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Some Result of Stability and Spectra Properties on Semigroup of Linear Operator

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Abstract

This paper consists of some properties of a new subclass of semigroup of linear operator. The stability and spectra analysis of ω-order preserving partial contraction mapping (ω-OCP_n) are obtained. The results show that operators on the proposed ω-OCP_n are densely defined and closed. Several existing results in the literature are contained in this work.

Keywords

Contraction Mapping, Semigroup, Banach Space, Resolvent and Bounded Operator

1. Introduction

The theory of stability is important since stability plays a central role in the structural theory of operators such as semigroup of linear operator, contraction semigroup, invariant subspace theory and to mention but few. The theory of stability is rich in which concerns the methods and ideas, and this shall be one of the main points of this paper. The recent advances deeply interact with modern topics from complex function theory, harmonic analysis, the geometry of Banach spaces, and spectra theory [1].

Another main focus of this paper is spectra analysis of a semigroup of linear operator, in which we use the resolvent to describe the relationship between the spectrum of A and of the semigroup operator $\left(T(t)\right)_{t\geq 0}$ and also determine the bounded linear operator A as the generators of one-parameter semigroups. Resolvent operators are particularly useful in the analysis of Sturm-Liouville operators and several others operators both bounded and unbounded.

Let X be a Banach space, $X_n \subseteq X$ be a finite set, $\left(T(t)\right)_{t\geq 0}$ the C_0-semigroup which is strongly continuous one parameter semigroup of bounded linear operator in X, ω-OCP_n be ω-order-preserving partial contraction mapping (semigroup of linear operator) which is an example of C_0-semigroup. Similarly, let $Mm(\mathbb{N})$ be a matrix, $L(X)$ be a bounded linear operator on X, P_n a partial transformation semigroup, $\rho(A)$ a resolvent set, $\sigma(A)$ be spectrum and A is a generator of C_0-semigroup.

This paper will focus on results of stability and spectra analysis of ω-OCP_n on Banach space as an example of a semigroup of linear called C_0-semigroup, and thereby establish the relationship between a semigroup, its generator and the resolvent as in **Figure 1**.

In [2], Batty obtained some spectral conditions for stability of one-parameter semigroup and also revealed some asymptotic behaviour of semigroup of operator, see also, Batty *et al.* [3]. Chill and Tomilov [4] established some resolvent approach to stability operator semigroup. Räbiger and Wolf in [5] deduced some spectral and asymptotic properties of dominated operator. For relevant work on non-linear and one-parameter semigroups, see ([6] and [7]). The aim of this work is, therefore, to obtain stability and spectra analysis on a new subclass of semigroup of linear operator.

2. Preliminaries

The following definitions are crucial to the proof of our main results.

Definition 2.1: (Stable Semigroup [8])

A strongly continuous semigroup $\left(T(t)\right)_{t\geq 0}$ is called

1) Uniformly exponentially stable if there exists $\epsilon > 0$ such that

$$\lim_{t\to\infty} e^{\epsilon t}\left\|T(t)\right\| = 0 \tag{2.1}$$

2) Uniformly stable if

$$\lim_{t\to\infty}\left\|T(t)\right\| = 0 \tag{2.2}$$

3) Strongly stable if

$$\lim_{t\to\infty}\left\|T(t)_x\right\| = 0 \quad \forall x\in X. \tag{2.3}$$

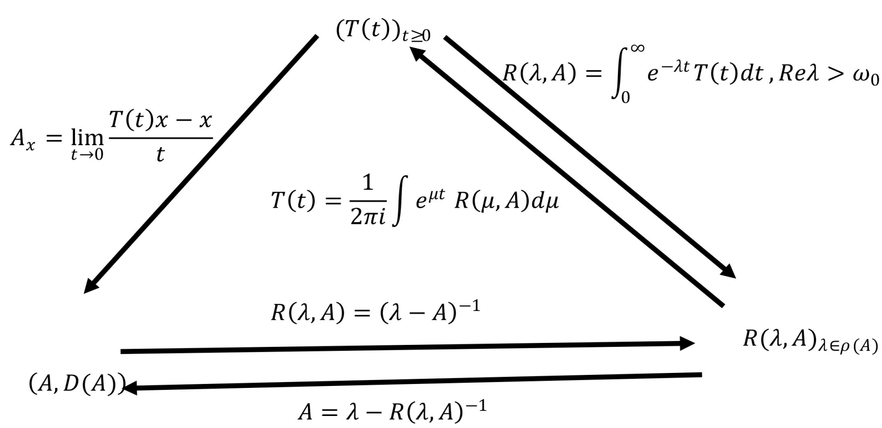

Figure 1. Diagrammatical representation of relationship between a semigroup, its generator and its resolvent [8].

Definition 2.2: (C_0-Semigroup [9])

A C_0-Semigroup is a strongly continuous one parameter semigroup of bounded linear operator on Banach space.

Definition 2.3: (ω-OCP_n [10])

A transformation $\alpha \in P_n$ is called ω-order-preserving partial contraction mapping if $\forall x, y \in \text{Dom}\alpha : x \le y \Rightarrow \alpha x \le \alpha y$ and at least one of its transformation must satisfy $\alpha y = y$ such that $T(t+s) = T(t)T(s)$ whenever $t, s > 0$ and otherwise for $T(0) = I$.

Definition 2.4: (Core [8])

Let A be a closed linear operator with domain $D(A)$ and range $R(A)$ in a Banach space X. A subspace D of $D(A)$ is called a core if A is the closure of its restriction to D.

Definition 2.5: (Resolvent Set [11])

We define the resolvent set of A denoted by $\rho(A)$ set of all $\lambda \in \mathbb{C}$ such that $\lambda I - A$ is one-to-one with range equal to X.

Definition 2.6: (Spectrum [11])

The spectrum of A denoted by $\sigma(A)$ is defined as the complement of the resolvent set.

Definition 2.7: (Hyperbolic [12])

A semigroup $(T(t))_{t\ge0}$ on a Banach space X is called hyperbolic if X can be written as direct sum $X = X_s \oplus X_u$ of two $(T(t))_{t\ge0}$-invariant, closed subspaces X_s, X_u such that the restricted semigroups $(T_s(t))_{t\ge0}$ on X_s and $(T_u(t))_{t\ge0}$ on X_u satisfy the following conditions:

1) The semigroup $(T_s(t))_{t\ge0}$ is uniformly exponentially stable on X_s.

2) The operator $T_u(t)$ are invertible on X_u, and $\left(T_u(t)^{-1}\right)_{t\ge0}$ is uniformly exponentially stable on X_u.

Some Basic Spectral Properties

1) To any linear operator A we associate its spectral bound defined by

$$\text{Spec}(A) = \text{Sup}\{\text{Re}\,\lambda : \lambda \in \sigma(A)\}.$$

2) Resolvent set: $\rho(A) = \{\lambda \in \mathbb{C} : \lambda - A : D(A) \to X \text{ is bijective}\}$.

3) Spectrum: $\sigma(A) = \mathbb{C} / \rho(A)$.

4) Resolvent: $R(\lambda; A) = (\lambda - A)^{-1}\ \forall \lambda \in \rho(A)$.

5) Resolvent equation: $R(\lambda; A) - R(\mu; A) = (\mu - \lambda) R(\lambda; A) R(\mu; A)$.

Example 1:

2×2 matrix $[M_m(\mathbb{R}_+)]$

Suppose

$$A = \begin{pmatrix} 1 & 2 \\ 2 & 2 \end{pmatrix}$$

and let $T(t) = e^{tA}$, then

$$e^{tA} = \begin{pmatrix} e^{t} & e^{2t} \\ e^{2t} & e^{2t} \end{pmatrix}$$

3×3 matrix $\left[M_m\left(\mathbb{R}_+\right)\right]$

Suppose

$$A=\begin{pmatrix}1 & 2 & 3\\ 1 & 2 & 2\\ - & 2 & 3\end{pmatrix}$$

and let $T(t)=e^{tA}$, then

$$e^{tA}=\begin{pmatrix}e^{t} & e^{2t} & e^{3t}\\ e^{t} & e^{2t} & e^{2t}\\ I & e^{2t} & e^{3t}\end{pmatrix}$$

Example 2:

2×2 matrix $\left[M_m\left(\mathbb{C}\right)\right]$, we have for each $\lambda>0$ such that $\lambda\in\rho(A)$ where $\rho(A)$ is a resolvent set on *X*.

Suppose we have

$$A=\begin{pmatrix}1 & 2\\ - & 2\end{pmatrix}$$

and let $T(t)=e^{tA_\lambda}$, then

$$e^{tA_\lambda}=\begin{pmatrix}e^{t\lambda} & e^{2t\lambda}\\ I & e^{2t\lambda}\end{pmatrix}$$

Example 3:

Let $X=C_{ub}\left(\mathbb{R}_+\right)$ be the space of all bounded and uniformly continuous function from $\mathbb{R}_+$ to $\mathbb{R}$, endowed with the sup-norm $\|\cdot\|_\infty$ and let $\{T(t);t\geq0\}\leq L(X)$ be defined by

$$\left[T(t)f\right](s)=f(t+s)$$

For each $f\in X$ and each $t,s\in\mathbb{R}_+$, one may easily verify that $\{T(t);t\geq0\}$ satisfies the example 1 and 2 above.

3. Main Results

In this section, results of stability and spectral properties on ω-OCP_n in Banach space and on C_0-semigroup are considered:

Theorem 3.1

Suppose X is a Banach space. Then a linear operator $A:D(A)\subseteq X\to X$ is an infinitesimal generator of a strongly continuous semigroup $\left(T(t)\right)_{t\geq0}$ on X is uniformly exponentially stable if and only if for all $p\in[1,\infty)$ one has

$$\int_0^\infty\left\|T(t)x\right\|^p\mathrm{d}t<\infty$$

for all $x\in X$ and $A\in\omega$-OCP_n.

Proof

If the semigroup is exponentially stable, then, the integral above is satisfied.

In order to show the converse implication, it suffices to verify that

$$\lim_{t\to\infty}\|T(t)\| = 0. \tag{3.1}$$

So, we define for $n \in \mathbb{N}$, the operators $\Im \in \left(X, L^p\left(\mathbb{R}_+, X\right)\right)$ by

$$\Im_n x = \phi_{[0,n]}(\cdot)T(\cdot) \tag{3.2}$$

Then by assumption, the set $\{\Im_n x : n \in \mathbb{N}\} \subset L^p\left(\mathbb{R}_+, X\right)$ is bounded for each $x \in X$, hence by the uniform boundedness principle, there exists $C > 0$ such that

$$\int_0^t \|T(r)x\|^p \,\mathrm{d}r \le C^p \|x\|^p \quad \text{for all } x \in X, t \ge 0.$$

On the other hand, there exist $M \ge 1$ and $w > 0$ such that

$$\|T(t)\| \le Me^{wt} \quad \text{for all } t \ge 0$$

From the previous two inequalities, we obtain

$$\begin{aligned}\frac{1-e^{-pwt}}{p}\|T(t)x\|^p &= \int_0^t e^{-pwt}\|T(r)T(t-r)x\|^p \,\mathrm{d}r \\ &\le \int_0^t M^p \|T(t-r)x\|^p \,\mathrm{d}r \\ &\le M^p C^p \|x\|^p \quad \forall x \in X, t \ge 0.\end{aligned} \tag{3.3}$$

Hence, there exists a constant $L > 0$ such that

$$\|T(t)\| \le L \quad \forall t \ge 0$$

Considering this, we conclude that

$$\begin{aligned}t\|T(t)x\|^p &= \int_0^t \|T(t-r)T(r)x\|^p \,\mathrm{d}r \\ &\le \int_0^t L^p \|T(r)x\|^p \,\mathrm{d}r \\ &\le L^p C^p \|x\|^p \quad \forall x \in X, t \ge 0\end{aligned} \tag{3.4}$$

and therefore

$$\|T(t)\| \le LCt^{\frac{-1}{p}} \quad \forall t > 0.$$

This implies

$$\lim_{t\to\infty}\|T(t)\| = 0 \tag{3.5}$$

Hence the proof is complete.

Proposition 3.2

Suppose X is a Banach space and $A : D(A) \subseteq X \to X$ where $A \in \omega\text{-}OCP_n$ is the infinitesimal generator for a strongly continuous semigroup $(T(t))_{t\ge0}$, then the following assertions are equivalent.

1) $(T(t))_{t\ge0}$ is hyperbolic.

2) $(T(t))_{t\ge0} \cap \Gamma = \phi$ for all $t > 0$.

Proof

The proof of implication 1) $\Rightarrow$ 2) starts from the observation that $\sigma(T(t)) = \sigma(A_s) \cup \sigma(A_u)$ because of the direct sum decomposition.

By assumption, $(T_s(t))_{t\geq 0}$ is uniformly exponentially stable; hence $r(T_s(t))<1$ for $t>0$, and therefore

$$\sigma(T_s(t))\cap\Gamma=\phi \tag{3.6}$$

By the same argument, we obtain that $r\left(T_s(t)^{-1}\right)<1$. Suppose

$$\sigma(T_u(t))=\left\{\lambda^{-1}:\lambda\in\sigma(A)^{-1}\right\}, \tag{3.7}$$

we conclude that $|\lambda|>1$ for each $\lambda\in\sigma(A_u)$; hence $\sigma(T_u(t))\cap\Gamma=\phi$.

To prove 2) ⇒ 1), we fix $s>0$ such that $\sigma(T(s))\cap\Gamma=\phi$ and we use the existence at a spectral projection P corresponding to the spectral set

$$\sigma(T(s))=\{\lambda\in\sigma(A):|\lambda|<1\}. \tag{3.8}$$

Then the space X is the direct sum $X=X_s\oplus X_u$ of the $(T(t))_{t\geq 0}$-invariant subspaces $X_s=rgP$ and $X_u=kerP$, where $A_s\subseteq X_s$ and $A_u\subseteq X_u$. Then the restriction $T_s(t)\in L(X_s)$ of T(s) has spectrum

$$\sigma(T_s(s))=\{\lambda\in\sigma(A):|\lambda|<1\} \tag{3.9}$$

hence, spectral radius $r(T_s(s))<1$. It follows that the semigroup $(T_s(t))_{t\geq 0}=(PT(t))_{t\geq 0}$ is uniformly exponentially stable on X_s.

Similarly, the restriction $T_u(s)\in L(X_u)$ of $T(s)$ in X_u has spectrum

$$\sigma(T_u(s))=\{\lambda\in\sigma(A):|\lambda|>1\} \tag{3.10}$$

hence is invertible on X_u. Clearly this implies that $T_u(t)$ is invertible for $0\leq t\leq s$, while for $t>s$ we choose $n\in N$ such that $ns>t$. Then

$$T_u(s)^n=T_u(ns)=T(ns-t)T_u(t)=T_u(t)T_u(ns-t) \tag{3.11}$$

hence $T_u(t)$ is invertible, since $T_u(s)$ is bijective.

Moreover, for the spectral radius, we have $r\left(T_u^{-1}(s)\right)<1$, and again this implies uniformly exponentially stable for the semigroup $\left(T_u(t)^{-1}\right)_{t\geq 0}$. Hence the proof.

Theorem 3.3

Suppose $A\in\omega\text{-}OCP_n$ and $\omega\text{-}OCP_n\in L(X)$. Let $A:D(A)\subseteq X\to X$ be a linear operator which satisfies:

a) A is densely defined and closed; and

b) $(0,+\infty)\subseteq\rho(A)$ and for each $\lambda>0$, we have

$$\|R(\lambda,A)\|_{L(X)}\leq\frac{1}{\lambda}.$$

Then:

1) $\lim_{\lambda\to\infty}\lambda R(\lambda,A)x=x$ for each $x\in X$,

2) $A_\lambda x=\lambda^2R(\lambda,A)x-\lambda x$ for each $x\in X$,

3) $\lim_{\lambda\to\infty}A_\lambda x=Ax$ for each $x\in D(A)$ and,

4) A_λ is the infinitesimal generator of a uniformly continuous semigroup $\{e^{tA\lambda};t\geq 0\}$ satisfying

$$\left\| e^{tA\lambda} \right\|_{L(X)} \le 1$$

for each $t \ge 0$. In addition for each $x \in X$ and $\lambda, \mu > 0$, we have

$$\left\| e^{tA\lambda} x - e^{tA\mu} x \right\| \le t \left\| A_\lambda x - A_\mu x \right\|.$$

Proof

Let $x \in D(A)$ and $\lambda > 0$. Then we have

$$\left\| \lambda R(\lambda, A) x - x \right\| = \left\| AR(\lambda, A) Ax \right\| \le \frac{1}{\lambda} \left\| Ax \right\| \tag{3.12}$$

and as a result

$$\lim_{\lambda \to \infty} \lambda R(\lambda, A) x = x$$

for each $x \in D(A)$.

Since $D(A)$ is dense in X and

$$\left\| \lambda R(\lambda, A) \right\|_{L(X)} \le 1,$$

and from (3.12), we deduce

$$\lim_{\lambda \to \infty} \lambda R(\lambda, A) x = x.$$

To show 2). Let us remark that we have successively

$$\lambda^2 R(\lambda, A) - \lambda I = \lambda^2 R(\lambda, A) - \lambda(\lambda I - A) R(\lambda, A) = \lambda AR(\lambda, A) = A_\lambda \tag{3.13}$$

So, if $x \in D(A)$, by 1), we have

$$\lim_{\lambda \to \infty} A_\lambda x = \lim_{\lambda \to \infty} \lambda R(\lambda, A) x = \lim_{\lambda \to \infty} \lambda R(\lambda, A) Ax = Ax, \tag{3.14}$$

which complete the proof of 2) and 3).

To show that $\left\| e^{tA\lambda} \right\|_{L(X)} \le 1$ for each $t \ge 0$. Since $A_\lambda \in \omega\text{-}OCP_n$ and $\omega\text{-}OCP_n \in L(X)$, then by theorem of uniformly continuous semigroup, it follows that its generates a uniformly semigroup $\left\{ e^{tA_\lambda}; t \ge 0 \right\}$.

In order to show that $\left\| e^{tA_\lambda} \right\|_{L(X)} \le 1$, let us remark that, by virtue of $A_\lambda x = \lambda^2 R(\lambda, A) x - \lambda x$ for each $x \in X$ and b), we have

$$\begin{aligned} \left\| e^{tA_\lambda} \right\|_{L(X)} &= \left\| e^{t\lambda^2 R(\lambda, A) - t\lambda I} \right\|_{L(X)} \le \left\| e^{t\lambda^2 R(\lambda, A)} \right\|_{L(X)} \left\| e^{-t\lambda I} \right\|_{L(X)} \\ &\le e^{t\lambda^2 \left\| R(\lambda, A) \right\|_{L(X)}} e^{-t\lambda I} \le e^{t\lambda I} e^{-t\lambda I} = 1 \end{aligned} \tag{3.15}$$

Since A_λ, A_μ, e^{tA_λ} and e^{tA_μ} commute each to another for each $\lambda, \mu \in \rho(A)$ and $\lambda, \mu > 0$, we have

$$\begin{aligned} \left\| e^{tA_\lambda} x - e^{tA_\mu} x \right\| &= \left\| \int_0^1 \frac{\mathrm{d}}{\mathrm{d}s} \left(e^{stA_\lambda} e^{(1-s)tA_\mu} x \right) \mathrm{d}s \right\| \\ &\le \int_0^1 t \left\| e^{stA_\lambda} e^{(1-s)tA_\mu} \left(A_\lambda x - A_\mu x \right) \right\| \mathrm{d}s \\ &\le t \left\| A_\lambda x - A_\mu x \right\| \end{aligned} \tag{3.16}$$

Hence the proof is complete.

Theorem 3.4

For $A \in \omega\text{-}OCP_n$, we have $A: D(A) \subseteq X \to X$ to be a linear operator satisfying both $(0,+\infty)\subseteq\sigma(A)$ and

$$\left\|\lambda^n R(\lambda,A)^n\right\|_{L(X)}\leq M$$

for each $n\in N$ and $\lambda>0$ and if λ,μ are regular values, *i.e.* $\lambda,\mu\in\rho(A)$ and $R(\lambda,A),R(\mu,A)\in L(X)$, then there exist:

1) $R(\lambda,A)-R(\mu,A)=(\mu-\lambda)R(\lambda,A)R(\mu,A)$.
2) $\|x\|\leq|x|\leq M\|x\|$.
3) $|\lambda R(\lambda,A)x|\leq|x|$ for each $x\in X$ and $\lambda>0$.

Proof

To prove 1), let us observe that

$$\begin{aligned}R(\lambda,A)&=R(\lambda,A)(\mu I-A)R(\mu,A)\\&=R(\lambda,A)\{(\mu-\lambda)I+(\lambda I-A)\}R(\mu,A)\\&=(\mu-\lambda)R(\lambda,A)R(\mu,A)+R(\mu,A)\end{aligned}$$

so that

$$R(\lambda,A)-R(\mu,A)=(\mu-\lambda)R(\lambda,A)R(\mu,A) \tag{3.17}$$

and this complete the proof of 1).

To prove 2), we assume for $\mu>0$ and let us define $|\cdot|_\mu:D(A)\subseteq X\to\mathbb{R}_+$ by

$$|x|_\mu=\text{Sup}_{n\in\mathbb{N}}\left\|\mu^n R(\mu,A)^n x\right\|, \tag{3.18}$$

it's obvious that

$$\|x\|\leq|x|_\mu\leq M\|x\| \tag{3.19}$$

and

$$|\mu R(\mu,A)x|_\mu\leq|x|_\mu.$$

We want to prove that

$$|\lambda R(\lambda,A)x|_\mu\leq|x|_\mu \tag{3.20}$$

for each $\lambda\in(0,\mu]$.

So by resolvent Equation (3.17), we have

$$R(\lambda,A)x=R(\mu,A)(x+(\mu-\lambda)R(\mu,A)x)$$

and therefore

$$|R(\lambda,A)x|_\mu\leq\frac{1}{\mu}|x|_\mu+\left(1-\frac{\lambda}{\mu}\right)|R(\lambda,A)x|_\mu.$$

Consequently $\lambda|R(\lambda,A)x|_\mu\leq|x|_\mu$ which proves (3.20).

From (3.19) and (3.20), we deduced that, for each $n\in\mathbb{N}$ and $\lambda\in(0,\mu]$, we have

$$\left\|\lambda^n R(\lambda,A)^n x\right\|\leq\left|\lambda^n R(\lambda,A)^n x\right|\leq|x|_\mu \tag{3.21}$$

Passing to the Sup for $n\in\mathbb{N}$ on the left hand side of the inequality above, we now get $|x|_\lambda \le |x|_\mu$ for each $\lambda \in (0, \mu]$. We can now define

$$|x| = \lim_{\mu \to 0} |x|_\mu \tag{3.22}$$

Since 2) readily follows from (3.19), and 3) from (3.21) by taking $n = 1$, we have

$$|\lambda R(\lambda, A)x| \le |x|$$

Hence the proof.

Conflicts of Interest

The authors declare no conflicts of interest regarding the publication of this paper.

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