Some Inequalities on $T_3$ Tree

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Abstract

The article proves several inequalities derived from nodal multiplication on $T_3$ tree. The proved inequalities are helpful to estimate certain quantities related with the $T_3$ tree as well as examples of proving an inequality embedded with the floor functions.

Keywords

Inequality, Floor Function, Binary Tree

1. Introduction

The $T_3$ tree, which first appeared in [1] and was formerly introduced in [2], is a perfect complete binary tree that is considered to be a new tool to study integers. The tree can reveal many new properties of integers such as the symmetric properties discovered in [3] and [4], the genetic property found in [5], and other properties introduced in [6] and [7]. The tree also shows its big potentiality in factorization of big semiprimes, as seen in [8] and [9]. A recent study found several inequalities related with estimation of multiplication on the tree. This article introduces the main results.

2. Preliminaries

This section lists for later sections the necessary preliminaries, which include definitions, notations and lemmas.

2.1. Definitions and Notations

Symbol $T_3$ is the $T_3$ tree that was introduced in [1] and [2] and symbol $N_{(k,j)}$ is by default the node at position $j$ on level $k$ of $T_3$, where $k \geq 0$ and $0 \leq j \leq 2^{k} - 1$. Number of the level by default begins at zero and index of the po-
sition also by default begins at zero. Symbol \( \lfloor x \rfloor \) is the floor function, an integer function of real number \( x \) that satisfies inequality \( x - 1 < \lfloor x \rfloor \leq x \), or equivalently \( \lfloor x \rfloor \leq x < \lfloor x \rfloor + 1 \). Symbol \( A \Rightarrow B \) means conclusion \( B \) can be derived from condition \( A \).

For convenience in deduction of a formula, comments are inserted by symbols that express their related mathematical foundations. For example, the following deduction

\[
A = B \quad (L) = C \quad (P) \leq D
\]

means that, lemma \((L)\) supports the step from \(B\) to \(C\), and proposition \((P)\) supports the step from \(C\) to \(D\).

### 2.2. Lemmas

**Lemma 1.** (See in [1]) \( T_3 \) tree has the following fundamental properties.

(P1). Every node is an odd integer and every odd integer bigger than 1 must be on the \( T_3 \) tree. Odd integer \( N \) with \( N > 1 \) lies on level \( \left\lfloor \log_2 N \right\rfloor - 1 \).

(P2). On level \( k \) with \( k = 0, 1, \ldots \), there are \( 2^k \) nodes starting by \( 2^{k+1} + 1 \) and ending by \( 2^{k+2} - 1 \), namely, \( N_{(k,j)} \in [2^{k+1} + 1, 2^{k+2} - 1] \) with \( j = 0, 1, \ldots, 2^k - 1 \).

(P3). \( N_{(k,j)} \) is calculated by

\[
N_{(k,j)} = 2^{k+1} + 1 + 2j, \ j = 0, 1, \ldots, 2^k - 1
\]

(P4). Multiplication of arbitrary two nodes of \( T_3 \), say \( N_{(m,\alpha)} \) and \( N_{(n,\beta)} \), is a third node of \( T_3 \). Let \( J = 2^m (1 + 2\beta) + 2^n (1 + 2\alpha) + 2\alpha\beta + \alpha + \beta \); the multiplication \( N_{(m,\alpha)} \times N_{(n,\beta)} \) is given by

\[
N_{(m,\alpha)} \times N_{(n,\beta)} = 2^{m+n+2} + 1 + 2J
\]

If \( J < 2^{m+n+1} \), then \( N_{(m,\alpha)} \times N_{(n,\beta)} = N_{(m+n+1,\chi)} \) lies on level \( m+n+1 \) of \( T_3 \); whereas, if \( J \geq 2^{m+n+1} \), \( N_{(m,\alpha)} \times N_{(n,\beta)} = N_{(m+n+2,\chi)} \) with \( \chi = J - 2^{m+n+1} \) lies on level \( m+n+2 \) of \( T_3 \).

**Lemma 2.** (See in [10]) Let \( \alpha \) and \( x \) be a positive real numbers; then it holds

\[
\alpha \lfloor x \rfloor - 1 < \lfloor \alpha x \rfloor < \alpha \left( \lfloor x \rfloor + 1 \right)
\]

Particularly, if \( \alpha \) is a positive integer, say \( \alpha = n \), then it yields

\[
n \lfloor x \rfloor \leq \lfloor nx \rfloor \leq n \left( \lfloor x \rfloor + 1 \right) - 1
\]

### 3. Main Results with Proofs

**Proposition 1.** For positive integer \( k \) and real number \( x > 0 \), it holds

\[
0 \geq 2^k \left\lfloor \frac{x}{2^k} \right\rfloor - \lfloor x \rfloor \geq \left\lfloor -\frac{1}{2^k}, 0 \leq k \leq \left\lfloor \log_2 x \right\rfloor \right\rfloor
\]

\[
\left\lfloor -\frac{1}{2^k}, k > \left\lfloor \log_2 x \right\rfloor \right\rfloor
\]

**Proof:** It can see by Lemma 2 that,
and 

\[ 2^{k} \left\lfloor \frac{x}{2^k} \right\rfloor - \left\lfloor x \right\rfloor \geq \left( 2^{k} \left\lfloor \frac{x}{2^k} \right\rfloor + 1 - 2^k \right) - \left\lfloor x \right\rfloor = 1 - 2^k \]

Meanwhile, when \( 2^k > x \), or \( k > \log_2 x \), \( \left\lfloor \log_2 x \right\rfloor \geq 0 \), \( \left\lfloor \frac{x}{2^k} \right\rfloor = 0 \); thus 

\[ 2^{k} \left\lfloor \frac{x}{2^k} \right\rfloor - \left\lfloor x \right\rfloor = -\left\lfloor x \right\rfloor . \]

Consequently (1) holds.

**Proposition 2.** Let \( N_{(m,\alpha)} \) and \( N_{(n,\beta)} \) be nodes of \( T_3 \) with \( 0 \leq m \leq n \); let 

\[ J = \frac{N_{(m,\alpha)} \times N_{(n,\beta)} - 1}{2} - 2^{m+n+1} \tag{2} \]

then when \( J < 2^{m+n+1} \)

\[ 2 \leq \left\lfloor \frac{N_{(m,\alpha)} \times N_{(n,\beta)} - 1}{2^{m+n+1}} \right\rfloor \leq 3 \]

\[ \left\lfloor \frac{N_{(m,\alpha)} \times N_{(n,\beta)} - 1}{2^{m+n+2}} \right\rfloor = 1 \]

\[ \left\lfloor \frac{N_{(m,\alpha)} \times N_{(n,\beta)} - 1}{2^{m+n+2+\sigma}} \right\rfloor = 0 \]

and when \( J \geq 2^{m+n+1} \)

\[ 2 \leq \left\lfloor \frac{N_{(m,\alpha)} \times N_{(n,\beta)} - 1}{2^{m+n+2}} \right\rfloor \leq 3 \]

\[ \left\lfloor \frac{N_{(m,\alpha)} \times N_{(n,\beta)} - 1}{2^{m+n+3}} \right\rfloor = 1 \]

\[ \left\lfloor \frac{N_{(m,\alpha)} \times N_{(n,\beta)} - 1}{2^{m+n+3+\sigma}} \right\rfloor = 0 \]

**Proof.** By Lemma 1 (P4), it knows, when \( J < 2^{m+n+1} \), \( N_{(m,\alpha)} \times N_{(n,\beta)} \) lies on level \( m+n+1 \) of \( T_3 \) and thus \( 2^{m+n+2} + 1 \leq N_{(m,\alpha)} \times N_{(n,\beta)} \leq 2^{m+n+3} - 1 \); hence it holds 

\[ 2 = \frac{2^{m+n+2}}{2^{m+n+1}} \leq \frac{N_{(m,\alpha)} \times N_{(n,\beta)} - 1}{2^{m+n+1}} \leq \frac{2^{m+n+3} - 2}{2^{m+n+1}} < 4 \]

and 

\[ 1 = \frac{2^{m+n+2}}{2^{m+n+2}} \leq \frac{N_{(m,\alpha)} \times N_{(n,\beta)} - 1}{2^{m+n+2}} \leq \frac{2^{m+n+3} - 2}{2^{m+n+2}} = 2 - \frac{1}{2^{m+n+1}} < 2 \]

Thus
\begin{align*}
2 \leq \left\lfloor \frac{N_{(m,\alpha)} \times N_{(n,\beta)}}{2^{m+n+1}} \right\rfloor & \leq 3 \\
\text{and} \quad \left\lfloor \frac{N_{(m,\alpha)} \times N_{(n,\beta)}}{2^{m+n+2}} \right\rfloor & = 1 \\
\text{and thus} \quad \left\lfloor \frac{N_{(m,\alpha)} \times N_{(n,\beta)}}{2^{m+n+2+\sigma}} \right\rfloor_{(\sigma \geq 1)} & = 0
\end{align*}

Similarly, when \( J \geq 2^{m+1} \), \( N_{(m,\alpha)} \times N_{(n,\beta)} \) lies on level \( m+n+2 \) of \( T_3 \) and \( 2^{m+3}+1 \leq N_{(m,\alpha)} \times N_{(n,\beta)} \leq 2^{m+4}-1 \) and it holds
\begin{align*}
2 & = \frac{2^{m+3}}{2^{m+2}} < \frac{N_{(m,\alpha)} \times N_{(n,\beta)} - 1}{2^{m+2}} \leq \frac{2^{m+4}-2}{2^{m+2}} = 4 - \frac{1}{2^{m+1}} < 4 \\
\Rightarrow 2 & \leq \left\lfloor \frac{N_{(m,\alpha)} \times N_{(n,\beta)} - 1}{2^{m+n+2}} \right\rfloor \leq 3
\end{align*}

and
\begin{align*}
1 & = \frac{2^{m+3}}{2^{m+3}} < \frac{N_{(m,\alpha)} \times N_{(n,\beta)} - 1}{2^{m+3}} \leq \frac{2^{m+4}-2}{2^{m+3}} = 2 - \frac{1}{2^{m+2}} < 2 \\
\Rightarrow \left\lfloor \frac{N_{(m,\alpha)} \times N_{(n,\beta)} - 1}{2^{m+n+3}} \right\rfloor = 1 \Rightarrow \left\lfloor \frac{N_{(m,\alpha)} \times N_{(n,\beta)} - 1}{2^{m+n+3+\sigma}} \right\rfloor_{(\sigma \geq 1)} = 0
\end{align*}

**Proposition 3.** Let \( N_{(m,\alpha)} \) be a node of \( T_3 \) and \( n \) be an integer with \( 0 \leq m \leq n \); then it holds
\begin{align*}
-1 < \frac{N_{(m,\alpha)} - 1}{2^{m+1}} & < 1 \quad (3) \\
0 < \frac{N_{(m,\alpha)} + 1}{2^{m+1}} & \leq 2 \quad (4)
\end{align*}

Thus for arbitrary integer \( \sigma \geq 0 \)
\begin{align*}
-\frac{1}{2^\sigma} < \frac{N_{(m,\alpha)} - 1}{2^{m+1+\sigma}} - \frac{1}{2^\sigma} < \frac{1}{2^\sigma} \quad (5) \\
0 < \frac{N_{(m,\alpha)} + 1}{2^{m+1+\sigma}} \leq 2^{1-\sigma} \quad (6)
\end{align*}

**Proof.** Considering that \( 2^{m+1} + 1 \leq N_{(m,\alpha)} \leq 2^{m+2} - 1 \) holds for arbitrary \( m \geq 0 \), it yields
\begin{align*}
-1 + \frac{1}{2^{m+1}} = \frac{2^{m+1}}{2^{m+1}} - 1 \leq \frac{N_{(m,\alpha)} - 1}{2^{m+1}} - 1 \leq \frac{2^{m+2}}{2^{m+1}} - 1 - 1 = \frac{1}{2^{m+1}} - \frac{1}{2^m} \quad (7)
\end{align*}

and
\begin{align*}
0 < \frac{1}{2^{m+1}} + \frac{1}{2^\sigma} = \frac{2^{m+1}}{2^{m+1}} + \frac{1}{2^\sigma} \leq \frac{N_{(m,\alpha)} + 1}{2^{m+1}} \leq \frac{2^{m+2}}{2^{m+1}} = \frac{2}{2^{m+1}} \leq 2 \quad (8)
\end{align*}
Consider in (7)

\[
\frac{1}{2^{n-m}} - 1 = \begin{cases} 
1 - \frac{1}{2^n} < 1, n = m \\
- \frac{1}{2^n} < 0, n = m + 1 \\
2^{n-m-1} - \frac{1}{2^n} - 1 < 0, n > m + 1 
\end{cases}
\]

and

\[
\frac{1}{2^{n-m}} - 1 = \begin{cases} 
0, n = m \\
-1 + \frac{1}{2^{n-m}} > -1, n > m 
\end{cases}
\]

it knows (3) and (4) hold and consequently (5) and (6) hold.

**Proposition 4.** Let \( N_{(m,\alpha)} \) and \( N_{(n,\beta)} \) be nodes of \( T_3 \) with \( 0 \leq m \leq n \); then it holds

\[
N_{(m,\alpha)} + \frac{N_{(m,\alpha)} - 1}{2^{m+1}} \leq \frac{N_{(m,\alpha)} \times N_{(n,\beta)} - 1}{2^{m+1}} \leq 2N_{(m,\alpha)} - \frac{N_{(m,\alpha)} + 1}{2^{m+1}} \tag{9}
\]

and thus for arbitrary integer \( \sigma \geq 0 \) it holds

\[
\frac{N_{(m,\alpha)} \times N_{(n,\beta)} - 1}{2^{m+1}} \leq \frac{N_{(n,\beta)} + 1}{2^{m+1+\sigma}} \leq \frac{N_{(m,\alpha)} \times N_{(n,\beta)} - 1}{2^{\sigma-1}} - \frac{N_{(m,\alpha)} + 1}{2^{m+1+\sigma}} \tag{10}
\]

Consequently, it yields

\[
N_{(m,\alpha)} \leq \left[ \frac{N_{(m,\alpha)} \times N_{(n,\beta)} - 1}{2^{m+1}} \right] \leq 2N_{(m,\alpha)} - 1 \tag{11}
\]

\[
\frac{N_{(m,\alpha)} - 1}{2} - 1 \leq \left[ \frac{N_{(m,\alpha)} \times N_{(n,\beta)} - 1}{2^{m+1}} \right] \leq N_{(m,\alpha)} - 1 \tag{12}
\]

and

\[
\frac{N_{(m,\alpha)} - 1}{2} - 2 \leq \left[ \frac{N_{(m,\alpha)} \times N_{(n,\beta)} - 1}{2^{m+1}} \right] \leq \frac{N_{(m,\alpha)} - 1}{2} \tag{13}
\]

\[
\frac{N_{(m,\alpha)} - 1}{2} - 2 \leq \left[ \frac{N_{(m,\alpha)} \times N_{(n,\beta)} - 1}{2^{m+1}} \right] \leq N_{(m,\alpha)} - 1 \tag{13^*}
\]

**Proof.** The condition that \( N_{(n,\beta)} \) is a node of \( T_3 \) leads to

\[
2^{m+1} + 1 \leq N_{(n,\beta)} \leq 2^{m+2} - 1
\]

Then direct calculation shows

\[
\frac{N_{(m,\alpha)} \times N_{(n,\beta)} - 1}{2^{m+1}} - N_{(m,\alpha)} - \frac{N_{(m,\alpha)} - 1}{2^{m+1}} = N_{(m,\alpha)} \times N_{(n,\beta)} - 2^{m+1} N_{(m,\alpha)} - N_{(m,\alpha)} + 1
\]

\[
= \frac{N_{(m,\alpha)} \times N_{(n,\beta)} - (2^{m+1} + 1)}{2^{m+1}} \geq 0
\]
and

\[ \frac{N_{m,n} \times N_{n,f} - 1}{2^{n+1}} - 2N_{m,n} + \frac{N_{m,n} + 1}{2^{n+1}} \]
\[ = \frac{N_{m,n} \times N_{n,f} - 1 - 2^{n+2} N_{m,n} + N_{m,n} + 1}{2^{n+1}} \]
\[ = \frac{N_{m,n} \times (N_{n,f} - (2^{n+2} - 1))}{2^{n+1}} \leq 0 \]

Hence (9) holds.

Multiplying each item in (9) by \( \frac{1}{2^{n+1}} \) for integer \( \sigma \geq 1 \) immediately yields (10).

By definition of the floor function, it holds

\[ \left\lfloor \frac{N_{m,n} \times N_{n,f} - 1}{2^{n+1}} \right\rfloor - 1 < \left\lfloor \frac{N_{m,n} \times N_{n,f} - 1}{2^{n+1}} \right\rfloor \leq \left\lfloor \frac{N_{m,n} \times N_{n,f} - 1}{2^{n+1}} \right\rfloor \leq 2N_{m,n} - 1 \]

By the Inequalities (3), (4) and (9) it yields

\[ \Rightarrow N_{m,n} - 1 < \left\lfloor \frac{N_{m,n} \times N_{n,f} - 1}{2^{n+1}} \right\rfloor \leq 2N_{m,n} - 1 \]

\[ \Rightarrow N_{m,n} \leq \left\lfloor \frac{N_{m,n} \times N_{n,f} - 1}{2^{n+1}} \right\rfloor \leq 2N_{m,n} - 1 \]

which says (11) holds.

Likewise, by definition of the floor function and referring to the Inequalities (5), (6) and (10), it yields

\[\begin{align*}
\frac{N_{m,n} \times N_{n,f} - 1}{2^{n+2}} - 1 &< \left\lfloor \frac{N_{m,n} \times N_{n,f} - 1}{2^{n+2}} \right\rfloor \leq \left\lfloor \frac{N_{m,n} \times N_{n,f} - 1}{2^{n+2}} \right\rfloor \leq N_{m,n} - \frac{N_{m,n} + 1}{2^{n+2}} \\
\Rightarrow \frac{N_{m,n}}{2} + \frac{N_{m,n} - 1}{2^{n+2}} - 1 &< \frac{N_{m,n} \times N_{n,f} - 1}{2^{n+2}} - 1 < \left\lfloor \frac{N_{m,n} \times N_{n,f} - 1}{2^{n+2}} \right\rfloor \leq N_{m,n} - \frac{N_{m,n} + 1}{2^{n+2}} \\
\Rightarrow \frac{N_{m,n}}{2} + \frac{N_{m,n} - 1}{2^{n+2}} - 1 &< \frac{N_{m,n} \times N_{n,f} - 1}{2^{n+2}} \leq N_{m,n} - 1 \\
\Rightarrow \frac{N_{m,n}}{2} + \frac{N_{m,n} - 1}{2^{n+2}} - 1 &< \left\lfloor \frac{N_{m,n} \times N_{n,f} - 1}{2^{n+2}} \right\rfloor \leq N_{m,n} - 1 \\
\Rightarrow \frac{N_{m,n}}{2} - \frac{1}{2} &< \left\lfloor \frac{N_{m,n} \times N_{n,f} - 1}{2^{n+2}} \right\rfloor \leq N_{m,n} - 1 \\
\Rightarrow \frac{N_{m,n} - 1}{2} &\leq \left\lfloor \frac{N_{m,n} \times N_{n,f} - 1}{2^{n+2}} \right\rfloor \leq N_{m,n} - 1
\end{align*}\]

which is the (12).

Similarly, the Inequalities (10) and the definition of the floor function lead to
\[
\frac{N(m, \alpha)}{2^2} + \frac{N(n, \beta)}{2^{n+3}} - 1 \leq \frac{N(m, \alpha) \times N(n, \beta) - 1}{2^{n+3}} \leq \frac{N(m, \alpha)}{2} - \frac{N(n, \beta) + 1}{2^{n+3}}
\]

and
\[
\frac{N(m, \alpha) \times N(n, \beta) - 1}{2^{n+3}} - 1 < \left\lfloor \frac{N(m, \alpha) \times N(n, \beta) - 1}{2^{n+3}} \right\rfloor \leq \frac{N(m, \alpha) \times N(n, \beta) - 1}{2^{n+3}}
\]

Then referring to the Inequalities (5) and (6), it immediately results in
\[
\frac{N(m, \alpha)}{2^2} + \frac{N(n, \beta)}{2^{n+3}} - 1 < \left\lfloor \frac{N(m, \alpha) \times N(n, \beta) - 1}{2^{n+3}} \right\rfloor \leq \frac{N(m, \alpha) \times N(n, \beta) - 1}{2^{n+3}} < \frac{N(m, \alpha)}{2} - \frac{N(n, \beta) + 1}{2^{n+3}}
\]

\[
\frac{N(m, \alpha)}{2^2} - 1 < \left\lfloor \frac{N(m, \alpha) \times N(n, \beta) - 1}{2^{n+3}} \right\rfloor \leq \frac{N(m, \alpha) - 1}{2}
\]

\[
\frac{N(m, \alpha)}{2} - \frac{1}{2} \leq \left\lfloor \frac{N(m, \alpha) \times N(n, \beta) - 1}{2^{n+3}} \right\rfloor \leq \frac{N(m, \alpha) - 1}{2}
\]

which is just the (13).

**Proposition 5.** Let \( N(m, \alpha) \) and \( N(n, \beta) \) be nodes of \( T_3 \) with \( 0 \leq m \leq n \); then it holds for integer \( 0 \leq s \leq m \)
\[
N(m, \alpha) - 2^{s+2} + 1 \leq 2^{s+2} \left\lfloor \frac{N(m, \alpha) \times N(n, \beta) - 1}{2^{n+2+s}} \right\rfloor - 2 \left\lfloor \frac{N(m, \alpha) \times N(n, \beta) - 1}{2^{n+2}} \right\rfloor \leq 2N(m, \alpha) - 1 \tag{16}
\]

and
\[
\frac{N(m, \alpha) - 1}{2} - 2^{s+2} \leq 2^{s+2} \left\lfloor \frac{N(m, \alpha) \times N(n, \beta) - 1}{2^{n+3+s}} \right\rfloor - 2 \left\lfloor \frac{N(m, \alpha) \times N(n, \beta) - 1}{2^{n+3}} \right\rfloor \leq N(m, \alpha) - 1 \tag{17}
\]

**Proof.** By Lemma 2 and Proposition 1, it holds when \( 0 \leq s \leq m \)
\[
2^{s+2} \left\lfloor \frac{N(m, \alpha) \times N(n, \beta) - 1}{2^{n+2+s}} \right\rfloor - 2 \left\lfloor \frac{N(m, \alpha) \times N(n, \beta) - 1}{2^{n+2}} \right\rfloor \tag{P1}
\]

\[
N(m, \alpha) - 1 - 2^{s+2} \leq 2^{s+2} \left\lfloor \frac{N(m, \alpha) \times N(n, \beta) - 1}{2^{n+3+s}} \right\rfloor - 2 \left\lfloor \frac{N(m, \alpha) \times N(n, \beta) - 1}{2^{n+3}} \right\rfloor \leq N(m, \alpha) - 1 \tag{L2}
\]

and
\[ 2^{s_{i_2}} \left\lfloor N_{(m,a)} \times N_{(n,b)} \right\rfloor - 2 \left( N_{(m,a)} \times N_{(n,b)} \right) - 1 \]

\[ (L2) \leq 2^{s_{i_2}} \left\lfloor N_{(m,a)} \times N_{(n,b)} \right\rfloor - 2 \left( \left\lfloor N_{(m,a)} \times N_{(n,b)} \right\rfloor \right) - 1 \]

\[ = 2 \left( N_{(m,a)} \times N_{(n,b)} \right) - 2 \left( N_{(m,a)} \times N_{(n,b)} \right) - 1 \]

\[ (L2) \leq 2 \left( N_{(m,a)} \times N_{(n,b)} \right) - 2 \left( N_{(m,a)} \times N_{(n,b)} \right) - 1 \]

That is

\[ \left\lfloor N_{(m,a)} \times N_{(n,b)} \right\rfloor + 1 - 2^{s_{i_2}} \]

\[ \leq 2^{s_{i_2}} \left\lfloor N_{(m,a)} \times N_{(n,b)} \right\rfloor - 2 \left( N_{(m,a)} \times N_{(n,b)} \right) - 1 \]

By (11) it holds

\[ N_{(m,a)} + 1 - 2^{s_{i_2}} \leq 2^{s_{i_2}} \left( N_{(m,a)} \times N_{(n,b)} \right) - 2 \left( N_{(m,a)} \times N_{(n,b)} \right) - 1 \]

which is just the (16).

Similarly it holds

\[ \left\lfloor N_{(m,a)} \times N_{(n,b)} \right\rfloor + 1 - 2^{s_{i_2}} \]

\[ \leq 2^{s_{i_2}} \left\lfloor N_{(m,a)} \times N_{(n,b)} \right\rfloor - 2 \left( N_{(m,a)} \times N_{(n,b)} \right) - 1 \]

and by (12) it yields

\[ \frac{N_{(m,a)} - 1}{2} - 2^{s_{i_2}} \leq 2^{s_{i_2}} \left( N_{(m,a)} \times N_{(n,b)} \right) - 2 \left( N_{(m,a)} \times N_{(n,b)} \right) - 1 \]

\[ \leq N_{(m,a)} - 1 \]

4. Conclusion

The \( T_i \) tree is emerging its value in studying integers. A lot of equations and inequalities will be research objectives. Since most of the inequalities on the \( T_i \) tree are in the form of floor functions, their proofs are often skillful. The inequalities proved in this article are not only quite useful for knowing the \( T_i \) tree, but also excellent samples for proving inequalities with the floor functions. Hope it helpful to the readers of interests.

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Conflicts of Interest

The authors declare no conflicts of interest regarding the publication of this paper.

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