sum_j T^j(1+\varepsilon)\, d(1+\varepsilon) \tag{1.7}$$

for Re $(1+\varepsilon)$ large enough. The sequence of generators are densely defined, *i.e.*, its domain $\operatorname{dom}(A_j)$ is dense in *X*. Exclusively deal with semigroups satisfying a polynomial growth $\sum_j \left\|T^j(1+\varepsilon)\right\| \lesssim (2+\varepsilon)^{\beta+\varepsilon}$ for some $\varepsilon>-\beta$, and hence (1.7) holds at least for all Re $\varepsilon>-1$. One writes $T^j(1+\varepsilon)=e^{-(1+\varepsilon)A_j}$ for $\varepsilon>-1$ and, more generally,

$$\sum_j \left(\mathcal{L}\mu^j\right)(A_j) = \int_{\mathbb{R}_+} \sum_j T^j(1+\varepsilon)\mu^j(d(1+\varepsilon))$$

where

$$\sum_j \left(\mathcal{L}\mu^j\right)(A_j) = \int_{\mathbb{R}_+} e^{-z(1+\varepsilon)} \sum_j \mu^j(d(1+\varepsilon))$$

is the Laplace transform of μ^j. So in the continuous case the Laplace transform takes the role of the Taylor series in the discrete case. Asking for good estimates for the sequence of operators of the form (1.6) is as asking for functional calculus estimates for the sequence of operators A_j. The continuous version of von Neumann's inequality states that if $X=H$ is a Hilbert space and if $\left\|\sum_j T^j(1+\varepsilon)\right\| \le 1$ for all $\varepsilon>-1$ (*i.e.*, if T^j are contraction semigroup), then

$$\sum_j \left\|f_j(A_j)\right\| \le \left\|\sum_j f_j\right\|_\infty \quad \left(f_j \in \mathcal{L}\mu^j\right)$$

where $\left\|f_j\right\|_\infty$ are the norms of f_j in the Banach algebra $H^\infty(\mathbb{C}_+)$ of bounded analytic functions on the open half place $\mathbb{C}_+ := \{z\in\mathbb{C} \mid \operatorname{Re} z>0\}$, see ([13], Theorem 7.1.7).

There are similarities in the discrete and in the continuous case, but also characteristic differences. The discrete case is usually a little more general, shows more irregularities, and often it is possible to transfer results from the discrete to the continuous case. (However, this may become quite technical, and prefer direct proofs in the continuous case whenever possible.) In the continuous case, the role of power-bounded operators is played by bounded semigroups, and similar to the discrete case, the class of bounded semigroups on Hilbert spaces appears to be rather enigmatic. In particular, there is a continuous analogue of Lebow's result due to Le Merdy [14], cf. also ([13], Section 9.1.3). And there remain some notorious open questions involving the functional calculus, e.g., the power-boundedness of the Cayley transform of the generator, cf. [15] and the references therein.

The strongest results in the discrete case obtained so far can be found in there markable [16] by Peller from 1982. One of Peller's results are that if T^j is a power-bounded of sequence operators on a Hilbert space H, then

$$\sum_j \left\| f_j\left(T^j\right) \right\| \lesssim \left\| \sum_j f_j \right\|_{B^0_{\infty,1}} \qquad \left(f_j \in \mathbb{C}[z]\right)$$

where is $B^0_{\infty,1}(\mathbb{D})$ is the so-called analytic Besov algebra on the disc .

In 2005, Vitse [17] made a major advance in showing that Peller's Besov class estimate still holds true on general Banach spaces if the power-bounded sequence of operators T^j is actually of Tadmor-Ritt type, *i.e.*, satisfies the "analyticity condition"

$$\sup_{n\ge 0} \sum_j \left\| n\left(\left(T^j\right)^{n+1} - \left(T^j\right)^n\right) \right\| < \infty.$$

She moreover established in [18] an analogue for strongly continuous bounded analytic semigroups. Whereas Peller's results rest on Grothendieck's inequality (and hence are particular to Hilbert spaces) Vitse's approach is based on repeated summation/integration by parts, possible because of the analyticity assumption.

Shall complement Vitse's result by devising an entirely new approach, using the transference methods. In doing so, avoid Grothendieck's inequality and reduce the problem to certain Fourier multipliers on vector-valued function spaces. By Plancherel's identity, on Hilbert spaces these are convenient to estimate, but one can still obtain positive results on $L^{1+\varepsilon}$-spaces or on UMD spaces. The approach works simultaneously in the discrete and in the continuous case, and hence do not only recover Peller's original result (Theorem 5.1) but only establish a complete continuous analogue (Theorm 5.3), conjectured in [18]. Moreover, we establish an analogue of the Besov-type estimates for $L^{1+\varepsilon}$-spaces and for UMD spaces (Theorem (5.7)). These results, however, are less satisfactory since the algebras of Fourier multipliers on the spaces $L^2(\mathbb{R};X)$ and $L^2(\mathbb{Z};X)$ are not thoroughly understood if X is not a Hilbert space.

Show how the transference methods can also be used to obtain "γ-versions" of the Hilbert space results. The central notion here is the so-called γ-boundedness of sequence of operators family, a strengthening of operator norm boundedness. It is related to the notion of *R*-boundedness and plays a major role in Kalton and Weis' work [19] on the H^∞-calculus. The "philosophy" behind this theory is that to each Hilbert space result based on Plancherel's theorem there is a corresponding Banach space version, when operator norm boundedness is replaced by γ-boundedness.

Give evidence to this philosophy by showing how the transference results enables one to prove γ-versions of functional calculus estimates on Hilbert spaces. As examples, we recover the γ-version of a result of Boyadzhiev and deLaubenfels, first proved by Kalton and Weis in [19] (Theorem 6.5). Then derive γ-versions of the Besov calculus theorems in both the discrete and the continuous forms. The simple idea consists of going back to the original factorization in the transference method, but exchanging the function spaces on which the Fourier multiplier sequence of operators act from an L^2-space into a γ-space. This idea is implicit in the original proof from [19] and has also been employed in a similar fashion recently by Le Merdy [20].

Finally, discuss consequences of the estimates for full functional calculi and singular integrals for discrete and continuous semigroups. For instance, prove that if $\left(T^j\left(1+\varepsilon\right)\right)_{(\varepsilon>-1)}$ is any strongly continuous semigroup on a UMD space *X*, then for all $0<a<a+\varepsilon$ the principal value integral

$$\lim_{\varepsilon\searrow 0}\int_{\epsilon<|1-a|<a}\frac{\sum_j T^j\left(1+\varepsilon\right)x}{1-a}$$

exists for all $x\in X$. For C_0-groups this is well-known, cf. [7], but for semi-groups which are not groups, this is entirely new.

Terminology: Use the common symbols $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{R}$, $\mathbb{C}$ for the sets of natural, integer, real and complex numbers. In our understanding 0 is not a natural number, and write

$$\mathbb{Z}_+ := \left\{n\in\mathbb{Z}\mid n\ge 0\right\}=\mathbb{N}\cup\left\{0\right\}\ \text{and}\ \mathbb{R}_+ := \left\{\left(1+\varepsilon\right)\in\mathbb{R}\mid\varepsilon>-1\right\}.$$

Moreover, $:=\left\{z\in\mathbb{C}\mid\left|z\right|<1\right\}$ is the open unit disc, $\mathbb{T}:=\left\{z\in\mathbb{C}\mid\left|z\right|=1\right\}$ is the torus, and $\mathbb{C}_+ := \left\{z\in\mathbb{C}\mid\operatorname{Re}z>0\right\}$ is the open right half plane.

Use *X*, *Y*, *Z* to denote (complex) Banach spaces, and A_j, *B*, *C* to denote closed possibly unbounded sequence of operators on them. By $\mathcal{L}\left(X\right)$ denote the Banach algebra of all bounded linear sequence of operators on the Banach space *X*, endowed with the ordinary sequence of operators norm. The domain, kernel and range of the sequence of operators A_j are denoted by $\operatorname{dom}\left(A_j\right)$, $\ker\left(A_j\right)$ and $\operatorname{ran}\left(A_j\right)$, respectively.

The Bochner space of equivalence classes of $1+\varepsilon$-integrable *X*-valued functions is denoted by $L^{(1+\varepsilon)}\left(R;X\right)$. If Ω is a locally compact space, then $M\left(\Omega\right)$ denotes the space of all bounded regular Borel measures on Ω. If $\mu^j\in M\left(\Omega\right)$ then $\sup\mu^j$ denotes its topological support. If $\Omega\subset\mathbb{C}$ is an open subset of the complex plane, $H^\infty\left(\Omega\right)$ denotes the Banach algebra of bounded holomorphic functions on Ω, endowed with the supremum norm

$$\sum_j\left\|f_j\right\|_{H^\infty(\Omega)}=\sup\left\{\sum_j\left|f_j\left(z\right)\right|\middle|z\in\Omega\right\}.$$

Shall need notation and results from Fourier analysis as collected in [13]. In particular, use the symbol $\mathcal{F}$ for the Fourier transform acting on the space of (possibly vector-valued) tempered distributions on $\mathbb{R}$, where agree that

$$\sum_j\mathcal{F}\mu^j\left(1+\varepsilon\right)=\int_{\mathbb{R}}\mathrm{e}^{-i(1+\varepsilon)(1+\varepsilon)}\sum_j\mu^j\left(d\left(1+\varepsilon\right)\right)$$

is the Fourier transform of a bounded measure $\mu^j\in M\left(R\right)$. A function $m\in L^\infty\left(\mathbb{R}\right)$ is called a bounded Fourier multiplier on $L^{(1+\varepsilon)}\left(\mathbb{R};X\right)$ if there is a constant $\varepsilon>-1$ such that

$$\sum_j \left\| \mathcal{F}^{-1}\left(m \cdot \mathcal{F} f_j\right) \right\|_{(1+\varepsilon)} \le (1+\varepsilon) \left\| \sum_j f_j \right\|_{(1+\varepsilon)} \tag{1.8}$$

holds true for all $f^j \in L^{(1+\varepsilon)}(\mathbb{R}; X) \cap \mathcal{F}^{-1}\left(L^1(\mathbb{R}; X)\right)$. The smallest $(1+\varepsilon)$ that can be chosen in (1.8) is denoted by $\|\cdot\|_{\mathcal{M}_{(1+\varepsilon),x}}$. This turns the space $\mathcal{M}_{(1+\varepsilon),X}(\mathbb{R})$ of all bounded Fourier multipliers on $L^{(1+\varepsilon)}(\mathbb{R}; X)$ into a unital Banach algebra.

A Banach space X is a UMD space, if and only if the function $(1+\varepsilon) \mapsto \text{sgn}(1+\varepsilon)$ is a bounded Fourier multiplier on $L^2(\mathbb{R}; X)$. Such spaces are the right ones to study singular integrals for vector-valued functions. In particular, by results of Bourgain, McConnel and Zimmermann, a vector-valued version of the classical Mikhlin theorem holds, see ([13], Appendix E.6) as well as Burkholder's article [21]. Each Hilbert space is UMD, and if X is UMD, then $L^{(1+\varepsilon)}\left(\Omega, \Sigma, \mu^j; X\right)$ is also UMD whenever $1 < 1+\varepsilon < \infty$ and $\left(\Omega, \Sigma, \mu^j\right)$ is a measure space.

The Fourier transform of $\mu^j \in \ell^1(\mathbb{Z})$ are

$$\sum_j \widehat{\mu^j}(z) = \sum_{n \in \mathbb{Z}} \sum_j \mu^j(n) z^n \quad (z \in \mathbb{T})$$

Analogously to the continuous case, form the algebra $\mathcal{M}_{(1+\varepsilon),X}(\mathbb{T})$ of functions $m \in L^{\infty}(\mathbb{T})$ which induce bounded Fourier multiplier sequence of operators on $\ell^{(1+\varepsilon)}(\mathbb{Z}; X)$, endowed with its natural norm.

Finally, given sets A_j and two real-valued functions $f_j, g^j : A_j \to \mathbb{R}$ write

$$f_j(a) \lesssim g^j(a) \quad \left(a \in A_j\right)$$

to abbreviate the statement that there is $\varepsilon > -1$ such that $f_j(a) \le (1+\varepsilon) g^j(a)$ for all $a \in A_j$.

2. Transference Identities

Introduce the basic idea of transference. Let G be a locally compact group with left Haar measure ds. Let $S \subseteq G$ be a closed sub-semigroup of G and let

$$T^j : S \to \mathcal{L}(X)$$

be a strongly continuous representation of S on a Banach space X. Let μ^j be a (scalar) Borel measure on S such that

$$\int_S \sum_j \left\| T^j(1+\varepsilon) \left|\mu^j\right| \left(d(1+\varepsilon)\right) \right\| < \infty$$

and let the sequence of operators $T^j_{\mu^j} \in \mathcal{L}(X)$ be defined by

$$\sum_j T^j_{\mu^j} x = \int_S \sum_j T^j(1+\varepsilon) x \mu^j\left(d(1+\varepsilon)\right) \quad (x \in X) \tag{2.1}$$

The aim of transference is an estimate of $\sum_j \left\| T^j_{\mu^j} \right\|$ in terms of a convolution sequence of operators involving μ^j. The idea to obtain such an estimate is, in a first step, purely formal.

For a (measurable) functions $\varphi_j : S \to \mathbb{C}$ denote by $\varphi_j T^j$ the pointwise products

$$\left(\varphi_j T^j\right): S \to \mathcal{L}(X), \quad (1+\varepsilon) \mapsto \varphi_j(1+\varepsilon) T^j(1+\varepsilon)$$

and by $\varphi_j \mu^j$ the measure

$$\left(\varphi_j T^j\right)\left(d(1+\varepsilon)\right) = \varphi_j(1+\varepsilon)\mu^j\left(d(1+\varepsilon)\right)$$

In the following do not distinguish between a function/measure defined on S and its extension to G by 0 on $G \backslash S$. Also, for Banach spaces X, Y, Z and sequence of operators-valued functions $F: G \to \mathcal{L}(Z, Y)$ and $H: G \to \mathcal{L}(Y, X)$ define the convolution $H * F: G \to \mathcal{L}(Z, X)$ formally by

$$(H * F)(1+\varepsilon) = \int_G H(1+\varepsilon) F\left((1+\varepsilon)^{-1}(1+\varepsilon)\right) d(1+\varepsilon) \quad \left((1+\varepsilon) \in G\right) \tag{2.2}$$

in the strong sense, as long as this is well defined. (Actually, as argue purely formally, at this stage do not bother too much about whether all things are well defined.) Instead, shall establish formulate first and then explore conditions under which they are meaningful.

The first lemma expresses the fact that a semigroup representation induces representations of convolution algebras on S (see, e.g., [1]).

Lemma 2.1. Let G, S, T^j, X as above and let $\varphi_j = (\varphi_j)_1 + (\varphi_j)_2$, $\psi_j = (\psi_j)_1 + (\psi_j)_2 : S \to \mathbb{C}$ be functions. Then, formally,

$$\left((\varphi_j)_1 + (\varphi_j)_2\right) T^j * \left((\psi_j)_1 + (\psi_j)_2\right) T^j = \left(\left((\varphi_j)_1 + (\varphi_j)_2\right)\left((\psi_j)_1 + (\psi_j)_2\right)\right) T^j.$$

Proof. Fix $(1+\varepsilon) \in G$. If $(1+\varepsilon) \in G$ is such that $(1+\varepsilon) \notin S \cap (1+\varepsilon) S^{-1}$ then $\left((\psi_j)_1 + (\psi_j)_2\right)(1+\varepsilon) = 0$ (in case $(1+\varepsilon) \notin S$ or $\left((\psi_j)_1 + (\psi_j)_2\right)\left((1+\varepsilon)^{-1}(1+\varepsilon)\right) = 0$ (in case $(1+\varepsilon) \notin (1+\varepsilon) S^{-1}$). On the other hand, if $(1+\varepsilon) \in S \cap (1+\varepsilon) S^{-1}$ then $(1+\varepsilon)$, $\left(\frac{1+\varepsilon}{\varepsilon}\right)(1+\varepsilon) \in S$, which implies that $(1+\varepsilon) \in S$ and $T^j(1+\varepsilon) T^j\left(\left(\frac{1+\varepsilon}{\varepsilon}\right)(1+\varepsilon)\right) = T^j(1+\varepsilon)$. Hence, formally

$$\begin{aligned}
&\sum_j \left(\left((\varphi_j)_1 + (\varphi_j)_2\right) T^j * \left((\psi_j)_1 + (\psi_j)_2\right) T^j\right)(1+\varepsilon) \\
&= \int_G \sum_j \left(\left((\varphi_j)_1 + (\varphi_j)_2\right) T^j\right)(1+\varepsilon)\left(\left((\psi_j)_1 + (\psi_j)_2\right) T^j\right)\left(\left(\frac{1+\varepsilon}{\varepsilon}\right)(1+\varepsilon)\right) d(1+\varepsilon) \\
&= \int_G \sum_j \left((\varphi_j)_1 + (\varphi_j)_2\right)(1+\varepsilon)\left((\psi_j)_1 + (\psi_j)_2\right)\left(\left(\frac{1+\varepsilon}{\varepsilon}\right)(1+\varepsilon)\right) \\
&\quad \times \left(T^j(1+\varepsilon) T^j\left(\left(\frac{1+\varepsilon}{\varepsilon}\right)(1+\varepsilon)\right) d(1+\varepsilon)\right) \\
&= \int_{S \cap (1+\varepsilon) S^{-1}} \sum_j \left((\varphi_j)_1 + (\varphi_j)_2\right)(1+\varepsilon)\left((\psi_j)_1 + (\psi_j)_2\right)\left(\left(\frac{1+\varepsilon}{\varepsilon}\right)(1+\varepsilon)\right) \\
&\quad \times \left(T^j(1+\varepsilon) T^j\left(\left(\frac{1+\varepsilon}{\varepsilon}\right)(1+\varepsilon)\right) d(1+\varepsilon)\right)
\end{aligned}$$

$$= \int_{S\cap(1+\varepsilon)S^{-1}} \sum_j \left((\varphi_j)_1 + (\varphi_j)_2\right)(1+\varepsilon)\left((\psi_j)_1 + (\psi_j)_2\right)\left(\left(\frac{1+\varepsilon}{\varepsilon}\right)(1+\varepsilon)\right)$$

$$\times d(1+\varepsilon)T^j(1+\varepsilon)$$

$$= \int_G \sum_j \left((\varphi_j)_1 + (\varphi_j)_2\right)(1+\varepsilon)\left((\psi_j)_1 + (\psi_j)_2\right)\left((1+\varepsilon)^{-1}(1+\varepsilon)\right)$$

$$\times d(1+\varepsilon)T^j(1+\varepsilon) \quad ■$$

$$= \sum_j \left(\left(\left((\varphi_j)_1 + (\varphi_j)_2\right) * \left((\psi_j)_1 + (\psi_j)_2\right)\right)T^j\right)(1+\varepsilon)$$

For a function $F: G \to X$ and measures μ^j on G let us abbreviate

$$\left\langle F, \sum_j \mu^j \right\rangle = \int_G F(1+\varepsilon)\sum_j \mu^j\left(d(1+\varepsilon)\right)$$

Defined in whatever weak sense. Stretch this notation to apply to all cases where it is reasonable. For example, μ^j could be a vector measure with values in X' or in $\mathcal{L}(X)$.

The reflection $F^{\sim}$ of F is defined by

$$\mathcal{F}^{\sim}: G \to X, \mathcal{F}^{\sim}(1+\varepsilon) := F\left(\left(\frac{1+\varepsilon}{\varepsilon}\right)\right).$$

If $H: G \to \mathcal{L}(X)$ is the sequence of operators-valued function, write $H * F$ for the convolution $H * F$ is defined also by (2.2). Furthermore, let

$$\left(\mu^j * F\right)(1+\varepsilon)\int_G F\left(\left(\frac{1+\varepsilon}{\varepsilon}\right)(1+\varepsilon)\right)\mu^j d(1+\varepsilon), \ \left((1+\varepsilon) \in G\right)$$

which is in coherence with the definitions above if μ^j has density and scalars are identified with their induced dilation sequence of operators. ■

The next lemma is almost a trivially.

Lemma 2.2. Let $H: G \to \mathcal{L}(X)$, $F: G \to X$ and μ^j a measure on G. Then $\left\langle H * F, \mu^j \right\rangle = \left\langle H, \mu^j * F^{\sim} \right\rangle$ formally.

Proof. Writing out the brackets into integrals, it is just Fubini's theorem:

$$\left\langle H * F, \sum_j \mu^j \right\rangle$$

$$= \int_G \int_G H(1+\varepsilon) F\left(\left(\frac{1+\varepsilon}{\varepsilon}\right)(1+\varepsilon)\right) d(1+\varepsilon) \sum_j \mu^j\left(d(1+\varepsilon)\right)$$

$$= \int_G \int_G H(1+\varepsilon) F\left(\left(1\left(\frac{1+\varepsilon}{\varepsilon}\right)\varepsilon\right)^{-1}(1+\varepsilon)\right) \sum_j \mu^j\left(d(1+\varepsilon)\right) d(1+\varepsilon) \quad ■$$

$$= \int_G H(1+\varepsilon) \int F^{\sim}\left(\left(\frac{1+\varepsilon}{\varepsilon}\right)(1+\varepsilon)\right) \sum_j \mu^j\left(d(1+\varepsilon)\right) d(1+\varepsilon)$$

$$= \int_G H(1+\varepsilon) \sum_j \left(\mu^j * F^{\sim}\right)(1+\varepsilon) d(1+\varepsilon) = \left\langle H, \sum_j \mu^j * F^{\sim} \right\rangle$$

If combine Lemmas 2.1 and 2.2 obtain the following.

Proposition 2.3. Let S be a closed sub-semigroup of G and let $T^j: S \to \mathcal{L}(X)$ be a strongly continuous representation. Let $\varphi_j, \psi_j: S \to \mathbb{C}$ and let μ^j be a measure on *S*. Then, writing $\eta := \varphi_j * \psi_j$,

$$T^j_{\eta\mu^j} = \left\langle T^j, \left(\varphi_j * \psi_j\right)\mu^j \right\rangle = \left\langle \varphi_j T^j, \mu^j * \left(\psi_j T^j\right)^{\sim} \right\rangle$$

formally. This result can be interpreted as a factorization of the sequence of operators $T^j_{\eta\mu^j}$ as

$$\begin{array}{ccc} \Phi(G;X) & \xrightarrow{L_\mu} & \Psi(G;X) \\ \iota\uparrow & & \downarrow P \\ X & \xrightarrow{T_{\eta\mu}} & X \end{array} \tag{2.3}$$

$T^j_{\eta\mu^j} = P \circ L_{\mu^j} \circ \iota$, where

1) ι maps $x \in X$ to the weighted orbit

$$(\iota x)(1+\varepsilon) := \psi_j\left(\frac{1+\varepsilon}{\varepsilon}\right) T^j\left(\frac{1+\varepsilon}{\varepsilon}\right) x \quad \left((1+\varepsilon) \in G\right)$$

2) L_{μ^j} are the convolution sequence of operators with μ^j

$$L_{\mu^j} F := \mu^j * F;$$

3) P maps an X-valued function on G back to an element of X by integrating against $\varphi_j T^j$:

$$PF = \left\langle \sum_j \varphi_j T^j, F \right\rangle = \int_G \sum_j \varphi_j (1+\varepsilon) T^j (1+\varepsilon) F(1+\varepsilon) d(1+\varepsilon),$$

4) $\Phi(G;X)$, $\Psi(G;X)$ are function spaces such that $\iota : X \to \Phi(G;X)$ and $P : \Psi(G;X) \to X$ are meaningful and bounded.

Call a factorization of the form (2.3) a transference identity. It induces a transference series estimates

$$\sum_j \left\| T^j_{\eta\mu^j} \right\|_{\mathcal{L}(X)} \le \|P\| \left\| \sum_j L_{\mu^j} \right\|_{\mathcal{L}(\Phi(G,X),\Psi(G,X))} \|\iota\| \tag{2.4}$$

3. Transference Principles for Groups

Shall explain that the classical transference principle of Berkson-Gillespie-Muhly [5] for uniformly bounded groups and the recent one for general C_0-groups [8] are instances of the explained technique (see, e.g., [1]).

3.1. Unbounded C_0-Groups

Take $G = S = \mathbb{R}$ and let $U = \left(U(1+\varepsilon)\right)_{(1+\varepsilon)\in\mathbb{R}} : \mathbb{R} \to \mathcal{L}(X)$.

Be a strongly continuous representation on the Banach space *X*. Then U is exponentially bounded, *i.e.*, its exponential type

$$\theta(U) := \left\{ \inf \varepsilon > -1 \mid \exists M \ge 0 : U(1+\varepsilon) \le M e^{(1+\varepsilon)|(1+\varepsilon)|} \left((1+\varepsilon) \in \mathbb{R}\right) \right\}$$

is finite. Choose $\beta + \varepsilon > (1+\varepsilon) > \theta(U)$ and take a measure μ^j on $\mathbb{R}$ such that

$$\left(\mu^j\right)_{(1+\varepsilon)} := \cosh\left((1+\varepsilon)\cdot\right)\mu^j \in M(\mathbb{R})$$

is a finite measure. Then $\sum_j U_{\mu^j} = \int_{\mathbb{R}}\left(\sum_j \mu^j\right)$ is well-defined. It turns out [8] that one can factorize

$$\eta = \frac{1}{\cosh\left((1+\varepsilon)\cdot\right)} = \varphi_j * \psi_j$$

where $\psi_j = 1/\cosh\left((\beta+\varepsilon)\cdot\right)$ and $\cosh\left((1+\varepsilon)\cdot\right)\varphi_j = O(1)$. Obtain $\mu^j = \eta\pi_{(1+\varepsilon)}$ and, writing $\left(\mu^j\right)_{(1+\varepsilon)}$ for μ^j in Proposition 2.3,

$$U_{\mu^j} = U_{\eta\left(\mu^j\right)_{(1+\varepsilon)}} = \varphi_j\left\langle U, \left(\mu^j\right)_{(1+\varepsilon)} * \left(\varphi_j U\right)^{\sim}\right\rangle = P \circ L_{\left(\mu^j\right)_{(1+\varepsilon)}} \circ \iota \tag{3.1}$$

If $-iA_j$ is the sequence generators of U and $f_j = F\mu^j$ is the Fourier transform of μ^j, one writes

$$\sum_j f_j\left(A_j\right) = \sum_j U_{\left(\mu^j\right)} = \int_{\mathbb{R}} U(1+\varepsilon)\sum_j \mu^j\left(d(1+\varepsilon)\right)$$

which is well-defined because the Fourier transform is injective. Applying the transference estimate (2.4) with $\phi(\mathbb{R};X) = \Psi(\mathbb{R};X) := L^{(1+\varepsilon)}(\mathbb{R};X)$ as the function spaces as in [8] leads to the series estimates

$$\sum_j \left\| f_j\left(A_j\right)\right\| \le \frac{1}{2}\left(\left\|\sum_j f_j\left(\cdot + i(1+\varepsilon)\right)\right\|_{\mathcal{M}_{(1+\varepsilon),X}(\mathbb{R})} + \left\|\sum_j f_j\left(\cdot - i(1+\varepsilon)\right)\right\|_{\mathcal{M}_{(1+\varepsilon),X}(\mathbb{R})}\right)$$

where $\mathcal{M}_{(1+\varepsilon),X}(\mathbb{R})$ denotes the space of all (scalar-valued) bounded Fourier multipliers on $L^{1+\varepsilon}(\mathbb{R};X)$. In the case that X is a UMD space one can use the Mikhlin type result for Fourier multipliers on $L^{1+\varepsilon}(\mathbb{R};X)$ to obtain a generalization of the Hieber Pruss theorem [22] to unbounded groups, see ([8], Theorem 3.6).

If $\varepsilon = 1$ and $X = H$, this Fourier multiplier norm coincides with the sup-norm by Plancherel's theorem, and by the maximum principle one obtains the H^∞-series estimates

$$\sum_j \left\| f_j\left(A_j\right)\right\| \le \left\|\sum_j f_j\right\|_{H^\infty\left(St(1+\varepsilon)\right)}, \tag{3.2}$$

where

$$St(1+\varepsilon) := \left\{z \in \mathbb{C} \mid \left|\operatorname{Im} z\right| < (1+\varepsilon)\right\}$$

Is the vertical strip of height $2(1+\varepsilon)$, symmetric about the real axis. This result is originally due to Boyadzhiev and De Laubenfels [23] and is closely related to McIntosh's theorem on H^∞-calculus for sectorial operators with bounded imaginary powers from [24], see ([8], Corollary 3.7) and ([13], Chapter 7).

3.2. Bounded Groups: The Classical Case

The classical transference principle, in the form put forward by Berkson, Gillespie and Muhly in [3] read as follows Let G be a locally compact amenable group, let $U = \left(U\left(1+\varepsilon\right)\right)_{(1+\varepsilon)\in G}$ be a uniformly bounded, strongly continuous representation of G on a Banach space X, and let $0 < \varepsilon < \infty$, $\varepsilon > 0$. Then

$$\left\| \int_G U\left(1+\varepsilon\right) \sum_j \mu^j\left(d\left(1+\varepsilon\right)\right) \right\| \le M^2 \left\| \sum_j L_{\left(\mu^j\right)} \right\|_{\mathcal{L}\left(L^{(1+\varepsilon)}(G,X)\right)}$$

for every bounded measures $\mu^j \in M(G)$. (Here $:= \sup_{(1+\varepsilon)\in G} \left\|U\left(1+\varepsilon\right)\right\|$.)

Shall review its proof in the special case of $G = \mathbb{R}$ (but the general case is analogous using Følner's condition, see ([3], p.10)). First, fix $n, N > 0$ and suppose that $\operatorname{supp}\left(\mu^j\right) \subset [-N, N]$. Then

$$\eta = \varphi_j * \psi_j = \frac{1}{2n} \mathbf{1}_{[-n,n]} * \mathbf{1}_{[-N-n,N+n]} = 1 \text{ on } [-N, N]$$

So $\eta\mu^j = \mu^j$; applying the transference estimate (2.4) with the function space $\phi(\mathbb{R}; X) = \Psi(\mathbb{R}; X) \cdot L^{(1+\varepsilon)}(\mathbb{R}; X)$ together with Holder's inequality yields

$$\begin{aligned} \sum_j \left\| T^j_{\mu^j} \right\| &\le M^2 \left\|\varphi_j\right\|_{\frac{1+\varepsilon}{\varepsilon}} \left\|\psi_j\right\|_{(1+\varepsilon)} \left\| L_{\left(\mu^j\right)} \right\|_{\mathcal{L}\left(L^{1+\varepsilon}(\mathbb{R},X)\right)} \\ &= M^2 (2n)^{-\left(\frac{1}{1+\varepsilon}\right)} (2N+2n)^{\left(\frac{1}{1+\varepsilon}\right)} \sum \left\| L_{\left(\mu^j\right)} \right\|_{\mathcal{L}\left(L^{1+\varepsilon}(\mathbb{R},X)\right)} \\ &= M^2 \left(1 + \frac{N}{n}\right)^{\left(\frac{1}{1+\varepsilon}\right)} \sum_j \left\| L_{\left(\mu^j\right)} \right\|_{\mathcal{L}\left(L^{1+\varepsilon}(\mathbb{R},X)\right)} \end{aligned}$$

Finally, let $n \to \infty$ and approximate a general $\mu^j \in M(\mathbb{R})$ by measures of finite support.

Remark 3.1. This proof shows a feature to which pointed already in the introduction, but which was not represent in the case of unbounded groups treated above. Here, an additional optimization argument appears which is based on some freedom in the choice of the auxiliary functions φ_j and ψ_j. Indeed, φ_j and ψ_j can vary as long as $\mu^j = \left(\varphi_j * \psi_j\right)\mu^j$ which amounts to $\varphi_j * \psi_j = 1$ on $\operatorname{supp}\left(\mu^j\right)$.

Remark 3.2. A transference principle for bounded cosine functions instead of groups was for the first time established and applied in [25].

4. A Transference Principle for Discrete and Continuous Operator Semigroup

Shall apply the transference method to the sequence of operators semi groups, *i.e.*, strongly continuous representations of the semigroup $\mathbb{R}_+$ (continuous case) or $\mathbb{Z}_+$ (discrete case) (see, e.g., [1]).

4.1. The Continuous Case

Let $T^j = \left(T^j\left(1+\varepsilon\right)\right)_{\varepsilon > -1}$ be strongly continuous (*i.e.* C_0-) one-parameter semigroup on a (non-trivial) Banach space X. By standard semigroup theory [12], T^j is exponentially bounded, *i.e.*, there exists $\varepsilon > -1$ such that

$$\sum_j \left\|T^j\left(1+\varepsilon\right)\right\| \le (1+\varepsilon) e^{-(1+\varepsilon)(1+\varepsilon)}$$

for all $\varepsilon > -1$. Consider complex measures μ^j on $\mathbb{R}_+ := [0,\infty)$ such that

$$\int_0^\infty \sum_j \left\|T^j(1+\varepsilon)\right\| \left|\mu^j\right| \left(d(1+\varepsilon)\right) < \infty$$

If μ^j are Laplace transformable and if $f_j = \mathcal{L}\mu^j$ are its Laplace (-Stieltjes) transform

$$\sum_j \mathcal{L}\mu^j(z) = \int_0^\infty \mathrm{e}^{-z(1+\varepsilon)} \sum_j \mu^j\left(d(1+\varepsilon)\right)$$

then use (similar to the group case) the abbreviation

$$\sum_j f_j(A_j) = \sum_j T^j_{\mu^j} = \int_0^\infty \sum_j T^j(1+\varepsilon)\mu^j\left(d(1+\varepsilon)\right)$$

where $-A_j$ are the sequence of generators of the semigroup T^j. The mapping $f_j \mapsto f_j(A_j)$ are well-defined since the Laplace transform is injective, and is called the Hille-Phillips functional calculus for A_j, see ([13], Section 3.3) and ([26], Chapter XV).

Theorem 4.1. Let $(0 < \varepsilon < \infty)$, $\varepsilon > 0$. Then there is a constant $\varepsilon > -1$ such that

$$\sum_j \left\|T^j_{\mu^j}\right\| \le (1+\varepsilon)\left(1+\log(a+\varepsilon/a)\right)(1+\varepsilon)(a+\varepsilon)^2 \left\|\sum_j L_{\mu^j}\right\|_{\mathcal{L}\left(L^{(1+\varepsilon)}(\mathbb{R},X)\right)} \tag{4.1}$$

whenever the following hypotheses are satisfied:

1) $T^j = \left(T^j(1+\varepsilon)\right)_{\varepsilon > -1}$ is a C_0-semigroup on the Banach space X;

2) $0 < a < a+\varepsilon < \infty$; $\varepsilon > 0$;

3) $(1+\varepsilon)(a+\varepsilon) := \sup_{0 \le (1+\varepsilon) \le a+\varepsilon} \sum_j \left\|T^j(1+\varepsilon)\right\|$;

4) $\mu^j \in (1+\varepsilon)(\mathbb{R}_+)$ such that $\operatorname{supp}(\mu^j) \subset [a, a+\varepsilon]$.

Proof. Take $\varphi_j \in L^{\frac{\varepsilon+1}{\varepsilon}}(0, a+\varepsilon)$, $\psi_j \in L^{1+\varepsilon}(0, a+\varepsilon)$ such that $\varphi_j * \psi_j = 1$ on $[a, a+\varepsilon]$, and let $\eta := \varphi_j * \psi_j$. Then $\eta\mu^j = \mu^j$ and Proposition 2.3 yields

$$T^j_{\mu^j} = T^j_{\eta\mu^j} = \left\langle \varphi_j T^j, \mu^j * \left(\psi_j T^j\right)^{\sim} \right\rangle.$$

Holder's inequality leads to series norm estimates

$$\sum_j \left\|T^j_{\mu^j}\right\| \le (1+\varepsilon)(a+\varepsilon)^2 \left\|\sum_j \varphi_j\right\|_{\frac{1+\varepsilon}{\varepsilon}} \left\|\psi_j\right\|_{1+\varepsilon} \left\|L_{(\mu^j)}\right\|_{\mathcal{L}\left(L^{1+\varepsilon}(\mathbb{R},X)\right)}$$

Hence, to prove the theorem it suffices to show that

$$\begin{aligned} C(a, a+\varepsilon) &= \inf\left\{ \left\|\varphi_j\right\|_{\frac{1+\varepsilon}{\varepsilon}} \left\|\psi_j\right\|_{1+\varepsilon} : \varphi_j, \psi_j = 1 \text{ on } [a, a+\varepsilon] \right\} \\ &\le (1+\varepsilon)\log(1+a+\varepsilon/a) \end{aligned}$$

with $(1+\varepsilon)$ independent of a and $a+\varepsilon$. This is done in Lemma A.1. ■

Remarks 4.2. The conclusion of the theorem is also true in the case $\varepsilon = 0$ or $1+\varepsilon = \infty$, but in this case

$$\sum_j \left\| L_{(\mu^j)} \right\|_{\mathcal{L}\left(L^{1+\varepsilon}(\mathbb{R},X)\right)} = \sum_j \left\| \mu^j \right\|_{M(\mathbb{R}_+)}$$

is just the total variation norm of μ^j. And clearly

$$\sum_j \left\| T^j_{\mu^j} \right\| \le (1+\varepsilon)(a+\varepsilon) \left\| \sum_j \mu^j \right\|_M$$

which is stronger than (4.1).

2) In functional calculus terms, (4.1) takes the form

$$\sum_j \left\| f_j\left(A_j\right) \right\| \le (1+\varepsilon)\left(1+\log\left(a+\varepsilon/a\right)\right) M\left(a+\varepsilon\right)^2 \left\| \sum_j f_j \right\|_{\mathcal{AM}_{(1+\varepsilon),X}(\mathbb{C}_+)}$$

where $f_j = \mathcal{L}\mu^j$ and

$$\mathcal{AM}_{(1+\varepsilon),X}\left(\mathbb{C}_+\right) := \left\{ f_j \in H^\infty\left(\mathbb{C}_+\right) \mid f_j\left(i\,\cdot\right) \in \mathcal{M}_{(1+\varepsilon),X}\left(\mathbb{R}\right) \right\}$$

is the (scalar) analytic $L^{1+\varepsilon}(\mathbb{R};X)$-Fourier multiplier algebra, endowed with the series norms

$$\sum_j \left\| f_j \right\|_{\mathcal{AM}_{(1+\varepsilon),X}(\mathbb{C}_+)} := \sum_j \left\| f_j\left(i\,\cdot\right) \right\|_{\mathcal{M}_{(1+\varepsilon),X}(\mathbb{R})}$$

Let us state a corollary for semigroups with polynomial growth type.

Corollary 4.3. Let ($0<\varepsilon<\infty$), $\varepsilon>0$. Then there is a constant $\varepsilon>-1$ such that the following is true. If $-A_j$ sequence of generates a C_0-semigroup $T^j = \left(T^j\left(1+s\right)\right)_{\varepsilon>-1}$ on a Banach space X such that there is $\varepsilon>-1$, $\varepsilon>-\beta$ with

$$\sum_j \left\| T^j\left(1+\varepsilon\right) \right\| \le (1+\varepsilon)(1+s)^{\beta+\varepsilon}, (\varepsilon>-1)$$

Then

$$\begin{aligned} \sum_j \left\| f_j\left(A_j\right) \right\| &\le (1+\varepsilon)(1+\varepsilon)^2 \left(1+\left(a+\varepsilon\right)\right)^{2(\beta+\varepsilon)} \\ &\times \left(1+\log\left(a+\frac{\varepsilon}{a}\right)\right) \left\| \sum_j f_j \right\|_{\mathcal{AM}_{(1+\varepsilon),X}(\mathbb{C}_+)} \end{aligned} \tag{4.2}$$

for $0<a<a+\varepsilon<\infty$, $f_j = \mathcal{L}\mu^j$ and $\mu^j \in (1+\varepsilon)\left[a, a+\varepsilon\right]$.

The case that $\varepsilon=-\beta$, *i.e.*, the case of a bounded semigroup, is particularly important, hence state it separately.

Corollary 4.4. Let $\left(0<\varepsilon<\infty\right)$, $\varepsilon>0$. Then there is a constant $\varepsilon>-1$ such that the following is true. If $-A_j$ sequence of generates uniformly bounded C_0-semigroup $T^j = \left(T^j\left(1+\varepsilon\right)\right)_{\varepsilon>-1}$ on a Banach space X then, with

$$M := \sup_{\varepsilon>-1} \sum_j \left\| T^j\left(1+\varepsilon\right) \right\|$$

$$\sum_j \left\| f_j\left(A_j\right) \right\| \le (1+\varepsilon) M^2 \left(1+\log\left(a+\varepsilon/a\right)\right) \left\| \sum_j f_j \right\|_{\mathcal{AM}_{(1+\varepsilon),X}(\mathbb{C}_+)} \tag{4.3}$$

for $0<a<a+\varepsilon<\infty$, $f_j = \mathcal{L}\mu^j$ and $\mu^j \in M\left[a, a+\varepsilon\right]$.

Remark 4.5. If $X=H$ is a Hilbert space and $\varepsilon=1$, by Plancherel's theorem and the maximum principle, Equation (4.3) becomes

$$\sum_j \left\| f_j\left(A_j\right) \right\| \lesssim M^2 \left(1+\log\left(a+\varepsilon/a\right)\right) \left\| \sum_j f_j \right\|_{H^\infty(\mathbb{C}_+)} \tag{4.4}$$

where $f_j = \mathcal{L}\mu^j$ is the Laplace-Stieltjes transform of μ^j. A similar estimate has been established by Vitse ([18], Lemma 1.5) on a general Banach space *X*, but with the semigroup being holomorphic and bounded on a sector.

4.2. The Discrete Case

Turn to the situation of a discrete operator semigroup *i.e.*, the powers of a bounded operator. Let $T^j \in \mathcal{L}(X)$ be bounded sequence of operators and $T^j = \left(\left(T^j\right)^n\right)_{n\in\mathbb{Z}_+}$ the corresponding semigroup representation. If $\mu^j \in \ell^1(\mathbb{Z}_+)$ is such that $\sum_{n=0}^{\infty}\sum_j \left|\mu^j(n)\right| \left\|\left(T^j\right)^n\right\| < \infty$ then (2.1) takes the form such that

$$\sum_j T^j_{\mu^j} = \sum_{n=0}^{\infty}\sum_j \mu^j(n)\left(T^j\right)^n$$

Denoting $\widehat{\mu^j}(z) := \sum_{n=0}^{\infty}\mu^j(n)z^n$ for $|z|\le 1$ also write $\widehat{\mu^j}\left(T^j\right) := T^j_{\mu^j}$.

Theorem 4.6. Let $(0<\varepsilon<\infty)$, $\varepsilon>0$. Then there is a constant $\varepsilon>-1$ such that

$$\sum_j \left\|\widehat{\mu^j}\left(T^j\right)\right\| \le (1+\varepsilon)\left(1+\log(a+\varepsilon/a)\right)M(a+\varepsilon)^2\left\|\sum_j L_{\mu^j}\right\|_{\mathcal{L}\left(L^{1+\varepsilon}(\mathbb{Z},X)\right)}$$

whenever the following hypotheses are satisfied:

1) T^j are bounded sequence of operators on a Banach space *X*;
2) $a, a+\varepsilon \in \mathbb{Z}_+$ with $\varepsilon>0$;
3) $M(a+\varepsilon) := \sup_{0\le n\le a+\varepsilon}\sum_j \left\|\left(T^j\right)^n\right\|$;
4) $\mu^j \in \ell^1(\mathbb{Z}_+)$ such that $\operatorname{supp}\left(\mu^j\right)\subset[a, a+\varepsilon]$.

Proof. This is completely analogous to the continuous situation. Take $\varphi_j \in \ell^{\left(\frac{1+\varepsilon}{\varepsilon}\right)}(\mathbb{Z}_+)$, $\psi_j \in \ell^{1+\varepsilon}(\mathbb{Z}_+)$ such that $\varphi_j * \psi_j = 1$ on $[a, a+\varepsilon]$, and let $\eta := \varphi_j * \psi_j$. Then $\eta\mu^j = \mu^j$ and Proposition 2.3 yields

$$\widehat{\mu^j}\left(T^j\right) = T^j_{\mu^j} = T^j_{\eta\mu^j} = \left\langle \varphi_j, T^j\mu^j * \left(\psi_j T^j\right)^{\sim}\right\rangle.$$

Holder's inequality leads to series norms estimate

$$\sum_j T^j_{\mu^j} \le M(a+\varepsilon)^2\left\|\sum_j \varphi_j\right\|_{\frac{1+\varepsilon}{\varepsilon}}\left\|\psi_j\right\|_{1+\varepsilon}\sum\left\|L_{\mu^j}\right\|_{\mathcal{L}\left(l^{1+\varepsilon}(\mathbb{Z},X)\right)}$$

So, similar to the continuous case, one is interested in estimating

$$c(a, a+\varepsilon) = \inf\left\{\sum_j \left\|\varphi_j\right\|_{\frac{1+\varepsilon}{\varepsilon}}\left\|\psi_j\right\|_{1+\varepsilon} : \varphi_j \in L^{\frac{1+\varepsilon}{\varepsilon}}(\mathbb{Z}_+), \psi_j \in L^{1+\varepsilon}(\mathbb{Z}_+),\right.$$
$$\left.\varphi_j * \psi_j = 1 \text{ on } [a, a+\varepsilon]\right\}$$

Applying Lemma A.2 concludes the proof. ■

Remarks 4.7. As in the continuous case, the assertion remains true for $\varepsilon = 0, \infty$, but is weaker than the obvious series estimates $\sum_j \left\|\widehat{\mu^j}\right\| \le M(a+\varepsilon)\left\|\sum_j \widehat{\mu^j}\right\|_{\ell^1}$

2) If write $f_j = \widehat{\mu^j}$, (4.5) takes the form

$$\sum_j \left\| f_j\left(T^j\right)\right\| \le (1+\varepsilon)\left(1+\log\left(a+\varepsilon/a\right)\right) M\left(a+\varepsilon\right)^2 \left\|\sum_j f_j\right\|_{\mathcal{AM}_{(1+\varepsilon),X}(\mathbb{D})}$$

Here

$$\mathcal{AM}_{(1+\varepsilon),X}(\mathbb{D}) := \left\{ f_j \in \mathcal{H}_\infty(\mathbb{D}) \ \left(f_j\right)_{|\mathbb{T}} \in \mathcal{M}_{(1+\varepsilon),X}(\mathbb{T}) \right\}$$

is the (scalar) analytic $L^{1+\varepsilon}(\mathbb{Z};X)$-Fourier multiplier algebra, endowed with theorem

$$\sum_j \left\| f_j \right\|_{\mathcal{AM}_{(1+\varepsilon),X(\mathbb{D})}} = \left\| \sum_j \left(f_j\right)_{|\mathbb{T}} \right\|_{\mathcal{M}_{(1+\varepsilon),X(\mathbb{T})}}$$

Similar to the continuous case state sequence for operators with polynomially growing powers.

Corollary 4.8. Let $(0<\varepsilon<\infty)$, $\varepsilon>0$. Then there is a constant $\varepsilon>-1$ such that the following is true. If T^j are bounded sequence of operators on a Banach space X such that there is $\varepsilon>-1$ with

$$\sum_j \left\| \left(T^j\right)^n \right\| \le (1+\varepsilon)(1+n)^{\beta+\varepsilon}, (n \ge 0),$$

then

$$\sum_j \left\| f_j\left(T^j\right)\right\| \le (1+\varepsilon)^2 (1+n)^{2(\beta+\varepsilon)} \left(1+\log\left(a+\frac{\varepsilon}{a}\right)\right) \left\| \sum_j f_j \right\|_{\mathcal{AM}_{(1+\varepsilon),X}(\mathbb{D})} \tag{4.6}$$

for $f_j = \widehat{\mu^j}$, $\mu^j \in \ell^1\left(\left[a, a+\varepsilon\right] \cap \mathbb{Z}\right)$, $a, a+\varepsilon \in \mathbb{Z}$ with $1 \le a \le a+\varepsilon$.

Remark 4.9. For the applications to Peller's theorem in the next section the extract asymptotics of $c(a,a+\varepsilon)$ is irrelevant, and one can obtain an effective estimate with much less effort. In the continuous case, the identity

$$c(a,a+\varepsilon) = c(1, a+\varepsilon/a)$$

already shows that $c(a,a+\varepsilon)$ only depends on $(a+\varepsilon/a)$. For the special choice of functions

$$\varphi_j = \mathbf{1}_{[0,1]},\ \psi_j = \mathbf{1}_{[0,a+\varepsilon]}$$

one has $\sum_j \left\|\varphi_j\right\|_{\frac{1+\varepsilon}{\varepsilon}} = 1$ and $\sum_j \left\|\varphi_j\right\|_{1+\varepsilon} = (a+\varepsilon)^{\left(\frac{1}{\varepsilon+1}\right)}$ and symmetrizing yields

$$\left\| f_j\left(A_j\right)\right\| (1+\varepsilon)(a+\varepsilon)^2 (a+\varepsilon/a)^{1/\max\left(1+\varepsilon, \frac{1+\varepsilon}{\varepsilon}\right)} \left\| \sum_j f_j \right\|_{\mathcal{AM}_{(1+\varepsilon),X}(\mathbb{C})}$$

In the discrete case take η as in the proof of Lemma A.2 and factorize

$$\hat{\eta} = \widehat{\varphi_j} \cdot \widehat{\psi_j} = \frac{1-z^a}{1-z} \cdot \frac{z}{a(1-z)}$$

Then

$$\sum_j \left\| \left(\varphi_j\right) \mathbf{1}_{[0,a+\varepsilon]} \right\|_{\left(\frac{1+\varepsilon}{\varepsilon}\right)}^{\left(\frac{1+\varepsilon}{\varepsilon}\right)} = a \quad \text{and} \quad \sum_j \left\| \left(\psi_j\right) \mathbf{1}_{[0,a+\varepsilon]} \right\|_{1+\varepsilon}^{1+\varepsilon} = \left(a+\varepsilon/a^{(1+\varepsilon)}\right) \quad \text{hence}$$

$$c(a,a+\varepsilon) \le \left\| \sum_j (\varphi_j) \mathbf{1}_{[0,a+\varepsilon]} \right\|_{\frac{1+\varepsilon}{\varepsilon}} \left\| (\psi_j) \mathbf{1}_{[0,a+\varepsilon]} \right\|_{1+\varepsilon}$$

$$= a^{\left(\frac{\varepsilon}{1+\varepsilon}\right)} (a+\varepsilon)^{\frac{1}{\varepsilon+1}} a^{-1} = (a+\varepsilon/a)^{\frac{1}{\varepsilon+1}}$$

$$\sum_j \left\| f_j(A_j) \right\| \le (1+\varepsilon)^2 (a+\varepsilon/a)^{1/\max\left(1+\varepsilon,\frac{1+\varepsilon}{\varepsilon}\right)} \left\| \sum f_j \right\|_{\mathcal{AM}_{(1+\varepsilon),X}(\mathbb{D})}$$

similar to the continuous case.

5. Peller's Theorems

The results can be used to obtain a new proof of some classical results of Peller's about Besov class functional calculi for bounded Hilbert space operators with polynomially growing powers from [16]. In providing the necessary notions essentially follow Peller's original work, changing the notation slightly (cf. also [17]) (see, e.g., [1]).

For an integer $n \ge 1$ let

$$(\varphi_j)_n(k) = \begin{cases} 0, & k \le 2^{n-1} \\ \frac{1}{2^{n-1}} \cdot (k - 2^{n-1}), & 2^{n-1} \le k \le 2^n \\ \frac{1}{2^n} \cdot (2^{n+1} - k), & 2^n \le k \le 2^{n+1} \\ 0, & 2^{n+1} \le k \end{cases}$$

That is, $(\varphi_j)_n$ are supported in $[2^{n-1}, 2^{n+1}]$, zero at the endpoints, $(\varphi_j)_n(2n) = 1$ and linear on all of the intervals $[2^{n-1}, 2^n]$ and $[2n, 2^{n+1}]$. Let $(\varphi_j)_0 := (1,1,0,\cdots)$, then

$$\sum_{n=0}^{\infty} \sum_j (\varphi_j)_n = \mathbf{1}_{\mathbb{Z}_+}$$

the sum being locally finite. For $\varepsilon > -1$ the Besov class $B_{n=0}^{(1+\varepsilon)}(\mathbb{D})$ is defined as the class of analytic functions f_j on the unit disc $\mathbb{D}$ satisfying

$$\sum_j \left\| f_j \right\|_{B_{\infty,1}^{(1+\varepsilon)}} = \sum_{n=0}^{\infty} 2^{n(1+\varepsilon)} \sum \left\| \widehat{(\varphi_j)_n} * f_j \right\|_{H^\infty(\mathbb{D})} < \infty$$

That is, if $f_j = \sum_{k \ge 0} (\beta+\varepsilon)_k z^k, \beta+\varepsilon = \left((\beta+\varepsilon)_k\right)_{k \ge 0}$, then

$$\sum_j \left\| f_j \right\|_{B_{\infty,1}^{(1+\varepsilon)}} = \sum_{n=0}^{\infty} 2^{n(1+\varepsilon)} \sum_j \left\| \widehat{(\varphi_j)_n} (\beta+\varepsilon) \right\|_{H^\infty(\mathbb{D})} < \infty$$

Following Peller ([16], p.347), one has

$$f_j \in B_{\infty,1}^{(1+\varepsilon)}(\mathbb{D}) \Leftrightarrow \int_0^1 (1-r)^{m-\varepsilon-2} \sum_j \left\| f_j^m \right\|_{L^\infty(r\mathbb{T})} \mathrm{d}r < \infty$$

where m is an arbitrary integer such that $m > (1+\varepsilon)$. Since only consider $\varepsilon > -1$, have

$$B_{\infty,1}^{(1+\varepsilon)}(\mathbb{D}) \subseteq H^\infty(\mathbb{D})$$

and it is known that $B_{\infty,1}^{(1+\varepsilon)}(\mathbb{D})$ is a Banach algebra in which the set of polynomials is dense. The following is essentially ([16], p.354, bottom line); give a new proof.

Theorem 5.1. (Peller 1982). There exists a constant $\varepsilon > -1$ such that the following holds: Let X be a Hilbert space, and let $T^j \in \mathcal{L}(X)$ such that

$$\sum_j \left\| \left(T^j\right)^n \right\| \le M(1+n)^{\beta+\varepsilon} \quad (n \ge 0)$$

with $\varepsilon > -\beta$ and $\varepsilon > -1$. Then

$$\sum_j \left\| f_j\left(T^j\right) \right\| \le (1+\varepsilon) 9^{(\beta+\varepsilon)} (1+\varepsilon)^2 \left\| \sum_j f_j \right\|_{B_{\infty,1}^{2(\beta+\varepsilon)}(\mathbb{D})}$$

for every polynomial f_j.

Proof. Let $f_j = \hat{v} = \sum_{k \ge 0} v_n z^n$, and v has finite support. If $n \ge 1$, then $(\varphi_j)_n v$ has support in $\left[2^{n-1}, 2^{n+1}\right]$, so can apply Corollary 4.8 with $\varepsilon = 1$ to obtain

$$\sum_j \left\| \widehat{(\varphi_j)_n v}\left(T^j\right) \right\| \le (1+\varepsilon)_2 (1+\varepsilon)^2 \left(1+2^{n+1}\right)^{2(\beta+\varepsilon)} (1+\log 4) \sum_j \left\| \sum_j \widehat{(\varphi_j)_n v} \right\|_{\mathcal{AM}_{2,X(\mathbb{D})}}$$

Since X is a Hilbert space, Plancherel's theorem (and standard Hardy space theory) yields that $\mathcal{AM}_{2,X(\mathbb{D})} = H^\infty(\mathbb{D})$ with equal norms. Moreover,

$$1 + 2^{n+1} \le 3 \cdot 2^n,$$

and hence obtain

$$\sum_j \left\| \widehat{(\varphi_j)_n v}\left(T^j\right) \right\| \le (1+\varepsilon)_2 9^{(\beta+\varepsilon)} (1+\varepsilon)^2 \cdot 2^{n(2(\beta+\varepsilon))} \left\| \sum_j \widehat{(\varphi_j)_n v} \right\|_{H^\infty(\mathbb{D})}$$

Summing up, arrive at

$$\begin{aligned} \sum_j \left\| f_j\left(T^j\right) \right\| &\le \sum_{n \ge 0} \left\| \sum_j \widehat{(\varphi_j)_n v}\left(T^j\right) \right\| \\ &\le |\upsilon_0| + |v_1| (1+\varepsilon) 2^{(\beta+\varepsilon)} + (1+\varepsilon)_2 9^{(\beta+\varepsilon)} (1+\varepsilon)^2 \sum_{n \ge 1} 2^{n(2(\beta+\varepsilon))} \\ \sum_j \left\| \widehat{(\varphi_j)_n v} \right\|_\infty &\le (1+\varepsilon) 9^{(\beta+\varepsilon)} M^2 \sum_j \left\| \sum_j f_j \right\|_{B_{\infty,1}^{2(\beta+\varepsilon)}(\mathbb{D})} \end{aligned}$$

For some constant $\varepsilon > -1$. ■

Remark 5.2. N. Nikolski has observed that Peller's Theorem 5.1 is only interesting if $(\beta+\varepsilon) \le 1/2$. Indeed, define

$$\left(A_j\right)_{(\beta+\varepsilon)}(\mathbb{D}) = \left\{ f_j = \sum_{k \ge 0} a_k z^k \mid \left\| f_j \right\|_{(A_j)_{(\beta+\varepsilon)}} = \sum_{k \ge 0} |a_k| (1+k)^{(\beta+\varepsilon)} < \infty \right\}$$

Then $\left(A_j\right)_{(\beta+\varepsilon)_j}(\mathbb{D})$ is a Banach algebra with respect to the norm $\|.\|_{(A_j)_{(\beta+\varepsilon)}}$ and one has the obvious series estimates

$$\sum_j \left\| f_j\left(T^j\right) \right\| \le (1+\varepsilon) \left\| \sum_j f_j \right\|_{(A_j)_{(\beta+\varepsilon)}} \quad \left(f_j \in \left(A_j\right)_{(\beta+\varepsilon)}(\mathbb{D}) \right)$$

If $\left(T^j\right)^k \le (1+\varepsilon)(1+k)^{(\beta+\varepsilon)}$, $k \in \mathbb{N}$. This is the "trivial" functional calculus for T^j mentioned in the Introduction, see (1.4). For $f_j \in B_{\infty,1}^{\beta+\varepsilon+1/2}(\mathbb{D})$ have

$$\sum_j \left\|f_j\right\|_{(A_j)_{(\beta+\varepsilon)}} = |a_0| + \sum_{k\ge 0} 2^{(k+1)(\beta+\varepsilon)}$$
$$\le |a_0| + \sum_{k\ge 0} 2^{(k+1)(\beta+\varepsilon)} \sum_{2^k \le n < 2^{k+1}} |a_n|$$
$$\le |a_0| + \sum_{k\ge 0} 2^{(k+1)(\beta+\varepsilon)} 2^{k/2} \left(\sum_{2^k \le n < 2^{k+1}} |a_n|^2 \right)^{1/2}$$
$$\le |a_0| + \sum_{k\ge 0} 2^{(\beta+\varepsilon)} 2^{(\beta+\varepsilon+1/2)k} \sum_j \left(\left\| \widehat{(\varphi_j)_{k-1}} * f_j \right\|_2 + \left\| \widehat{(\varphi_j)_k} * f_j \right\|_2 + \left\| \widehat{(\varphi_j)_{k+1}} * f_j \right\|_2 \right)$$
$$\lesssim \sum_{k=0}^{\infty} 2^{(\beta+\varepsilon+1/2)k} \sum_j \left\| \widehat{(\varphi_j)_k} * f_j \right\|_\infty = \left\| \sum_j f_j \right\|_{B_{\infty,1}^{(\beta+\varepsilon+1/2)/2}}$$

by the Cauchy-Schwarz inequality, Plancherel's theorem and the fact that $H^\infty(\mathbb{D}) \subset H^2(\mathbb{D})$. This shows that $B_{\infty,1}^{(\beta+\varepsilon)+1/2}(\mathbb{D}) \subseteq (A_j)_{(\beta+\varepsilon)}(\mathbb{D})$. Hence, if $\varepsilon \ge 1/2 - \beta$, then $2(\beta+\varepsilon) \ge (\beta+\varepsilon+1)/2$, and therefore

$$B_{\infty,1}^{2(\beta+\varepsilon)}(\mathbb{D}) \subseteq B_{\infty,1}^{(\beta+\varepsilon)+1/2}(\mathbb{D}) \subseteq (A_j)_{(\beta+\varepsilon)}(\mathbb{D}),$$

and the Besov calculus is weaker than the trivial $(A_j)_{(\beta+\varepsilon)}$-calculus.

On the other hand, for $\varepsilon > -\beta$, the example

$$\sum_j f_j(z) = \sum_{n=0}^{\infty} 2^{-2(\beta+\varepsilon)n} z^{2^n} \in (A_j)_{(\beta+\varepsilon)}(\mathbb{D}) / B_{\infty,1}^{2(\beta+\varepsilon)}(\mathbb{D})$$

shows that $(A_j)_{(\beta+\varepsilon)}(\mathbb{D})$ is not included into $B_{\infty,1}^{2(\beta+\varepsilon)}(\mathbb{D})$, and so the Besov calculus does not cover the trivial calculus. (By a straightforward argument one obtains the embedding $(A_j)_{(\beta+\varepsilon)}(\mathbb{D}) \subseteq B_{\infty,1}^{(\beta+\varepsilon)}(\mathbb{D})$.

5.1. An Analogue in the Continuous Case

Peller's theorem has an analogue for continuous one-parameter semigroups. The role of the unit disc $\mathbb{D}$ is taken by the right half-plane $\mathbb{C}_+$, the power-series representation of a function on $\mathbb{D}$ is replaced by a Laplace transform representation of a function on $\mathbb{D}$. However, a subtlety appears that is not present in the discrete case, namely the possibility (or even necessity) to consider also dyadic decompositions "at zero". This leads to so-called "homogeneous" Besov spaces, but due to the special form of the estimate (4.2) we have to treat the decomposition at 0 different from the decomposition at ∞.

To be more precise, consider the partition of unity

$$(\varphi_j)_n(k) = \begin{cases} 0, & 0 \le (1+\varepsilon) \le 2^{n-1} \\ \dfrac{1}{2^{n-1}} \cdot \left((1+\varepsilon) - 2^{n-1}\right), & 2^{n-1} \le (1+\varepsilon) \le 2^n \\ \dfrac{1}{2^n} \cdot \left(2^{n+1} - (1+\varepsilon)\right), & 2^n \le (1+\varepsilon) \le 2^{n+1} \\ 0, & 2^{n+1} \le (1+\varepsilon) \end{cases}$$

for $n \in \mathbb{Z}$. Then $\sum_{n\in\mathbb{Z}}(\varphi_j)_n = \mathbf{1}_{(0,\infty)}$, the sum being locally finite in $(0,\infty)$. For $\varepsilon > -1$, an analytic functions $f_j : \mathbb{C}_+ \to \mathbb{C}$ is in the (mixed-order homogeneous) Besov space $B^{0,(1+\varepsilon)}_{\infty,1}(\mathbb{C}_+)$ if $f_j(\infty) = \lim_{(1+\varepsilon)\to\infty} f_j(1+\varepsilon)$ exists and

$$\sum_j \|f_j\|_{B^{0,(1+\varepsilon)}_{\infty,1}} = \sum_j |f_j(\infty)| + \sum_{n<0}\sum_j \left\|\mathcal{L}(\varphi_j)_n * f_j\right\|_{H^\infty(\mathbb{C}_+)} + \sum_{n\ge 0} 2^{n(1+\varepsilon)} \sum_j \left\|\mathcal{L}(\varphi_j)_n * f_j\right\|_{H^\infty(\mathbb{C}_+)} < \infty$$

Here $\mathcal{L}$ denotes (as before) the Laplace transform

$$\sum_j \mathcal{L}(\varphi_j)(z) = \int_0^\infty \mathrm{e}^{-(1+\varepsilon)z} \sum_j (\varphi_j)(1+\varepsilon)\, d(1+\varepsilon) \qquad (\operatorname{Re} z > 0)$$

Since dealing with $\varepsilon > -1$ only, it is obvious that $B^{0,(1+\varepsilon)}_{\infty,1}(\mathbb{C}_+) \subseteq H^\infty(\mathbb{C}_+)$.

Clearly, definition of $B^{0,(1+\varepsilon)}_{\infty,1}(\mathbb{C}_+)$ is a little sloppy, and to make it rigorous would need to employ the theory of Laplace transforms of distributions. However, shall not need that here, because shall use only functions of the form $f_j = \mathcal{L}\mu^j$, where μ^j are bounded measure with compact support in $[0,\infty]$. In this case

$$\sum_j \mathcal{L}(\varphi_j)_n * f_j \sum_j \mathcal{L}(\varphi_j)_n * \mathcal{L}\mu^j = \sum_j \mathcal{L}\left((\varphi_j)_n \mu^j\right)$$

by a simple computation.

Theorem 5.3. There is an absolute constant $\varepsilon > -1$ such that the following holds: Let X be a Hilbert space, and let $-A_j$ be the sequence of generators of a strongly continuous semigroup $T^j = \left(T^j(1+\varepsilon)\right)_{(\varepsilon+1)\in\mathbb{R}_+}$ on X such that

$$\sum_j \left\|T^j(\varepsilon+1)\right\|(1+\varepsilon)(1+s)^{\beta+\varepsilon}, \ (n \ge 0)$$

with $\varepsilon > -\beta$ and $\varepsilon > -1$. Then

$$\sum_j \left\|f_j(A_j)\right\| \le (1+\varepsilon)9^{(\beta+\varepsilon)}(1+\varepsilon)^2 \left\|\sum_j f_j\right\|_{B^{0,2(\beta+\varepsilon)}_{\infty,1}(\mathbb{C}_+)}$$

for every $f_j = \mathcal{L}\mu^j$, μ^j being a bounded measure on $\mathbb{R}_+$ of compact support.

Proof. The proof is analogous to the proof of Theorem 5.1. One has

$$\mu^j = f_j(\infty)\delta_0 + \sum_{n<0}(\varphi_j)_n \mu^j + \sum_{n\ge 0}(\varphi_j)_n \mu^j$$

where the first series converges in $(1+\varepsilon)$ [0, 1] and the second is actually finite. Hence

$$\begin{aligned}
\sum_j \left\|f_j(A_j)\right\| &\le \sum_j |f_j(\infty)| + \sum_j\sum_{n\in\mathbb{Z}} \left\|\left[\mathcal{L}\left((\varphi_j)_n \mu^j\right)\right](A_j)\right\| \\
&\lesssim \left|\sum_j f_j(\infty)\right| + \sum_{n\in\mathbb{Z}} (1+\varepsilon)^2 \left(1+2^{n+1}\right)^{2(\beta+\varepsilon)} \sum_j \left\|\mathcal{L}(\varphi_j)_n * f_j\right\|_{\mathcal{L}(L^2(\mathbb{R},X))} \\
&= \left|\sum_j f_j(\infty)\right| + \sum_{n\in\mathbb{Z}} (1+\varepsilon)^2 \left(1+2^{n+1}\right)^{2(\beta+\varepsilon)} \sum_j \left\|\mathcal{L}(\varphi_j)_n * f_j\right\|_{H^\infty(\mathbb{C}_+)} \\
&\lesssim \sum_j |f_j(\infty)| + (1+\varepsilon)^2 \sum_{n\in\mathbb{Z}} \left(1+2^{n+1}\right)^{2(\beta+\varepsilon)} \sum_j \left\|\mathcal{L}(\varphi_j)_n * f_j\right\|_{H^\infty(\mathbb{C}_+)}
\end{aligned}$$

$$+(1+\varepsilon)^2 \sum_{n\geq 0}\left(3\cdot 2^n\right)^{2(\beta+\varepsilon)} \sum_j \left\| \mathcal{L}\left(\varphi_j\right)_n * f_j \right\|_{H^\infty(\mathbb{C}_+)}$$

$$\leq (1+\varepsilon)^2\, 9^{(\beta+\varepsilon)} \left\| \sum_j f_j \right\|_{B^{0,2(\beta+\varepsilon)}_{\infty,1}}$$

by Plancherel's theorem and Corollary 4.3. ■

Remark 5.4. The space $B^{0,0}_{\infty,1}(\mathbb{C}_+)$ has been considered by Vitse in [18] under the name $B^0_{\infty,1}(\mathbb{C}_+)$, and refer to that section for more information. In particular, Vitse proves that $f_j \in B^{0,0}_{\infty,1}(\mathbb{C}_+)$ if and only if $f_j \in H^\infty(\mathbb{C}_+)$ and

$$\int_0^\infty \sup_{(1+\varepsilon)\in\mathbb{R}} \sum_j \left| f_j'\left((1+\varepsilon)+i(1+\varepsilon)\right)\right| d(1+\varepsilon) < \infty$$

Let us formulate the special case $\varepsilon = -\beta$ as a corollary, with a slight generalization.

Corollary 5.5. There is a constant $\varepsilon > -1$ such that the following is true. Whenever $-A_j$ the sequence of generates a strongly continuous semigroup $\left(T^j(1+\varepsilon)\right)_{\varepsilon>-1}$ on a Hilbert space such that $\left\|\sum_j T^j(1+\varepsilon)\right\| \leq M$ for all $\varepsilon > -1$, then

$$\sum_j \left\| f_j\left(A_j\right)\right\| \leq (1+\varepsilon) M^2 \left\|\sum_j f_j\right\|_{B^{0,0}_{0,1}(\mathbb{C}_+)} \tag{5.1}$$

for all $f_j = \mathcal{L}\mu^j$, $\mu^j \in M(\mathbb{R}_+)$.

Proof. It is easy to see that the Laplace transform $\mathcal{L}: M(\mathbb{R}_+) \to B^{0,0}_{0,1}(\mathbb{C}_+)$ is bounded. Since (5.1) is true for measures with compact support and such measures are dense in $M(\mathbb{R}_+)$, a approximation argument proves the claim. ■

Remarks 5.6.

1) Vitse ([18], Introduction, p.248) in a short note suggests to prove corollary 5.5 by a discretization argument using Peller's Theorem 5.1 or $\varepsilon = \beta$. This is quite plausible. But no details are given in [18] and it seems that further work is required to make this approach rigorous.

2) (Cf. Remark 4.9.) To prove Theorem 5.1 and 5.3 did not make full use of the logarithmic factor $\log\left(1+(a+\varepsilon)/a\right)$ but only of the fact that it is constant in n if $[a, a+\varepsilon] = \left[2^{n-1}, 2^{n+1}\right]$. However, as Vitse notes in ([18], Remark 4.2), the logarithmic factor appears a fortiori, indeed. If $\operatorname{supp}\mu^j \subseteq [a, a+\varepsilon]$ then if we write

$$\sum_j \mu^j = \sum_j \sum_{n\in\mathbb{Z}} \left(\varphi_j\right)_n \mu^j$$

the number $N = card\left\{n \in \mathbb{Z} \mid \left(\varphi_j\right)_n \mu^j \neq 0\right\}$ of non-zero terms in the sum is proportional to $\log\left(1+(a+\varepsilon)/a\right)$. Hence, for the purposes of functional calculus estimates neither Lemma A.1 nor A.2 is necessary.

3) (Cf. Remark 5.2.) Different to the discrete case, the Besov estimates are not completely uninteresting in the case $\varepsilon > (1/2-\beta)$, because $(\beta+\varepsilon)$ affects only the decomposition at ∞.

5.2. Generalizations for UMD Spaces

The proofs of Peller's theorems use essentially that the underlying space is a Hilbert space. Indeed, have applied Plancherel's theorem in order to estimate the Fourier multiplier norm of a function by its L^∞-norm Hence do not expect Peller's theorem to be avoid on other Banach spaces without modifications. Show that replacing ordinary boundedness of sequence of operators families by the so-called γ-boundedness, Peller's theorems carry over to arbitrary Banach spaces. Here suggest a different path, namely to replace the algebra $H^\infty(\mathbb{C}_+)$ in the construction of the Besov $B^{0,(1+\varepsilon)}_{\infty,1}$ by the analytic multiplier algebra $\mathcal{AM}_{(\varepsilon+1),X}(\mathbb{C}_+)$, introduced in Remark 4.2(2).

To simplify notation, let us abbreviate $\mathcal{A}_{(1+\varepsilon)} = \mathcal{AM}_{(\varepsilon+1),X}(\mathbb{C}_+)$. For $\varepsilon > -1$ and $f_j : \mathbb{C}_+ \to \mathbb{C}$ we say $f_j \in B_1^{0,(1+\varepsilon)}\left[\mathcal{A}_{(1+\varepsilon)}\right]$ if $f_j \in H^\infty(\mathbb{C}_+)$, $f_j(\infty) = \lim_{(1+\varepsilon)\to\infty} f_j(1+\varepsilon)$ exists and

$$\sum_j \left\|f_j\right\|_{B_1^{0,(1+\varepsilon)}\left[\mathcal{A}_{(1+\varepsilon)}\right]} = \sum_j \left|f_j(\infty)\right| + \sum_{n<0}\sum_j \left\|\mathcal{L}(\varphi_j)_n * f_j\right\|_{\mathcal{A}_{(1+\varepsilon)}} + \sum_{n\ge 0} 2^{n(1+\varepsilon)} \sum_j \left\|\mathcal{L}(\varphi_j)_n * f_j\right\|_{\mathcal{A}_{(1+\varepsilon)}} < \infty$$

Then the following analogue of Theorem 3.5 holds, with a similar proof.

Theorem 5.7. $(0<\varepsilon<\infty), \varepsilon > 0$. Then there is a constant $\varepsilon > -1$ such that the following holds: Let $-A_j$ be the sequence of generators of a strongly continuous semigroup $T^j = \left(T^j(1+\varepsilon)\right)_{(1+\varepsilon)\in\mathbb{R}_+}$ on a Banach space X such that

$$\sum_j \left\|T^j(1+\varepsilon)\right\| \le (1+\varepsilon)(2+\varepsilon)^{\beta+\varepsilon} \quad (n \ge 0)$$

with $\varepsilon > -\beta$ and $\varepsilon > 0$. Then

$$\sum_j \left\|f_j(A_j)\right\| \le (1+\varepsilon) 9^{(\beta+\varepsilon)} (1+\varepsilon)^2 \left\|\sum_j f_j\right\|_{B_1^{0,2(\beta+\varepsilon)}\left[\mathcal{M}_{(1+\varepsilon)}\right]}$$

for every $f_j = \mathcal{L}\mu^j$, μ^j bounded measure on $\mathbb{R}_+$ of compact support.

For $X = H$ is a Hilbert space and $\varepsilon = 1$ one is back at Theorem 5.3. For special cases of X-typically if X is an L^1—or a $C(K)$-space—one has

$$B_1^{0,0}\left[M_{(1+\varepsilon)}\right] = (1+\varepsilon)(\mathbb{R}_+).$$

But if X is a UMD space, one has positive results. To formulate them let

$$B_1^\infty(\mathbb{C}_+) := \left\{f_j \in H^\infty(\mathbb{C}_+) \mid z f_j'(z) \in H^\infty(\mathbb{C}_+)\right\}$$

be the analytic Mikhlin algebra. This is a Banach algebra with respect to the series norms

$$\sum_j \left\|f_j\right\|_{H_1^\infty} = \sup_{z\in\mathbb{C}_+} \sum_j \left|f_j(z)\right| + \sup_{z\in\mathbb{C}_+} \sum_j \left|z f_j'(z)\right|$$

If X is a UMD space then the vector-valued version of the Mikhlin theorem ([13], Theorem E.6.2) implies that one has a continuous inclusion

$$H_1^\infty(\mathbb{C}_+) \subseteq \mathcal{AM}_{(\varepsilon+1),X}(\mathbb{C}_+)$$

where the embedding constant depends on $(\varepsilon+1)$ and (the UMD constant of) X. If one defines $B_1^{0,(1+\varepsilon)}\left[H_1^\infty\right]$ analogously to $B_1^{0,(1+\varepsilon)}\left[\mathcal{M}_{(1+\varepsilon)}\right]$ above, then obtain the following.

Corollary 5.8. If X is a UMD space, then Theorem 5.7 is still valid when $\mathcal{AM}_{(\varepsilon+1),X}(\mathbb{C}_+)$ is replaced by $H_1^\infty(\mathbb{C}_+)$ and the constant $(1+\varepsilon)$ is allowed to depend on (the UMD-constant of) X.

Fix $\theta\in(\pi/2,\pi)$ and consider the sector

$$\Sigma_\theta := \left\{z\in C\setminus\{0\} \mid |\arg z|<\theta\right\}.$$

Then $H^\infty(\Sigma_\theta)\subset H_1^\infty(\mathbb{C}_+)$, as follows from an application of the Cauchy integral formula, see [13]. Hence, if define $B_{\infty,1}^{0,(1+\varepsilon)}(\Sigma_\theta)$ by replacing the space $H^\infty(\mathbb{C}_+)$ in the definition of $H_1^\infty(\mathbb{C}_+)$ by $H(\Sigma_\theta)$.

Corollary 5.9. Let $\theta\in(\pi/2,\pi)$, let X be a UMD space, and let $(0<\varepsilon<\infty)$, $\varepsilon>0$. Then there is a constant $(1+\varepsilon)=c(\theta,X,1+\varepsilon)$ such that the following holds. Let $-A_j$ be the sequence of generators of a strongly continuous semigroup $T^j=T^j(1+\varepsilon)_{(1+\varepsilon)\in\mathbb{R}_+}$ on X such that

$$\sum_j\left\|T^j(1+\varepsilon)\right\|\le(1+\varepsilon)(1+s)^{(\beta+\varepsilon)}\quad(n\ge 0)$$

with $\varepsilon>-\beta$ and $\varepsilon>0$. Then

$$\sum_j\left\|f_j(A_j)\right\|\le(1+\varepsilon)9^{(\beta+s)}(1+\varepsilon)^2\left\|\sum_j f_j\right\|_{B_{\infty,1}^{0,2(\beta+s)}(\Sigma_\theta)}$$

for every $f_j=\mathcal{L}\mu^j$, μ^j a bounded measure on $\mathbb{R}_+$ of compact support.

Note that Theorem 5.3 above simply says that if X is a Hilbert space. One can choose $\theta=\pi/2$ in Corollary 5.9.

Remark 5.10. It is natural to ask whether $B_1^{0,(1+s)}\left[H_1^\infty\right]$ or $B_{\infty,1}^{0,2(\beta+\varepsilon)}(\Sigma_\theta)$ are actually Banach space algebras. This is probably not true, as the underlying Banach algebras $H_1^\infty(\mathbb{C}_+)$ and $H^\infty(\Sigma_\theta)$ are not true invariant under shifting along imaging axis, and hence are not $L^1(\mathbb{R})$-convolution modules.

6. Generalizations Involving γ-Boundedness

Discuss one possible generalization of Peller's theorem, involving still an assumption on the Banach space and a modification of the Besov algebra, but no additional assumption on the semigroup. Here follow a different path, strengthening the requirements on the semigroups under consideration. Vitse has shown in [17] [18] that the Peller-type results remain true without any restriction on the Banach space if the semigroup is bounded analytic (in the continuous case), or the sequence of operators is a Tadmor-Ritt operator (in the discrete case). (These two situations correspond to each other in a certain sense, see e.g. ([13], Section 9.2.4)).

The approach here is based on the ground-breaking work of Kalton and Weis of recent years, involving the concept of γ-boundedness. This is a stronger notion of boundedness of a set of the sequence of operators between two Banach spaces in. The "philosophy" of the Kalton-Weis approach is that every Hilbert space theorem which rests on Plancherel's theorem (and no other result specific for Hilbert spaces) can be transformed into a theorem on general Banach spaces, when the sequence of operators norm boundedness is replaced by γ-boundedness.

The idea is readily sketched. In the proof of Theorem 5.3 used the transference identity (2.3) with the function space $L^2(\mathbb{R};X)$ and factorized the sequence of operators $T^j_{\mu^j}$ over the Fourier multiplier L_{μ^j}. If X is a Hilbert space, the 2-Fourier multiplier norm of L_{μ^j} are just $\left\|\mathcal{L}\mu^j\right\|_\infty$ and this led to the Besov class estimate. Replace the function space $L^2(\mathbb{R};X)$ by the space $\gamma(\mathbb{R};X)$; in order to make sure that the transference identity (2.3) remains valid, need that the embedding ι and the projection P from (2.3) are well defined. And this is where the concept of γ-boundedness comes in. Once have established the transference identity, can pass to the transference estimate; and since $L^\infty(\mathbb{R})$ is also the Fourier multiplier algebra of $\gamma(\mathbb{R};X)$, recover the infinity norm as in the $L^2(\mathbb{R};H)$-case from above.

Shall pass to more rigorous mathematics, starting with a (very brief) introduction to the theory of γ-spaces. For a deeper account refer to [27] (see, e.g. [1]).

6.1. γ-Summing and γ-Radonifying Operators

Let H be a Hilbert space and X a Banach space. The sequence of operators $T^j : H \to X$ is called γ-summing if

$$\sum_j \left\|T^j\right\|_\gamma = \sup_F \mathbb{E}\sum_j \left(\left\|\sum_{e\in F}\gamma_e \otimes T^j e\right\|_X^2\right)^{1/2} < \infty$$

where the supremum is taken over all finite orthonormal systems $F \subseteq H$ and $(\gamma_e)_{e\in F}$ is an independent collection of standard Gaussian random variables on some probability space. It can be shown that in this definition it suffices to consider only finite subsets F of some fixed orthonormal basis of H. Let

$$\gamma_\infty(H;X) := \left\{T^j : H \to X \mid T^j \text{ is } \gamma\text{-summing}\right\}$$

the space of γ-summing sequence of operators of H into X. This is a Banach space with respect to the norm $\|\cdot\|_\gamma$. The closure in $\gamma_\infty(H;X)$ of the space of finite rank sequence of operators are denoted by $\gamma(H;X)$, and its elements $T^j \in \gamma(H;X)$ are called γ-radonifying. By a theorem of Hoffman-Jørgensen and Kwapie', if X does not contain c_0 then $\gamma(H;X) = \gamma_\infty(H;X)$, see ([27], Thm.6.2).

From the definition of the γ-norm the following important ideal property of the γ-spaces is quite straightforward [27].

Lemma 6.1. (Ideal Property). Let Y be another Banach space and K another Hilbert space, let $L : X \to Y$ and $R : K \to H$ be bounded linear sequence of operators, and let $T^j \in \gamma_\infty(H;X)$. Then

$$LT^jR \in \gamma_\infty(K,Y) \text{ and } \sum_j \left\|LT^jR\right\|_\gamma \le \|L\|_{\mathcal{L}(X,Y)} \left\|\sum_j T^j\right\|_\gamma \|R\|_{\mathcal{L}(K,H)}$$

If $T^j \in \gamma(H,X)$, then $LT^jR \in \gamma(K,Y)$.

If $g^j \in H$ we abbreviate $\overline{g^j} := \langle \cdot, g^j \rangle$, *i.e.*, $g^j \to \overline{g^j}$ s the canonical conjugate-linear bijection of H onto its dual $\overline{H}$. Every finite rank sequence of operators $T^j : H \to X$ has the form

$$\sum_j T^j = \sum_{j=1}^{n} \sum_j \overline{\left(g^j\right)_n} \otimes x_j$$

and one can view $\gamma(H;X)$ as a completion of the algebraic tensor product $\overline{H} \otimes X$ with respect to the γ-series norms. Since

$$\sum_j \left\| \overline{g^j} \otimes x \right\|_\gamma = \sum_j \left\| g^j \right\|_H \left\| x \right\|_X = \sum_j \left\| \overline{g^j} \right\|_{\overline{H}} \left\| x \right\|_X$$

for every $g^j \in H$, $x \in X$, the γ-series norms are cross-norm. Hence every nuclear the sequence of operators $T^j : H \to X$ is γ-radonifying and $\sum_j \left\| T^j \right\|_\gamma \le \left\| \sum_j T^j \right\|_{\text{nuc}}$. (Recall that T^j are nuclear the sequence operators if $T^j = \sum_{n\ge 0} \overline{\left(g^j\right)_n} \otimes X_n$ for some $\left(g^j\right)_n \in H$, $x_n \in X$ with $\sum_{n\ge 0} \sum_j \left\| \left(g^j\right)_n \right\|_H \left\| x_n \right\|_X < \infty$). The following application turns out to be quite useful.

Lemma 6.2. Let *H*, *X* as before, and let $\left(\Omega, \Sigma, \mu^j\right)$ be a measure space. Suppose that $f_j : \Omega \to H$ and $g^j : \Omega \to X$ are (strongly) μ^j-measurable and

$$\int_\Omega \sum_j \left\| f_j(1+\varepsilon) \right\|_H \left\| g^j(1+\varepsilon) \right\|_X \mu^j\left(d(1+\varepsilon)\right) < \infty$$

Then $\widehat{f_j} \otimes g^j \in L^1\left(\Omega, \gamma(H,X)\right)$ and

$$\sum_j T^j = \int_\Omega \sum_j \left(\widehat{f_j} \otimes g^j \right) d\left(\mu^j\right) \in \gamma(H,X)$$

satisfies

$$T^j h = \int_\Omega \sum_j \left\langle h, f_j(1+\varepsilon) \right\rangle g^j(1+\varepsilon) \mu^j\left(d(1+\varepsilon)\right) \quad (h \in H)$$

And

$$\sum_j \left\| T^j \right\|_\gamma \le \int_\Omega \left\| (1+\varepsilon) \right\|_H \sum_j \left\| g^j(1+\varepsilon) \right\|_X \mu^j\left(d(1+\varepsilon)\right)$$

Suppose that $H = L^2\left(\Omega, \Sigma, \mu^j\right)$ for some measure space $\left(\Omega, \Sigma, \mu^j\right)$. Every function $u \in L^2(\Omega; X)$ defines the sequence of operators T_u^j: $L^2(\Omega) \to X$ by integration:

$$T_u^j = L^2(\Omega) \to X, \ \sum_j T_u^j(h) = \int_\Omega \sum_j h(u) d\left(\mu^j\right)$$

(Actually, one can do this under weaker hypotheses on *u*, but shall have no occasion to use the more general version.) Identify the operator T_u^j with the function u and write $u \in \gamma(\Omega; X)$ in place of $T_u^j \in \gamma\left(L^2(\Omega); X\right)$.

Extending an idea of ([19], Remark 3.1) can use Lemma 6.2 to conclude that certain vector-valued functions define γ-radonifying operators. Note that $a=-\infty$ or $a+\varepsilon=\infty$ are allowed, moreover employ the convention that $\infty\cdot 0=0$.

Corollary 6.3. Let $(a,a+\varepsilon)\subseteq\mathbb{R}$, let $u\in W^{1,1}_{loc}((a,a+\varepsilon);X)$ and let $\varphi_j:(a,a+\varepsilon)\to\mathbb{C}$. Suppose that one of the following two conditions are satisfied:

1) $\sum_j\|\varphi_j\|_{L^2(a,a+\varepsilon)}\|u(a)\|_X<\infty$ and $\int_a^{a+\varepsilon}\sum_j\|\varphi_j\|_{L^2(1+\varepsilon,a+\varepsilon)}\left\|\acute{u}(1+\varepsilon)\right\|_X d(1+\varepsilon)<\infty$

2) $\sum_j\|\varphi_j\|_{L^2(a,a+\varepsilon)}\|u(a+\varepsilon)\|_X<\infty$ and $\int_a^{a+\varepsilon}\sum_j\|\varphi_j\|_{L^2(a,(1+\varepsilon))}\left\|\acute{u}(1+\varepsilon)\right\|_X d(1+\varepsilon)<\infty$

Then $\varphi_j\cdot u\in\gamma((a,a+\varepsilon);X)$ with respective estimates for $\sum_j\|\varphi_j\cdot u\|_\gamma$.

Proof. In case 1) use the representation

$$u(1+\varepsilon)=u(a)+\int_a^{(1+\varepsilon)}\acute{u}(1+\varepsilon)\,d(1+\varepsilon)$$

Leading to

$$\sum_j(\varphi_j\cdot u)=\sum_j\varphi_j\otimes u(a)+\int_a^{a+\varepsilon}\mathbf{1}_{(1+\varepsilon,a+\varepsilon)}\sum_j\left(\varphi_j\otimes\acute{u}\right)(1+\varepsilon)\,d(1+\varepsilon)$$

Then apply Lemma 6.2. In case 2) start with

$$u(1+\varepsilon)=u(a+\varepsilon)+\int_{(1+\varepsilon)}^{a+\varepsilon}\acute{u}\left(d(1+\varepsilon)\right)$$

and proceed similarly. ■

The space $\gamma(L^2(\Omega);X)$ can be viewed as space of generalized X-valued functions on Ω. Indeed, if $\Omega=\mathbb{R}$ with the Lebesgue measure, $\gamma_\infty(L^2(\mathbb{R});X)$ is a Banach space of X-valued tempered distributions. For such distributions their Fourier transform is coherently defined via its adjoint action: $\mathcal{F}T^j:=T^j\circ\mathcal{F}$, and the ideal property mentioned above shows that $\mathcal{F}$ restricts to almost isometric isomorphisms of $\gamma_\infty(L^2(\mathbb{R});X)$ and $\gamma(L^2(\mathbb{R});X)$. Similarly, the multiplication with some function $m\in L^\infty(\mathbb{R})$ extends via adjoint action coherently to $L(L^2(\mathbb{R});X)$, and the ideal property above yields that $\gamma_\infty(L^2(\mathbb{R});X)$ and $\gamma(L^2(\mathbb{R});X)$ are invariant. Furthermore,

$$\sum_j\left\|T^j\mapsto mT^j\right\|_{\gamma_\infty\to\gamma_\infty}=\|m\|_\infty$$

for every $m\in L^\infty(\mathbb{R})$. Combining these two facts obtain that for each $m\in L^\infty(\mathbb{R})$ the Fourier multiplier sequence of operators with symbol m

$$F_m(T^j):=\mathcal{F}^{-1}(m\mathcal{F}T^j)\left(T^j\in\mathcal{L}(L^2(\mathbb{R});X)\right)$$

is bounded on $\gamma_\infty(L^2(\mathbb{R});X)$ and $\gamma(L^2(\mathbb{R});X)$ with series norms estimates

$$\sum_j\left\|F_m(T^j)\right\|_\gamma\le\|m\|_{L^\infty(\mathbb{R})}\left\|\sum_j T^j\right\|_\gamma$$

Similar remarks apply in the discrete case $\Omega=\mathbb{Z}$.

An important result in the theory of γ-radonifying sequence of operators is the multiplier theorem. Here one considers a bounded sequence of operators-valued functions $T^j:\Omega\to\mathcal{L}(X;Y)$ and asks under what conditions the multiplier sequence of operators

$$\mathcal{M}_{T^j}:L^2(\Omega;X)\to L^2(\Omega;Y),\ \mathcal{M}_{T^j}f_j=T^j(\cdot)f_j(\cdot)$$

are bounded for the γ-norms. To formulate the result, one needs new notion.

Let X, Y be Banach spaces. Collections $\mathcal{T}^j\subset\mathcal{L}(X;Y)$ is said to be γ-bounded if there is a constant $\varepsilon>-1$ such that

$$\sum_j\mathbb{E}\left(\left\|\sum_{T^j\in\mathcal{T}'^j}\gamma_{T^j}T^jx_{T^j}\right\|_X^2\right)^{1/2}\le(1+\varepsilon)\sum_j\mathbb{E}\left(\left\|\sum_{T^j\in\mathcal{T}'^j}\gamma_{T^j}x_{T^j}\right\|_X^2\right)^{1/2}\tag{6.1}$$

for all finite subsets $\mathcal{T}'^j\subseteq\mathcal{T}^j$, $\left(x_{T^j}\right)_{T^j\in\mathcal{T}'^j}\subset X$. (Again, $\left(\gamma_{T^j}\right)_{T^j\in\mathcal{T}'^j}$ is an independent collection of standard Gaussian random variables on some probability space.) If $\mathcal{T}^j$ is γ-bounded, the smallest constant c such that (6.1) holds, is denoted by $\gamma\left(\mathcal{T}^j\right)$ and is called the γ-bound of $\mathcal{T}^j$. Ready to state the result, established by Kalton and Weis in [19].

Theorem 6.4 (Multiplier theorem). Let $H=L^2(\Omega)$ for some measure space $\left(\Omega,\Sigma,\mu^j\right)$ and let X, Y be Banach spaces. Let $T^j:\Omega\to\mathcal{L}(X;Y)$ be a strongly μ^j-measurable mapping such that

$$\mathcal{T}^j:=\left\{T^j(1+\varepsilon)\mid(1+\varepsilon)\in\Omega\right\}$$

is γ-bounded. Then the multiplication sequence of operators

$$\mathcal{M}_{T^j}:L^2(\Omega)\otimes X\to L^2(\Omega;Y),\ \ f_j\otimes x\mapsto f_j(\cdot)T^j(\cdot)x$$

extends uniquely to a bounded sequence of operators

$$\mathcal{M}_{T^j}:\gamma\left(L^2(\Omega);X\right)\to\gamma^\infty\left(L^2(\Omega);Y\right)$$

with

$$\sum_j\left\|\mathcal{M}_{T^j}S\right\|_\gamma\le\gamma\sum_j\left(\mathcal{T}^j\right)\|S\|_\gamma,\left(S\in\gamma\left(L^2(\Omega);X\right)\right).$$

It is unknown up to now whether such a multiplier $\mathcal{M}_{T^j}$ always must have its range in the smaller class $\gamma\left(L^2(\Omega);Y\right)$.

6.2. Unbounded C_0-Groups

Have applied the transference identities to unbounded C_0-groups in Banach spaces. In the case of a Hilbert space this yielded a proof of the Boyadzhiev-de Laubenfels theorem, *i.e.*, that all sequence of generators of a C_0-group on a Hilbert space has bounded H^∞-calculus on vertical strips, if the strip height exceeds the exponential type of the group. The analogue of this result for general Banach spaces but under γ-boundedness conditions is due to Kalton and Weis ([19], Thm.6.8). Give a new proof using the transference techniques (see, e.g., [1]).

Recall that the exponential type of a C_0-group on a Banach space X is

$$\theta(U):=\inf\left\{\varepsilon>-1\exists\mid\varepsilon>-1:\left\|U(1+\varepsilon)\right\|\le(1+\varepsilon)\mathrm{e}^{(1+\varepsilon)|(1+\varepsilon)|}\left((1+\varepsilon)\in\mathbb{R}\right)\right\}.$$

Let us call the number

$$\theta_\gamma(U) := \inf\left\{\varepsilon > -1 \mid \left\{e^{-(1+\varepsilon)|(1+\varepsilon)|}U(1+\varepsilon) \mid (1+\varepsilon)\in\mathbb{R}\right\} \text{ is } \gamma\text{-bounded}\right\}$$

the exponential γ-type of the group U. If $\theta_\gamma(U) < \infty$ call U exponentially γ-bounded. The following is the γ-analogue of the Boyadzhiev de-Laubenfels theorem, see Equation (3.2).

Theorem 6.5. (Kalton-Weis). Let $-iA_j$ be the sequence of generators of a C_0-group $(U(1+\varepsilon))_{(1+\varepsilon)\in\mathbb{R}}$ on a Banach space X. Suppose that U is exponentially γ-bounded. Then A_j has a bounded $H^\infty(St(1+\varepsilon))$-calculus for every $(1+\varepsilon) > \theta_\gamma(U)$.

Proof. Choose $\theta_\gamma(U) < (1+\varepsilon) < (\beta+\varepsilon)$. By usual approximation techniques ([8], Proof of Theorem 3.6) it suffices to show an estimate

$$\sum_j \left\|f_j(A_j)\right\| \lesssim \left\|\sum_j f_j\right\|_{H^\infty(St(1+\varepsilon))}$$

only for $f_j = \mathcal{F}\mu^j$ with μ^j a measure such that $(\mu^j)_{(1+\varepsilon)} \in (1+\varepsilon)(\mathbb{R})$. (Recall the $(\mu^j)_{(1+\varepsilon)}(d(1+\varepsilon)) = \cosh((1+\varepsilon)(1+\varepsilon))\mu^j(d(1+\varepsilon))$, so that $f_j = \mathcal{F}\mu^j$ has a bounded holomorphic extension to $St(1+\varepsilon)$). By the transference identity (3.1) the sequence of operators $f_j(A_j)$ factorizes as

$$f_j(A_j) = P \circ L_{(\mu^j)_{(1+\varepsilon)}} \circ \iota .$$

Here $L_{(\mu^j)_{(1+\varepsilon)}}$ is convolution with $(\mu^j)_{(1+\varepsilon)}$,

$$lx(1+\varepsilon) = \frac{1}{\cosh(\beta+\varepsilon)(1+\varepsilon)}U(-(1+\varepsilon))x \qquad (x\in X, (1+\varepsilon)\in\mathbb{R})$$

and

$$PF = \int_{\mathbb{R}}\sum_j \psi_j(1+\varepsilon)U(1+\varepsilon)F(1+\varepsilon)d(1+\varepsilon)$$

this factorization was considered to go via the space $L^2(\mathbb{R};X)$, *i.e.*,

$$\iota : X \to L^2(\mathbb{R};X), P : L^2(\mathbb{R};X) \to X.$$

However, the exponential γ-boundedness of U will allow us to replace the space $L^2(\mathbb{R};X)$ by $\gamma(L(\mathbb{R});X)$. Once this is ensured, the estimate is immediate, since convolution with $(\mu^j)_{(1+\varepsilon)}$ are the Fourier multiplier with symbol $\mathcal{F}(\mu^j)_{(1+\varepsilon)}$. Know that this is bounded on $\gamma(L^2(\mathbb{R});X)$ with a norm not exceeding $\sum_j \left\|\mathcal{F}(\mu^j)_{(1+\varepsilon)}\right\|_{L^\infty(\mathbb{R})}$, which by elementary computations and the maximum principle can be majorized by $\sum_j \left\|\mathcal{F}\mu^j\right\|_{H^\infty(st(1+\varepsilon))}$.

To see that indeed $\iota : X \to \gamma(L^2(\mathbb{R});X)$, write

9)
$$(1+\varepsilon)=\frac{1}{\cosh(\beta+\varepsilon)(1+\varepsilon)}(-U(1+\varepsilon))x$$
$$=\left(\mathrm{e}^{-(1+\varepsilon)|(1+\varepsilon)|}U(-(1+\varepsilon))\right)\left(\frac{\mathrm{e}^{(1+\varepsilon)|(1+\varepsilon)|}}{\cosh(\beta+\varepsilon)(1+\varepsilon)}x\right)$$

and use the Multiplier Theorem 6.4 to conclude that $\iota : X \to \gamma_\infty(\mathbb{R};X)$ is bounded. To see that $ran(\iota)\subset\gamma(\mathbb{R};X)$ we employ a density argument. If $x\in \mathrm{dom}(A_j)$, write $\iota x=\psi_j\cdot u$ with $\psi_j(1+\varepsilon)=\cosh((\beta+\varepsilon)(1+\varepsilon))-1$ and $u(1+\varepsilon)=U(-(1+\varepsilon))x((1+\varepsilon)\in\mathbb{R})$.

Then $u\in C^1(\mathbb{R};X), u'(1+\varepsilon)=iU(-(1+\varepsilon))A_jx, \psi_j\in L^2(\mathbb{R})$, and,

$$\int_0^\infty\sum_j\|\psi_j\|_{L^2((1+\varepsilon),\infty)}\left\|\acute{u}(1+\varepsilon)\right\|_X d(1+\varepsilon),$$
$$\int_{-\infty}^0\sum_j\|\psi_j\|_{L^2(-\infty,(1+\varepsilon))}\left\|\acute{u}(1+\varepsilon)\right\|_X d(1+\varepsilon)<\infty$$

Hence,

$$lx=\sum_j\psi_j\cdot u=\sum_j\psi_j\otimes x+\int_0^\infty\mathbf{1}_{(-\infty,(1+\varepsilon))}\sum_j\left(\psi_j\otimes\acute{u}\right)(1+\varepsilon)d(1+\varepsilon)$$
$$-\int_{-\infty}^0\mathbf{1}_{(-\infty,(1+\varepsilon))}\sum_j\psi_j\otimes\acute{u}(1+\varepsilon)d(1+\varepsilon)\in\gamma(\mathbb{R},X)$$

by Corollary 6.3 (One has to apply (1)) to the part of $\varphi_j u$ on $\mathbb{R}_+$ and (2) (to the part on $\mathbb{R}_-$) Since $\mathrm{dom}(A_j)$ is dense in *X*, conclude that $ran(\iota)\subseteq\gamma(L^2(\mathbb{R});X)$ as claimed.

Finally, show that $P:\gamma(L^2(\mathbb{R});X)\to X$ is well-defined. Clearly

$$P=\left(\text{integrate against }\mathrm{e}^{\theta|(1+\varepsilon)|}\varphi_j(1+\varepsilon)\right)\circ\left(\text{multiply with }\mathrm{e}^{-\theta|(1+\varepsilon)|}U(1+\varepsilon)\right)$$

where $\theta_\gamma(U)<\theta<(1+\varepsilon)$. Know that $\varphi_j(1+\varepsilon)=O\left(\mathrm{e}^{-(1+\varepsilon)|(1+\varepsilon)|}\right)$, so by the Multiplier Theorem 6.4, every things works out fine. Note that in order to be able to apply the multiplier theorem, have to start already in $\gamma(L^2(\mathbb{R});X)$. And this is why had to ensure that ι maps there in the first place. ■

Remark 6.6. Independently of us, Le Merdy [20] has recently obtained a γ-version of the classical transference principle for bounded groups. The method is similar, by re-reading the transference principle with the γ-space in place of a Bochner space.

6.3. Peller's Theorm-γ-Version, Discrete Case

Turn to the extension of Peller's theorems from Hilbert spaces to general spaces. Begin with the discrete case.

Theorem 6.7. There is an absolute constant $\varepsilon>-1$ such that the following holds: Let *X* be a Banach space, and let $T^j\in\mathcal{L}(X)$ such that the set

$$\mathcal{T}^j:=\left\{(1+n)^{(\beta+\varepsilon)}\left(\left(T^j\right)^n\right)\mid n\geq 0\right\}$$

is γ-bounded. Then

$$\sum_j \left\| f_j\left(T^j\right) \right\| \le (1+\varepsilon) 9^{(\beta+\varepsilon)} \gamma\left(T^j\right)^2 \left\| \sum_j f_j \right\|_{B^{2(\beta+\varepsilon)}_{\infty,0}(\mathbb{D})}$$

for every polynomial f_j.

The theorem is a consequence of the following lemma, the arguments being completely analogous to the proof of Theorem 5.1.

Lemma 6.8. There is a constant $\varepsilon > -1$ such that

$$\sum_j \left\| \widehat{\mu^j}\left(T^j\right) \right\| \le (1+\varepsilon)\left(1+\log(a+\varepsilon/a)\right) M(a+\varepsilon) \left\| \sum_j \widehat{\mu^j} \right\|_{H^\infty(\mathbb{D})}$$

whenever the following hypotheses are satisfied:

1) T^j are bounded sequence of operators on a Banach space *X*;

2) $a, a+\varepsilon \in \mathbb{Z}$ with $1 \le a \le a+\varepsilon$;

3) $M(a+\varepsilon) := \gamma\left\{\left(T^j\right)^n \mid 0 \le n \le a+\varepsilon\right\}$;

4) $\widehat{\mu^j} \in \ell^1(\mathbb{Z}_+)$ such that $\operatorname{supp}\left(\mu^j\right) \subset [a, ba+\varepsilon]$.

Proof. This is analogous to Theorem 4.6 Take $\varphi_j, \psi_j \in L^2(\mathbb{Z}_+)$ such that $\psi_j * \varphi_j = 1$ on $[a, a+\varepsilon]$ and $\operatorname{supp}\varphi_j, \operatorname{supp}\psi_j \subset [0, a+\varepsilon]$. Then

$$\widehat{\mu^j}\left(T^j\right) = \varphi_j T^j, \mu^j * \left(\psi_j T^j\right)^{\sim} = P \circ L_{\mu^j} \circ \iota,$$

see (2.3). Note that only functions of finite support are involved here, so $ran(\iota) \subset L^2(\mathbb{Z}) \otimes X$. Hence can take γ-norms and estimate

$$\sum_j \left\| \widehat{\mu^j}\left(T^j\right) \right\| \le \|P\|_{\gamma\left(L^2(\mathbb{Z}), X\right) \to X} \left\| \sum_j L_{\mu^j} \right\|_{\gamma \to \gamma} \|\iota\|_{X \to \gamma\left(L^2(\mathbb{Z}), X\right)}$$

Note that

$$\iota x = \left(T^j\right)^{\sim} \mathbf{1}_{[-a+\varepsilon, 0]} \left(\left(\psi_j\right)^{\sim} \otimes x\right)$$

so the multiplier theorem yields

$$\|\iota x\|_\gamma \le M(a+\varepsilon) \sum_j \left\| \left(\psi_j\right)^{\sim} \otimes x \right\|_\gamma = M(a+\varepsilon) \sum_j \left\|\psi_j\right\|_2 \|x\|$$

Similarly, P can be decomposed as

$$P = \left(\text{integrate against } \varphi_j\right) \circ \left(\text{multiply with } \mathbf{1}_{[0, a+\varepsilon]} T^j\right)$$

and hence the multiplier theorem yields

$$\|P\|_{\gamma \to X} \le \sum_j \left\|\varphi_j\right\|_2 M(a+\varepsilon)$$

Finally note that

$$\sum_j \left\| L_{\mu^j} \right\|_{\gamma \to \gamma} = \sum_j \left\| \widehat{\mu^j} \right\|_{H^\infty(\mathbb{D})}$$

since—similar to the continuous case—all bounded measurable functions on $\mathbb{T}$ define bounded Fourier multipliers on $\gamma\left(L^2(\mathbb{Z}); X\right)$. Putting the pieces together obtain

$$\sum_j \left\| \widehat{\mu^j}\left(T^j\right) \right\| \le M(a+\varepsilon)^2 \left\| \sum_j \varphi_j \right\|^2 \left\|\psi_j\right\|^2 \left\| \widehat{\mu^j} \right\|_{H^\infty(\mathbb{D})}$$

and an application of Lemma A.2 concludes the proof. ■

6.4. Peller's Theorem-γ-Version, Continuous Case

Turn to the continuous version(s) of Peller's theorem.

Theorem 6.9. There is an absolute constant $\varepsilon > -1$ such that the following holds: Let $-A_j$ be the sequence of generators of a strongly continuous semigroup $T^j = \left(T^j\left(1+\varepsilon\right)\right)_{\varepsilon \geq -1}$ on a Banach space *X*. Suppose that $\varepsilon > -\beta$ is such that the set

$$\mathcal{T}^j := \left\{\left(2+\varepsilon\right)^{-(\beta+\varepsilon)} T^j\left(1+\varepsilon\right) \mid \varepsilon > -1\right\}$$

is γ-bounded. Then

$$\sum_j \left\| f_j\left(A_j\right) \right\| \leq \left(1+\varepsilon\right) 9^{(\beta+\varepsilon)} \gamma\left(\mathcal{T}^j\right)^2 \left\| \sum_j f_j \right\|_{B^{0,2(\beta+\varepsilon)}_{\infty,1}(\mathbb{C}_+)}$$

for every $f_j = \mathcal{L}\mu^j$, $\mu^j \in M\left(\mathbb{R}_+\right)$ with compact support.

Corollary 6.10. There is an absolute constant $\varepsilon > -1$ such that the following holds: Let $-A_j$ be the sequence of generators of a strongly continuous semigroup $T^j = \left(T^j\left(1+\varepsilon\right)\right)_{\varepsilon > -1}$ on a Banach space X such that

$$\mathcal{T}^j := \left\{T^j\left(1+\varepsilon\right) \mid \varepsilon > -1\right\}$$

is γ-bounded. Then

$$\sum_j \left\| f_j\left(A_j\right) \right\| \leq \left(1+\varepsilon\right) \gamma\left(\mathcal{T}^j\right)^2 \left\| \sum_j f_j \right\|_{B^{0,0}_{\infty,1}(\mathbb{C}_+)}$$

for every $f_j = \mathcal{L}\mu^j$, μ^j bounded measure on $\mathbb{R}_+$.

The theorem is a consequence of the following lemma, the arguments being. The proofs are analogous to the proofs in the Hilbert space case, based on the following lemma.

Lemma 6.11. There is a constant $\varepsilon > -1$ such that

$$\sum_j \left\| f_j\left(A_j\right) \right\| \leq \left(1+\varepsilon\right)\left(1+\log\left(a+\varepsilon/a\right)\right) M\left(a+\varepsilon\right)^2 \left\| \sum_j f_j \right\|_{H^\infty(\mathbb{C}_+)} \tag{6.2}$$

whenever the following hypotheses are satisfied:

1) $T^j = \left(T^j\left(1+\varepsilon\right)\right)_{\varepsilon > -1}$ is a C_0-semigroup on the Banach space *X*;

2) $0 < a < a+\varepsilon < \infty$; $\varepsilon > 0$;

3) $M\left(a+\varepsilon\right) := \gamma\left\{T^j\left(1+\varepsilon\right) \mid 0 \leq 1+\varepsilon \leq a+\varepsilon\right\}$;

4) $f_j = \mathcal{L}\mu^j$, where $\mu^j \in M\left(\mathbb{R}_+\right)$ such that $\operatorname{supp}\left(\mu^j\right) \subseteq \left[a, a+\varepsilon\right]$.

Proof. Examine the proof of Theorem 4.1 Choose $\varphi_j, \psi_j \in L^2\left(a, a+\varepsilon\right)$ such that $\varphi_j * \psi_j = 1$ on $\left[a, a+\varepsilon\right]$. Then

$$f_j\left(A_j\right) = T^j_{\mu^j} = P \circ L_{\left(\mu^j\right)} \circ \iota,$$

where for $x \in X$ and $F : \mathbb{R} \to X$

$$\iota x = (\psi_j)^{\sim} (T^j)^{\sim} x, \quad PF = \int_0^{a+\varepsilon} \sum_j \varphi_j (1+\varepsilon) T^j (1+\varepsilon) F(1+\varepsilon) d(1+\varepsilon)$$

Claim that $\iota : X \to \gamma(\mathbb{R}; X)$ with

$$\|l\|_{X \to \gamma} \le M(a+\varepsilon) \left\| \sum_j \psi_j \right\|_{L^2(0,a+\varepsilon)}$$

As in the case of groups, the estimate follows from the multiplier theorem; and the fact that $ran(\iota) \subset \gamma(\mathbb{R}; X)$ (and not just $\gamma_\infty(\mathbb{R}; X)$) comes from a density argument. Indeed, if $x \in \mathrm{dom}(A_j)$ then $\iota x = (\psi_j)^{\sim} \cdot u$ with

$$u(1+\varepsilon) = T^j(-(1+\varepsilon))x \text{ for } \varepsilon > -1.$$

Since $u \in C^1[-(a+\varepsilon), 0]$ and $(\psi_j)^{\sim} \in L^2(-(a+\varepsilon), 0)$, Corollary 6.3 and the ideal property yield that

$$\iota x = (\psi_j)^{\sim} \cdot u \in \gamma((-(a+\varepsilon), 0); X) \subseteq \gamma(\mathbb{R}; X).$$

Since $\mathrm{dom}(A_j)$ is dense in *X*, $ran(\iota) \subseteq \gamma(\mathbb{R}; X)$, as claimed.

Note that P can be factorized as

$$P = (\text{integrate against } \varphi_j) \circ (\text{multiply with } \mathbf{1}(0,b) T^j)$$

and so $\|P\|_{\gamma \to X} \le M(a+\varepsilon) \|\varphi_j\|_{L^2(0,a+\varepsilon)}$ by the multiplier theorem. Combine these results to obtain

$$\sum_j \|f_j(A_j)\| \le M^2 (a+\varepsilon)^2 \left\| \sum_j \varphi_j \right\|_{L^2(0,a+\varepsilon)} \|\psi_j\|_{L^2(0,a+\varepsilon)} \|f_j\|_{H^\infty(\mathbb{C}_+)}$$

and an application of Lemma A.1 concludes the proof. ∎

7. Singular Integrals and Functional Calculus

7.1. Functional Calculus

Provided series estimates of the form

$$\sum_j \|f_j(A_j)\| \lesssim \left\| \sum_j f_j \right\|_{B^{0,2(\beta+\varepsilon)}_{(\beta+\varepsilon),1}}$$

Under various conditions on the Banach space *X*, the semigroup T^j is on the angle θ. However, to derive these estimates required $f_j = \mathcal{L}\mu^j$, μ^j some bounded measure of compact support. It is certainly natural to ask whether one can extend the results to all $f_j \in B^{0,2(\beta+\varepsilon)}_{(\beta+\varepsilon),1}(\Sigma_\theta)$, *i.e.*, to a proper Besov class functional calculus (see, e.g., [1]).

The major problem here is not the norm estimate, but the definition of $f_j(A_j)$ in the first place. (If $f_j = \mathcal{L}\mu^j$ for the measure μ^j with compact support, this problem does not occur). Of course one could pass to a closure with respect to the Besov norm, but this yields a too small function class in general. And it does not show how this definition of $f_j(A_j)$ relates with all the others in the literature, especially, with the functional calculus for sectorial sequence of operators [13] and the one for half-plane type operators [28].

7.2. Singular Integrals for Semigroups

A usual consequence of transference estimates is the convergence of certain singular integrals. It has been known for a long time that if $\left(U(1+\varepsilon)\right)_{(1+\varepsilon)\in\mathbb{R}}$ is a C_0-group on a UMD space X then the principal value integral

$$\int_{-1}^{1} U(1+\varepsilon)x\frac{d(1+\varepsilon)}{(1+\varepsilon)}$$

exists for every $x\in X$. This was the decisive ingredient in the Dore-Venni theorem and in Fattorini's theorem, as discussed in [7]. For semigroups, these proofs fail and this is not surprising as one has to profit from cancellation effects around 0 in order to have a principal value integral converging. The results imply that if one shifts the singularity away from 0 then the associated singular integral for a semigroup will converge, under suitable assumptions on the Banach space or the semigroup, for groups gave a fairly general statement in ([8], Theorem 4.4).

Theorem 7.1. Let $\left(T^j(1+\varepsilon)\right)_{\varepsilon>-1}$ be a C_0-semigroup on a UMD Banach space X, let $(0<a<a+\varepsilon)$, $\varepsilon>0$ and let $g^j\in BV[\varepsilon,2a+\varepsilon]$ be such that $g^j(\cdot+a+\varepsilon)$ are even. Then the principal value integral

$$\lim_{\varepsilon\searrow 0}\int_{\varepsilon a\le|1-a|\le a}\sum_j g^j(1+\varepsilon)x\frac{d(1+\varepsilon)}{(1-a)} \tag{7.1}$$

converges for every $x\in X$.

Proof. Define

$$\sum_j (f_j)_\epsilon(z)=\int_{\epsilon a\le|1-a|\le a}\sum_j g^j(1+\varepsilon)e^{-(1+\varepsilon)z}\frac{d(1+\varepsilon)}{(1-a)}\quad \left((1+\varepsilon)\in\mathbb{C}\right)$$

Then, since g^j are even about the singularity $(a+\varepsilon)$.

$$\sum_j (f_j)_\epsilon(z)=\int_{\epsilon a\le|1-a|\le a}\sum_j g^j(1+\varepsilon)T^j(1+\varepsilon)\frac{d(1+\varepsilon)}{(1-a)}=f_\epsilon(z)$$

$$=\int_{\epsilon a\le|1-a|\le a}\sum_j g^j(1+\varepsilon)\left(T^j(1+\varepsilon)x-T^j(a+\varepsilon)x\right)\frac{d(1+\varepsilon)}{(1-a)}$$

If $x\in\operatorname{dom}(A_j)$ then $T^j(\cdot)x$ are continuously differentiable and since g^j are even about the singularity $a+\varepsilon,a$ well-known argument shows that the limit (7.1) exists Hence, by density, one only has to show that $\sup_{0<\epsilon<1}\left\|(f_j)_\epsilon(A_j)\right\|<\infty$. In order to establish this, define $h(x)=g^j(ax+a+\varepsilon)$ and

$$\sum_j (f_j)_\epsilon(z)\le\int_{\varepsilon(a\le|0\le((a+1)|}\sum_j g^j(1+\varepsilon)e^{-(1+\varepsilon)(1+\varepsilon)}\frac{d(1+\varepsilon)}{(1+\varepsilon)-(a+\varepsilon)},\ (z\in\mathbb{Z})$$

Use Theorem 4.1 to estimate

$$\sum_j\left\|(f_j)_\epsilon(A_j)\right\|\lesssim\left(1+\log\left(\frac{2a+\varepsilon}{\varepsilon}\right)\right)\left\|\sum_j (f_j)_\epsilon(A_j)\right\|_{\mathcal{M}_{(1+\varepsilon))X}}$$

By a change of variables,

$$\sum_j (f_j)_\epsilon (i(1+\varepsilon)) = \mathrm{e}^{-i(1+\varepsilon)(a+\varepsilon)} \int_{\epsilon\le|1+\varepsilon|\le 1} \mathrm{e}^{-ia(1+\varepsilon)(1+\varepsilon)} \frac{h(1+\varepsilon)}{(1+\varepsilon)} d(1+\varepsilon)$$
$$= \mathrm{e}^{-i(1+\varepsilon)(a+\varepsilon)} \mathcal{F}\left(PV - \frac{h_\epsilon}{(1+\varepsilon)}\right)(a(1+\varepsilon))$$

where $h_\epsilon = (h)\mathbf{1}_{\{\epsilon\le|(1+\varepsilon)|\le 1\}}$. It is a standard fact from Fourier multiplier theory that the exponential factor in front and the dilation by a in the argument do not change Fourier multiplier norms. So one is reduced to estimate the $\mathcal{M}_{(1+\varepsilon),X}$ -norms of the functions

$$m_\epsilon = \mathcal{F}\left(PV - \frac{h_\epsilon}{(1+\varepsilon)}\right), \quad (0 < \epsilon < 1)$$

Remark 7.2. The result is also true on a general Banach space if $\{T^j(1+\varepsilon) \mid 0 \le (1+\varepsilon) \le 1\}$ is γ-bounded. The proof is analogous, but in place of Theorem 4.1 one has to employ Lemma 6.11.

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Conflicts of Interest

The authors declare no conflicts of interest regarding the publication of this paper.

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Appendix A. Two Lemmata

Provide two lemmata concerning an optimization problem for convolutions on the halfline or the positive integers.

Lemma A.1. (Haase-Hytonen) Let $(0<\varepsilon<\infty)$, $\varepsilon>0$, For $0<a<a+\varepsilon$, let

$$c(a,a+\varepsilon)=\inf\left\{\left\|(\varphi_j)_1+(\varphi_j)_2\right\|_{\frac{1+\varepsilon}{\varepsilon}}+\left\|(\psi_j)_1+(\psi_j)_2\right\|_{1+\varepsilon}:\right.$$
$$((\varphi_j)_1+(\varphi_j)_2)\in L^{\frac{1+\varepsilon}{\varepsilon}}(0,a+\varepsilon),(\psi_j)_1+(\psi_j)_2\in L^{1+\varepsilon}(0,a+\varepsilon),$$
$$\left.((\varphi_j)_1+\varphi_2)*((\psi_j)_1+(\psi_j)_2)=1\text{ on }[a,a+\varepsilon]\right\}$$

Then there are constants $\varepsilon>-1$ such that

$$(1+\varepsilon)(1+\log(a+\varepsilon/a))\le c(a,a+\varepsilon)\le(1+\varepsilon)(1+\log(a+\varepsilon/a))$$

for all $0<a<a+\varepsilon$.

Proof. Fix $(0<\varepsilon<\infty)$, $\varepsilon>0$ Suppose that $\varphi_j\in L^{\frac{1+\varepsilon}{\varepsilon}}(\mathbb{R}_+)$ and $(\psi_j)_1+(\psi_j)_2\in L^{1+\varepsilon}(\mathbb{R}_+)$ with $((\varphi_j)_1+(\varphi_j)_2)*((\psi_j)_1+(\psi_j)_2)=1$ on $[a,a+\varepsilon]$. Then, by Hölder's inequality,

$$1=\left|((\varphi_1+\varphi_2)*((\psi_j)_1+(\psi_j)_2))(a)\right|\le\left\|(\varphi_j)_1+(\varphi_j)_2\right\|_{\frac{1+\varepsilon}{\varepsilon}}\left\|(\psi_j)_1+(\psi_j)_2\right\|_{1+\varepsilon}$$

which implies $\varepsilon>0$. Secondly,

$$\log(a+\varepsilon/a)=\left|\int_a^{a+\varepsilon}\sum_j\left|((\varphi_j)_1+(\varphi_j)_2)((\psi_j)_1+(\psi_j)_2)(1+\varepsilon)\right|\frac{d(1+\varepsilon)}{(1+\varepsilon)}\right|$$
$$\le\int_a^{a+\varepsilon}\int_0^{(1+\varepsilon)}\left|\sum_j((\varphi_j)_1+(\varphi_j)_2)(0)\right|\left|((\psi_j)_1+(\psi_j)_2)(1+\varepsilon)\right|\frac{d(1+\varepsilon)}{(1+\varepsilon)}$$
$$\le\int_0^{\infty}\int_{(1+\varepsilon)}^{\infty}\sum_j\left(\frac{\left|((\varphi_j)_1+(\varphi_j)_2)(0)\right|}{(1+\varepsilon)}\right)d(1+\varepsilon)\left|(\psi_j)_1+(\psi_j)_2(1+\varepsilon)\right|d(1+\varepsilon)$$
$$=\int_0^{\infty}\int_0^{\infty}\sum_j\frac{\left|((\varphi_j)_1+(\varphi_j)_2)(1+\varepsilon)\right|\left|((\psi_j)_1+(\psi_j)_2)(1+\varepsilon)\right|}{2(1+\varepsilon)}$$
$$\le\frac{\pi}{\sin(\pi/1+\varepsilon)}\left\|\sum_j(\varphi_j)_1+(\varphi_j)_2\right\|_{\frac{1+\varepsilon}{\varepsilon}}\left\|(\psi_j)_1+(\psi_j)_2\right\|_{1+\varepsilon}$$

(This is "Hilbert's absolute inequality", see ([29], Chapter 5.10).) This yields

$$(1+\varepsilon)\ge\frac{\sin(\pi/1+\varepsilon)}{\pi}\log\left(\frac{a+\varepsilon}{a}\right)$$

Taking both we arrive at

$$\mathbf{1}\vee\frac{\sin(\pi/1+\varepsilon)}{\pi}\log\left(\frac{a+\varepsilon}{a}\right)\le c(a,a+\varepsilon)$$

Since $\sin(\pi/1+\varepsilon)\ne0$, one can find $\varepsilon>-1$ such that

$$(1+\varepsilon)(1+\log(a+\varepsilon/a)) \le 1 \vee \frac{\sin\left(\frac{\pi}{1+\varepsilon}\right)}{\pi}\log\left(\frac{a+\varepsilon}{a}\right)$$

and the lower estimate is established.

To prove the upper estimate note first that without loss of generality we may assume that $a=1$. Indeed, passing from $\left((\varphi_j)_1+(\varphi_j)_2,(\psi_j)_1+(\psi_j)_2\right)$ to

$$\left((a)^{\frac{\varepsilon}{1+\varepsilon}}\left(\left((\varphi_j)_1+(\varphi_j)_2\right)+\left((\psi_j)_1+(\psi_j)_2\right)\right)((a)\cdot), a^{1/(1+\varepsilon)}\left((\psi_j)_1+(\psi_j)_2\right)((a)\cdot)\right)$$

reduces the $(a,a+\varepsilon)$-case to the $(1,a+\varepsilon/a)$-case and shows that

$$c(a,a+\varepsilon)=c(1,a+\varepsilon/a).$$

The idea is to choose $(\varphi_j)_1+(\varphi_j)_2$, $(\psi_j)_1+(\psi_j)_2$ in such a way that

$$\left((\varphi_j)_1+(\varphi_j)_2,(\psi_j)_1+(\psi_j)_2\right)(1+\varepsilon)=\begin{cases}1+\varepsilon, & 0\le(1+\varepsilon)\le 1\\ 1, & \varepsilon>-1\end{cases}$$

and cut them after $(\mathrm{a}+\varepsilon)$ Taking Laplace transforms, this means

$$\left[\left(\mathcal{L}\left((\varphi_j)_1+(\varphi_j)_2\right)\right)\left(\mathcal{L}\left((\psi_j)_1+(\psi_j)_2\right)\right)\right](z)=\frac{1-\mathrm{e}^{-z}}{z^2}$$

for $\operatorname{Re} z>0$. Fix $\theta\in(0,1)$ and write

$$\frac{1-\mathrm{e}^{-z}}{z^2}=\frac{\left(1-\mathrm{e}^{-z}\right)^{(1-\theta)}}{z}\cdot\frac{\left(1-\mathrm{e}^{-z}\right)^{\theta}}{z}$$

by the binomial series,

$$\frac{1-\mathrm{e}^{-z}}{z^2}=\sum_{k=0}^{\infty}(\beta+\varepsilon)_k^{(\theta)}\frac{\mathrm{e}^{-kz}}{z}=\sum_{k=0}^{\infty}\mathcal{L}\left(\mathbf{1}_{(k,\infty)}\right)(z)$$

and writing $\mathbf{1}_{(k,\infty)}=\sum_{j=k}^{\infty}\mathbf{1}_{(j,j+1)}$ see that can take

$$\sum_j(\psi_j)_1+(\psi_j)_2=\sum_{k=0}^{\infty}\left(\sum_{j=k}^{\infty}(\beta+\varepsilon)_k^{(\theta)}\mathbf{1}_{(j,j+1)}\right)=\sum_{j=0}^{\infty}\left(\sum_{k=0}^{j}(\beta+\varepsilon)_k^{(\theta)}\right)\mathbf{1}_{(j,j+1)}$$

and likewise

$$\sum_j(\varphi_j)_1+(\varphi_j)_2=\sum_j\sum_{j=0}^{\infty}\left(\sum_{k=0}^{j}(\beta+\varepsilon)_k^{(1-\theta)}\right)1_{(j,j+1)}$$

Let $\beta^{(\theta)}=\sum_{k=0}^{j}(\beta+\varepsilon)_K^{(\theta)}$ By standard asymptotic analysis

$$(\beta+\varepsilon)_k^{(\theta)}=0\left(\frac{1}{k^{1+\theta}}\right)\text{ and }\beta^{(\theta)}=0\left(\frac{1}{(1+j)^{\theta}}\right)$$

It is clear that

$$c(1,a+\varepsilon)\le\left\|\sum_1\left((\varphi_j)_1+(\varphi_j)_2\right)\mathbf{1}_{(0,a+\varepsilon)}\right\|_{\frac{1+\varepsilon}{\varepsilon}}\left\|\left((\psi_j)_1+(\psi_j)_2\right)\mathbf{1}_{(0,a+\varepsilon)}\right\|_{1+\varepsilon}$$

Now,

$$\sum_j\left\|\left((\varphi_j)_1+(\varphi_j)_2\right)\mathbf{1}_{(0,a+\varepsilon)}\right\|_{1+\varepsilon}^{1+\varepsilon}$$

$$= \int_0^{a+\varepsilon} \sum_j \left| \left((\psi_j)_1 + (\psi_j)_2 \right)(1+\varepsilon) \right|^{1+\varepsilon} d(1+\varepsilon)$$

$$= \sum_{j=0}^{\infty} \left(\beta^\theta\right)^{1+\varepsilon} \int_0^{a+\varepsilon} \mathbf{1}_{(j,j+1)}(1+\varepsilon)\, d(1+\varepsilon) \lesssim \sum_{j=0}^{\infty} (1+j)^{-\theta(1+\varepsilon)} \gamma_{(j,a+\varepsilon)}$$

with

$$\gamma_{(j,a+\varepsilon)} = \begin{cases} 1, & j \le (a+\varepsilon)-1 \\ (a+\varepsilon)-j, & j \le (a+\varepsilon) \le j+1 \\ 0, & (a+\varepsilon) \le j \end{cases}$$

with $\theta := 1/(1+\varepsilon)$ this yields

$$\sum_j \left\| \left((\psi_j)_1 + (\psi_j)_2 \right) \mathbf{1}_{(0,a+\varepsilon)} \right\|_{1+\varepsilon}^{1+\varepsilon}$$

$$\le 1 + \sum_{j=1}^{\lfloor a+\varepsilon \rfloor -1} \int_j^{j+1} \frac{dx}{x} + \frac{(a+\varepsilon) - \lfloor a+\varepsilon \rfloor}{1+\lfloor a+\varepsilon \rfloor}$$

$$\le 2 + \log\left(\lfloor a+\varepsilon \rfloor\right) \le 2\left(1 + \log\lfloor a+\varepsilon \rfloor\right)$$

Analogously, noting that $1-\theta = 1-(1/1+\varepsilon) = \left(1 \Big/ \left(\frac{1+\varepsilon}{\varepsilon}\right)\right)$,

$$\sum_j \left\| \left((\psi_j)_1 + (\psi_j)_2 \right) \mathbf{1}_{(0,a+\varepsilon)} \right\|_{\left(\frac{1+\varepsilon}{\varepsilon}\right)}^{\left(\frac{1+\varepsilon}{\varepsilon}\right)} \lesssim 2\left(1 + \log(a+\varepsilon)\right)$$

which combines to

$$c(1, a+\varepsilon) \lesssim 2\left(1 + \log(a+\varepsilon)\right)$$

as was to prove. ■

Lemma A.2. Let $(0 < \varepsilon < \infty)$, For $0 \le a \le a+\varepsilon$,

$$c(a, a+\varepsilon)$$

$$= \inf \left\{ \left\| (\varphi_j)_1 + (\varphi_j)_2 \right\|_{\frac{1+\varepsilon}{\varepsilon}} \left\| (\psi_j)_1 + (\psi_j)_2 \right\|_{1+\varepsilon} : \left((\varphi_j)_1 + (\varphi_j)_2 \right) \in L^{\frac{1+\varepsilon}{\varepsilon}}(\mathbb{Z}_+), \right.$$

$$\left. \left((\varphi_j)_1 + (\varphi_j)_2 \right) * \left((\psi_j)_1 + (\psi_j)_2 \right) = 1 \text{ on } [a, a+\varepsilon] \right\}$$

Then there are constants $\varepsilon > -1$ such that

$$(1+\varepsilon)\left(1 + \log(a+\varepsilon/a)\right) \le c(a, a+\varepsilon) \le (1+\varepsilon)\left(1 + \log(a+\varepsilon/a)\right)$$

for all $0 < a < a+\varepsilon$.

Proof. The proof is similar to the proof of Lemma A.1. The lower estimate is obtained in a totally analogous fashion, making use of the discrete version of Hilbert's absolute inequality ([29], Thm. 5.10.2) and the estimate

$$\sum_{n=a}^{a+\varepsilon} \left(\frac{1}{n+1} \right) \ge \frac{1}{2} \log(a+\varepsilon/a)$$

For the upper estimate let

$$\eta(j) = \begin{cases} j/(a), & j = 0, 1, \cdots, a \\ 1, & j \ge (a)+1 \end{cases}$$

and look for factorizations $\left(\left(\varphi_j\right)_1+\left(\varphi_j\right)_2\right)*\left(\left(\psi_j\right)_1+\left(\psi_j\right)_2\right)=\eta$. Considering the Fourier transform find

$$\hat{\eta}(z)=\frac{z}{(a)}\frac{1-z^{(a)}}{(1-z)^2}$$

and so try (as in the continuous case) the "Ansatz"

$$\sum_j\left(\psi_j\right)_1+\left(\psi_j\right)_2=\frac{z}{(a)^{\theta}}\frac{\left(1-z^{(a)}\right)^{\theta}}{1-z} \text{ and } \left(\varphi_j\right)_1+\left(\varphi_j\right)_2=\frac{1}{(a)^{1-\theta}}\frac{\left(1-z^{(a)}\right)^{1-\theta}}{1-z}$$

for $\theta:=1/(1+\varepsilon)$. Note that

$$\sum_j\left(\left(\psi_j\right)_1+\left(\psi_j\right)_2\right)(z)=\frac{z}{(a)^{\theta}(1-z)}\sum_{j=0}^{\infty}\sum_j(\beta+\varepsilon)_j^{(\theta)}z^{(a)j}=\frac{z}{(a)^{\theta}(1-z)}\sum_{k=0}^{\infty}\gamma k^{z^k}$$

where

$$\gamma k=\gamma k(a,\theta)=\begin{cases}(\beta+\varepsilon)_{k/a}^{(\theta)} & \text{if } a/k\\ 0 & \text{else}\end{cases}$$

Consequently,

$$\sum_j\left(\left(\psi_j\right)_1+\left(\psi_j\right)_2\right)(z)=\frac{z}{(a)^{\theta}}\sum_{n=0}^{\infty}\left(\sum_{k=1}^{n}\gamma k\right)z^n=\frac{z}{(a)^{\theta}}\sum_{n=0}^{\infty}\beta_{\lfloor n/a\rfloor}^{(\theta)}$$

and, likewise,

$$\sum_j\left(\varphi_j\right)_1+\left(\varphi_j\right)_2(z)=\frac{z}{(a)^{1-\theta}}\sum_{n=0}^{\infty}\beta_{\lfloor n/a\rfloor^{z^n}}^{(1-\theta)}$$

As in the continuous case, it suffices to cut off $\left(\varphi_j\right)_1+\left(\varphi_j\right)_2$ and $\left(\psi_j\right)_1+\left(\psi_j\right)_2$ after $a+\varepsilon$, so

$$c(a,a+\varepsilon)\le\left\|\sum_j\left(\left(\varphi_j\right)_1+\left(\varphi_j\right)_2\right)\mathbf{1}_{[0,a+\varepsilon]}\right\|_{\frac{1+\varepsilon}{\varepsilon}}\left\|\left(\left(\psi_j\right)_1+\left(\psi_j\right)_2\right)\mathbf{1}_{[0,a+\varepsilon]}\right\|_{1+\varepsilon}$$

write $a+\varepsilon=k(a)+r$ with $0\le r<a$ and $k:=\lfloor a+\varepsilon/a\rfloor$; then

$$\begin{aligned}&\left\|\left(\left(\psi_j\right)_1+\left(\psi_j\right)_2\right)\mathbf{1}_{[0,a+\varepsilon]}\right\|_{1+\varepsilon}^{1+\varepsilon}\\ &\le\frac{1}{a}\sum_{n=0}^{a+\varepsilon}\left|\beta_{\lfloor n/a\rfloor}^{(\theta)}\right|^{1+\varepsilon}\lesssim\frac{1}{a}\sum_{n=0}^{a+\varepsilon}\left(1+\lfloor n/a\rfloor\right)^{-1}\\ &=\frac{1}{a}\sum\left(\frac{a}{1}+\frac{a}{2}+\cdots+\frac{a}{k}+\frac{r}{k+1}\right)\le\sum_{j=1}^{k+1}\frac{1}{j}\\ &\le\sum 1+\int_1^{k+1}\frac{dx}{x}=1+\log(k+1)\le 2\left(1+\log(a+\varepsilon/a)\right)\end{aligned}$$

A similar estimate holds for $\sum_j\left\|\left(\left(\varphi_j\right)_1+\left(\varphi_j\right)_2\right)\mathbf{1}_{[0,a+\varepsilon]}\right\|_{\frac{1+\varepsilon}{\varepsilon}}^{1+\varepsilon}$.