The Analyticity for the Product of Analytic Functions on Octonions and Its Applications

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Abstract

Given two left \(\mathbb{O}^c\)-analytic functions \(f, g\) in some open set \(\Omega\) of \(\mathbb{R}^8\), we obtain some sufficient conditions for \(fg\) is also left \(\mathbb{O}^c\)-analytic in \(\Omega\). Moreover, we prove that \(f\lambda\) is a left \(\mathbb{O}^c\)-analytic function for any constants \(\lambda \in \mathbb{O}^c\) if and only if \(f\) is a complex Stein-Weiss conjugate harmonic system. Some applications and connections with Cauchy-Kowalewski product are also considered.

Keywords

Octonions, \(\mathbb{O}^c\)-Analytic Functions, Stein-Weiss Conjugate Harmonic System, Cauchy-Kowalewski Product

1. Introduction

Let \(\Omega\) be an open set of \(\mathbb{R}^8\). A function \(f\) in \(C^1(\Omega, \mathbb{O})\) is said to be left (right) \(\mathbb{O}\)-analytic in \(\Omega\) when

\[
Df = \sum_{j=0}^{7} e_j \frac{\partial f}{\partial x_j} = 0 \quad \left(fD = \sum_{j=0}^{7} \frac{\partial f}{\partial x_j} e_j = 0\right),
\]

where the Dirac \(D\)-operator and its adjoint \(\bar{D}\) are the first-order systems of differential operators in \(C^1(\Omega, \mathbb{O})\) defined by \(D = \sum_{j=0}^{7} e_j \frac{\partial}{\partial x_j}\) and

\[
\bar{D} = e_0 \frac{\partial}{\partial x_0} - \sum_{j=1}^{7} e_j \frac{\partial}{\partial x_j}.
\]

If \(f\) is a simultaneously left and right \(\mathbb{O}\)-analytic function, then \(f\) is called an \(\mathbb{O}\)-analytic function. If \(f\) is a (left) \(\mathbb{O}\)-analytic function in \(\mathbb{R}^8\), then \(f\) is called a (left) \(\mathbb{O}\)-entire function.
Since octonions is non-commutative and non-associative, the product \( f(x)g(x) \) of two left \( \mathbb{O} \)-analytic functions \( f(x) \) and \( g(x) \) is generally no longer a left \( \mathbb{O} \)-analytic function. Furthermore, if \( g(x) \equiv \lambda \) becomes an octonionic constant function, the product \( f(x)\lambda \) is also probably not a left \( \mathbb{O} \)-analytic function; that is, the collection of left \( \mathbb{O} \)-analytic functions is not a right module (see [1]).

The purpose of this paper is to study the analyticity for the product of two left \( \mathbb{O}' \)-analytic functions in the framework of complexification of \( \mathbb{O}, \mathbb{O}' \). Especially, the analyticity for the product of left \( \mathbb{O}' \)-analytic functions and \( \mathbb{O}' \) constants will be consider more by us.

The rest of this paper is organized as follows. Section 2 is an overview of some basic facts concerning octonions and octonionic analysis. Section 3 we give some sufficient conditions for the product \( f(x)g(x) \) of two left \( \mathbb{O}' \)-analytic functions \( f(x) \) and \( g(x) \) is also a left \( \mathbb{O}' \)-analytic function. In Section 3, we prove that, \( f(x)\lambda \) is a left \( \mathbb{O}' \)-analytic function for any constants \( \lambda \in \mathbb{O}' \) if and only if \( f(x) \) is a complex Stein-Weiss conjugate harmonic system. This gives the solution of the problem in [2]. In the last section we give some applications for our results.

2. Preliminaries: Octonions and Octonionic Analysis

It is well known that there are only four normed division algebras [3] [4] [5]: the real numbers \( \mathbb{R} \), complex numbers \( \mathbb{C} \), quaternions \( \mathbb{H} \) and octonions \( \mathbb{O} \), with the relations \( \mathbb{R} \subseteq \mathbb{C} \subseteq \mathbb{H} \subseteq \mathbb{O} \). In other words, for any \( x = (x_1, \ldots, x_n) \), \( y = (y_1, \ldots, y_n) \in \mathbb{R}^n \), if we define a product “\( \times \)” such that \( xy \in \mathbb{R}^n \) and \( |x \cdot y| = |x||y| \), where \( |x| = \sqrt{\sum x_i^2} \), then the only four values of \( n \) are 1, 2, 4, 8.

Quaternions \( \mathbb{H} \) is not commutative and octonions \( \mathbb{O} \) is neither commutative nor associative. Unlike \( \mathbb{R}, \mathbb{C} \) and \( \mathbb{H} \), the non-associative octonions can not be embedded into the associative Clifford algebras [6].

Octonions stand at the crossroads of many interesting fields of mathematics, they have close relations with Clifford algebras, spinors, Bott periodicity, Projection and Lorentzian geometry, Jordan algebras, and exceptional Lie groups, and also, they have many applications in quantum logic, special relativity and supersymmetry [3] [4].

Denote the set \( \mathbb{W} \) by

\[
\mathbb{W} = \{ (1,2,3),(1,4,5),(1,7,6),(2,4,6),(2,5,7),(3,4,7),(3,6,5) \}.
\]

Then the multiplication rules between the basis \( e_0, e_1, \ldots, e_7 \) on octonions are given by [3] [7]:

\[
e_0^2 = e_0, \quad e_i e_0 = e_i, \quad e_i e_0 = e_i, \quad e_i^2 = -1, \quad i = 1, 2, \ldots, 7,
\]

and for any triple \( (\alpha, \beta, \gamma) \in \mathbb{W} \),

\[
e_\alpha e_\beta = e_\gamma = -e_\gamma e_\alpha, \quad e_\beta e_\gamma = e_\alpha = -e_\alpha e_\beta, \quad e_\alpha e_\gamma = e_\beta = -e_\beta e_\gamma.
\]

For each \( x = \sum_{i=0}^{7} x_i e_i \in \mathbb{O} (x_i \in \mathbb{R}, i = 0, 1, \ldots, 7) \), \( x_0 \) is called the scalar part of
Let $x = \sum_{i=0}^{7} x_i e_i$ be termed its vector part. Then the norm of $x$ is $|x| = \left( \sum_{i=0}^{7} x_i^2 \right)^{1/2}$ and its conjugate is defined by $\overline{x} = \sum_{i=0}^{7} \overline{x_i} e_i = x_0 - x$. We have $x \overline{x} = \sum_{i=0}^{7} x_i \overline{x_i} = \sum_{i=0}^{7} x_i \overline{x_i} = (x, y \in \mathbf{O})$. Hence, $x^{-1} = \overline{x} |x|^2$ is the inverse of $x (\neq 0)$.

Let $x = \sum_{i=0}^{7} x_i e_i, y = \sum_{i=0}^{7} y_i e_i \in \mathbf{O}(x_j, y_j \in \mathbb{R}, i = 0, 1, \cdots, 7)$, then

$$xy = x_0 y_0 - x_1 y_1 + y_0 x_1 + x_2 y_2 + y_2 x_0 - x_0 y_2,$$

(2.1)
where $\langle x, y \rangle := \sum_{i=0}^{7} x_i y_i$ is the inner product of vectors $x, y$ and

$$x \times y := e_1 (A_{23} + A_{45} - A_{67}) + e_2 \left( -A_{13} + A_{26} + A_{47} \right) + e_3 \left( A_{12} + A_{47} - A_{56} \right) + e_4 \left( -A_{15} - A_{26} - A_{37} \right) + e_5 \left( A_{14} - A_{27} + A_{56} \right) + e_6 \left( A_{17} + A_{24} - A_{35} \right) + e_7 \left( -A_{16} + A_{25} + A_{34} \right)$$
is the cross product of vectors $x, y$, with

$$A_y = \text{det} \begin{pmatrix} x_i & x_j \\ y_i & y_j \end{pmatrix}, \quad i, j = 1, 2, \cdots, 7.$$

For any $x, y \in \mathbf{O}$, the inner product and cross product of their vector parts satisfy the following rules [8]:

$$\langle x \times y \rangle \cdot z = 0, \quad \langle x \times y \rangle \cdot y = 0, \quad x \parallel y \iff x \times y = 0, \quad x \times y = -y \times x.$$

We usually utilize associator as an useful tool on octonions since its non-associativity. Define the associator $[x, y, z]$ of any $x, y, z \in \mathbf{O}$ by $[x, y, z] = \langle xy \rangle z - x \langle yz \rangle$. The octonions obey the following some weakened associative laws.

For any $x, y, z, u, v \in \mathbf{O}$, we have (see [7])

$$[x, y, z] = [y, z, x], \quad [x, z, y] = -[x, y, z], \quad [x, x, y] = 0 = [x, x, x]$$

(2.2) and the so-called Moufang identities [5]

$$(uvu)x = u(v(ux)), \quad x(uvu) = (xu)v)u, \quad u(xy)u = (ux)(uy).$$

**Proposition 2.1** ([7]). For any $i, j, k \in \{0, 1, \cdots, 7\}$, $[e_i, e_j, e_k] = 0 \iff ijk = 0$ or $(i-j)(j-k)(k-i) = 0$ or $\langle e_i, e_j \rangle e_k = \pm 1$.

**Proposition 2.2** ([7]). Let $e_i, e_j, e_k$ be three different elements of $\{e_i, e_j, \cdots, e_k\}$ and $\langle e_i, e_j \rangle e_k \neq \pm 1$. Then $\langle e_i, e_j \rangle e_k = -\langle e_j, e_k \rangle e_i$.

Since octonions is an alternative algebra (see [3] [9] [10]), we have the following power-associativity of octonions.

**Proposition 2.3.** Let $x_1, x_2, \cdots, x_k \in \mathbf{O}$, $(l_1, \cdots, l_n)$ be $n$ elements out of $\{1, \cdots, k\}$ repetitions being allowed and let $(x_1 x_2 \cdots x_k)_{\oplus_n}$ be the product of $n$ octonions in a fixed associative order $\oplus_n$. Then $\sum_{(l_1, \cdots, l_n)} (x_1 x_2 \cdots x_k)_{\oplus_n}$ is independent of the associative order $\oplus_n$, where the sum runs over all...
distinguishable permutations of \((l_1,\cdots,l_k)\)

**Proof.** Let \(x = \lambda_1 x_1 + \lambda_2 x_2 + \cdots + \lambda_n x_n\), then \(\sum_{\pi \in \mathcal{I}(l_1,\cdots,l_k)} (x_1 x_2 \cdots x_n)_{\pi,\alpha}\) is just the coefficient of \(\lambda_1 \lambda_2 \cdots \lambda_n\) in the product of \(x^n = (x_1 \cdots x_n)_{\alpha,\alpha}\). By induction and (2.2), one can easily prove that \(x^n = (x_1 \cdots x_n)_{\alpha,\alpha}\) is independent of the associative order \(\otimes\) for any \(x \in \mathbf{O}\). Hence \(\sum_{\pi \in \mathcal{I}(l_1,\cdots,l_k)} (x_1 x_2 \cdots x_n)_{\pi,\alpha}\) is also independent of the associative order \(\otimes\). □

\(
\mu = (\mu_0, \mu_1, \cdots, \mu_n) \)

is called a Stein-Weiss conjugate harmonic system if they satisfy the following equations (see [11]):

\[
\sum_{j=0}^n \frac{\partial \mu_i}{\partial x_j} = 0, \quad \frac{\partial \mu_i}{\partial x_j} = \frac{\partial \mu_j}{\partial x_i} \quad (0 \leq i < j \leq n).
\]

It is easy to see that if \(F(x_0, x_1, \cdots, x_n) = (f_0, f_1, \cdots, f_n)\) is a Stein-Weiss conjugate harmonic system in an open set \(\Omega\) of \(R^n\), then there exists a real-valued harmonic function \(\Phi\) in \(\Omega\) such that \(F\) is the gradient of \(\Phi\). Thus \(\overline{F} = f_0 e_0 - f_1 e_1 - \cdots - f_n e_n = \overline{D}\Phi\) is an \(\mathbf{O}\)-analytic function. But inversely, this is not true [12].

**Example.** Observe the \(\mathbf{O}\)-analytic function \(g(x) = \left(x_1^2 - x_2^2\right) e_2 - 2 x_0 x_3 e_1\). Since

\[
\frac{\partial g_2}{\partial x_0} = 2 x_0 \neq 0 = \frac{\partial g_2}{\partial x_2},
\]

\(g\) is not a Stein-Weiss conjugate harmonic system.

In [13] Li and Peng proved the octonionic analogue of the classical Taylor theorem. Taking account of Proposition 2.3, we obtain an improving of Taylor type theorem for \(\mathbf{O}\)-analytic functions (see [14] [15]).

**Theorem A (Taylor).** If \(f(x)\) is a left \(\mathbf{O}\)-analytic function in \(\Omega\) which containing the origin, then it can be developed into Taylor series

\[
f(x) = \sum_{k=0}^{\infty} \sum_{\pi \in \mathcal{I}(l_1,\cdots,l_k)} V_{\pi,\alpha} (x) \partial_{x_{l_1}} \cdots \partial_{x_{l_k}} f(0),
\]

and if \(f(x)\) is a right \(\mathbf{O}\)-analytic function, then the Taylor series of \(f\) at the origin is given by

\[
f(x) = \sum_{k=0}^{\infty} \sum_{\pi \in \mathcal{I}(l_1,\cdots,l_k)} \partial_{x_{l_1}} \cdots \partial_{x_{l_k}} f(0) V_{\pi,\alpha} (x),
\]

where \((l_1,\cdots,l_k)\) runs over all possible combinations of \(k\) elements out of \(\{1,\cdots,7\}\) repetitions being allowed.

The polynomials \(V_{\pi,\alpha}\) of order \(k\) in Theorem A is defined by

\[
V_{\pi,\alpha} (x) = \frac{1}{k!} \sum_{\pi \in \mathcal{I}(l_1,\cdots,l_k)} \left(\left(z_{l_1} z_{l_2}\right) z_{l_3} \cdots \right) z_{l_k},
\]

where the sum runs over all distinguishable permutations of \((l_1,\cdots,l_k)\) and
\[ z_j = x_j e_0 - x_0 e_j, \quad j = 1, \ldots, k. \]

We have the following uniqueness theorem for \( O \)-analytic functions [7].

**Proposition 2.4.** If \( f \) is left (right) \( O \)-analytic in an open connect set \( \Omega \subset \mathbb{R}^k \) and vanishes in the open set \( \mathcal{E} \subset \Omega \cap \{x_0 = a_0\} \neq \emptyset \), then \( f \) is identically zero in \( \Omega \).

**Proof.** Without loss of generality, we let \( \mathcal{E} \) which containing the origin and let \( x_0 = 0 \). Then \( f \) can be developed into Taylor series

\[
f(x) = \sum_{k=0}^{\infty} \sum_{i=0}^{7} f_{i_1 \ldots i_k} (0) \partial_{i_1} \cdots \partial_{i_k} x_0 = \sum_{k=0}^{\infty} \sum_{i=0}^{7} f_{i_1 \ldots i_k} (0) \partial_{i_1} \cdots \partial_{i_k} x_0.
\]

Thus we have

\[
f(x) = \sum_{k=0}^{\infty} \sum_{i=0}^{7} f_{i_1 \ldots i_k} (0) \partial_{i_1} \cdots \partial_{i_k} x_0 = 0.
\]

By the uniqueness of the Taylor series for the real analytic function, we have

\[
f(x) = \sum_{k=0}^{\infty} \sum_{i=0}^{7} f_{i_1 \ldots i_k} (0) \partial_{i_1} \cdots \partial_{i_k} x_0 = 0.
\]

For more references about octonions and octonionic analysis, we refer the reader to [7] [13]-[20].

**3. Sufficient Conditions**

In what follows we consider the complexification of \( O \), it is denoted by \( O' \). Thus, \( z \in O' \) is of the form \( z = \sum_{i=0}^{7} z_i e_i, z_i \in \mathbb{C} \). \( z_0 \) and \( z = \sum_{i=0}^{7} z_i e_i \) are still called the scalar part and vector part, respectively. The norm of \( z \in O' \) is

\[
|z| = \left( \sum_{i=0}^{7} |z_i|^2 \right)^{1/2}
\]

and its conjugate is defined by \( \bar{z} = \sum_{i=0}^{7} \bar{z}_i e_i \), where \( \bar{z}_i \) is of the conjugate in the complex numbers. We can easily show that for any \( z, z' \in O' \),

\[
|zz'| \leq \sqrt{2} |z| |z'|
\]

For any \( z \in O' \), we may rewrite \( z \) as \( z = x + iy \), where \( x, y \in \mathbb{O} \). The multiplication rules in \( O' \) is the same as in (2.1). Note that \( O' \) is no longer a division algebra. Finally, the properties of associator in (2.2) except that \( [z, u, v] = 0 \) are also true for any \( z, u, v \in O' \):

\[
[z, u, v] = [u, v, z], \quad [z, v, u] = -[z, u, v], \quad [z, z, u] = 0.
\]

**Example.** Let \( z = e_1 + ie_2, u = e_3 \), then

\[
[z, u] = [e_1 + ie_2, e_3] = \{e_1, e_2, e_3\} - i\{e_2, e_3, e_4\} = 4ie, \neq 0.
\]

By (3.1) we can get the following lemma, which is useful to deduce our results.

**Lemma 3.1.** Let \( z, u, v \in O' \) and there exists complex numbers \( \lambda \) and \( \mu \) such that \( |\lambda|, |\mu| \neq 0 \) such that \( \lambda z + \mu u = 0 \) or \( \lambda u + \mu z = 0 \) or \( \lambda u + \mu z = 0 \), then \( [z, u, v] = 0 \).

For functions, \( f \), under study will be defined in an open set \( \Omega \) of \( \mathbb{R}^k \) and take values in \( O' \), with the form \( f(x) = \sum_{i=0}^{7} f_i (x) e_i \), where \( f_i (x) (i = 0, 1, \ldots, 7) \) are the complex-valued functions.

Hence, we say that a function \( f(x) = g(x) + ih(x) \) is left \( O' \)-analytic in an
open set $\Omega$ of $\mathbb{R}^\delta$, if $g(x)$ and $h(x)$ are the left $O^\delta$-analytic functions, since

$$Df = 0 \iff Dg = Dh = 0,$$

where $D = \sum_{i=0}^\delta \frac{\partial}{\partial x_i}$ is the Dirac operator as in Section 1.

In the case of $O^\delta$, we call $f(x) = g(x) + ih(x)$ a complex Stein-Weiss conjugate harmonic system, if $g(x), h(x)$ are the Stein-Weiss conjugate harmonic systems. A left (right) $O^\delta$-analytic functions $g(x)$ also have the Taylor expansion as in Theorem A.

Now we consider the product $f(x)g(x)$ of two left $O^\delta$-analytic functions $f(x), g(x)$ in $\Omega$. In general, $f(x)g(x)$ is no longer left $O^\delta$-analytic in $\Omega$. But, in some particular cases, the product $f(x)g(x)$ can maintain the analyticity for two left $O^\delta$-analytic functions $f(x)$ and $g(x)$.

**Theorem 3.2.** Let $f(x), g(x)$ be two left $O^\delta$-analytic functions in $\Omega$. Then $f(x)g(x)$ is also left $O^\delta$-analytic in $\Omega$ if $f(x), g(x)$ satisfy one of the following conditions:

1) $f(x)$ or $g(x)$ is a complex constant function.

2) $f(x)$ is a complex Stein-Weiss conjugate harmonic system in $\Omega$ and $g(x)$ is an $O^\delta$-constant function.

3) $f(x)$ is of the form $f(x) = f_0 e_i + f_1 (i \in \{1, 2, \cdots, 7\})$ and $f(x), g(x)$ depend only on $x_0$ and $x_i$, where $f_0, f_1$ are the complex-valued functions.

4) $f(x)$ and $g(x)$ belong to the following class

$$\mathcal{G} = \left\{ h(x) \mid Dh(x) = 0, h(x) = \sum_{i=0}^\delta h_i(x) e_i, h_i(x) \in C^\delta(\Omega, \mathbb{C}) \right\}.$$ (3.2)

5) $f(x)$ is of the form $f(x) = f_0 e_0 + f_1 e_a + f_2 e_\beta + f_3 e_\gamma$, $g = c_0 e_0 + c_a e_a + c_\beta e_\beta + c_\gamma e_\gamma$ is a constant function, where $(\alpha, \beta, \gamma) \in \mathcal{O}^\delta$, $c_0, c_a, c_\beta, c_\gamma \in \mathbb{C}$ and $f(x)$ depends only on $x_0, x_a, x_\beta, x_\gamma$.

**Proof.**

1) The proof is trivial.

2) In view of Proposition 2.1 we have $[e_i, e_j, \lambda] = 0$ when $i = 0$ or $j = 0$ or $i = j$ for any $\lambda \in O^\delta$. Then we have

$$D(f\lambda) = \sum_{i,j=0}^\delta \frac{\partial f}{\partial x_i} e_i (e_j, \lambda)$$

$$= \sum_{i,j=0}^\delta \frac{\partial f}{\partial x_i} (e_i e_j) \lambda - \sum_{i,j=0}^\delta \frac{\partial f}{\partial x_i} [e_i, e_j, \lambda]$$

$$= (Df) \lambda - \sum_{i,j=0}^\delta \frac{\partial f}{\partial x_i} [e_i, e_j, \lambda]$$

$$= (Df) \lambda - \sum_{i\leq j \leq 7} \frac{\partial f}{\partial x_i} [e_i, e_j, \lambda].$$

Since $f$ is a complex Stein-Weiss conjugate harmonic system, thus $Df = 0$ and $\frac{\partial f}{\partial x_i} = \frac{\partial f}{\partial x_j}$ for $i, j \geq 1, i \neq j$. But $[e_i, e_j, \lambda] = -[e_j, e_i, \lambda]$, therefore
\[ D(f, \lambda) = - \sum_{i \in A, j \in T} \frac{\partial f_i}{\partial x_i} [e_i, e_j, \lambda] = \sum_{i \in A, j \in T} \left( \frac{\partial f_i}{\partial x_j} \frac{\partial f_j}{\partial x_i} \right) [e_i, e_j, \lambda] = 0. \]

3) Since \( f(x), g(x) \) are only related to variables \( x_0 \) and \( x_i \), we have

\[ D(fg) = \left( \frac{\partial}{\partial x_0} + \frac{\partial}{\partial x_i} \right) \left( (f_0 + f_i)g \right) \]

\[ = \frac{\partial f}{\partial x_0} g + e_i \left( \frac{\partial f_0}{\partial x_i} + \frac{\partial f_i}{\partial x_i} \right) g + f \frac{\partial g}{\partial x_0} + e_i \left( (f_0 + f_i) \frac{\partial g}{\partial x_i} \right). \]

By Lemma 3.1 it follows that

\[ e_i \left( \frac{\partial f_0}{\partial x_i} + \frac{\partial f_i}{\partial x_i} \right) g = \left( e_i (f_0 + f_i) \right) \frac{\partial g}{\partial x_i} = \left( (f_0 + f_i) e_i \right) \frac{\partial g}{\partial x_i}. \]

and

\[ e_i \left( (f_0 + f_i) \frac{\partial g}{\partial x_i} \right) = \left( e_i (f_0 + f_i) \right) \frac{\partial g}{\partial x_i} = \left( (f_0 + f_i) e_i \right) \frac{\partial g}{\partial x_i}. \]

Thus we get

\[ D(fg) = \frac{\partial f}{\partial x_0} g + \left( e_i \frac{\partial f}{\partial x_i} \right) g + f \left( e_i \frac{\partial g}{\partial x_0} \right) + \left( e_i \frac{\partial g}{\partial x_i} \right) = (Df)g + f(Dg) = 0. \]

4) Let \( f(x) = f_0 e_0 + \sum_{i=1}^{7} f_i e_i \) and \( g(x) = g_0 e_0 + \sum_{i=1}^{7} g_i e_i \), then we have

\[ D \left( f(x) g(x) \right) = \sum_{j=0}^{7} e_j \left( \frac{\partial}{\partial x_j} \left( f_0 e_0 + f_i \sum_{i=1}^{7} e_i \right) \left( g_0 e_0 + g_i \sum_{i=1}^{7} e_i \right) \right) \]

\[ = \sum_{j=0}^{7} e_j \left( \frac{\partial f_0}{\partial x_j} e_0 + \frac{\partial f_i}{\partial x_j} \sum_{i=1}^{7} e_i \right) \left( g_0 e_0 + g_i \sum_{i=1}^{7} e_i \right) + \sum_{j=0}^{7} e_j \left( f_0 e_0 + f_i \sum_{i=1}^{7} e_i \right) \left( \frac{\partial g_0}{\partial x_j} e_0 + \frac{\partial g_i}{\partial x_j} \sum_{i=1}^{7} e_i \right). \]

By Lemma 3.1 we get

\[ e_j \left( \frac{\partial f_0}{\partial x_j} e_0 + \frac{\partial f_i}{\partial x_j} \sum_{i=1}^{7} e_i \right) \left( g_0 e_0 + g_i \sum_{i=1}^{7} e_i \right) = e_j \left( \frac{\partial f}{\partial x_j} \right) g \]

and

\[ e_j \left( f_0 e_0 + f_i \sum_{i=1}^{7} e_i \right) \left( \frac{\partial g_0}{\partial x_j} e_0 + \frac{\partial g_i}{\partial x_j} \sum_{i=1}^{7} e_i \right) = e_j \left( \frac{\partial g}{\partial x_j} \right) f. \]

Hence we obtain
\[ D(f(x)g(x)) = \sum_{j=0}^{7} \left( e_j \frac{\partial f}{\partial x_j} \right) g + \left( e_j \frac{\partial g}{\partial x_j} \right) f = (Df)g + (Dg)f = 0. \]

5) This case is equivalent to a left quaternionic analytic function right-multiplying by a quaternionic constant, the analyticity is obvious since the multiplication of the quaternion is associative.

The proof of Theorem 3.2 is complete. \( \square \)

From Theorem 3.2(d), if \( f(x), g(x) \in \mathcal{G} \), then \( f(x)g(x) \in \mathcal{G} \); that is, the multiply operation in \( \mathcal{G} \) is closed. Also, the division operation is closed in \( \mathcal{G} \). Actually, let \( f(x) = f_0(x) + \sum_{i=1}^{7} f_i(x)e_i \in \mathcal{G} \), assume \( f_0^2 + 7f_1^2 \neq 0 \), then
\[
(f(x))^{-1} = \frac{f_0 - f_1(e_1 + e_2 + \cdots + e_7)}{f_0^2 + 7f_1^2}.
\]

Thus we have
\[
D(f(x))^{-1} = \sum_{i=0}^{7} e_i \frac{\partial (f(x))^{-1}}{\partial x_i} = \sum_{j=0}^{7} e_j \left( f_0^2 + 7f_1^2 \right)^{-1} \left( \frac{\partial f_0}{\partial x_j} + 14f_0 \frac{\partial f_1}{\partial x_j} \right) \left( f_0^2 + 7f_1^2 \right)^{-1} - \left( f_0 - f_1(e_1 + e_2 + \cdots + e_7) \right) \left( 2f_0 \frac{\partial f_0}{\partial x_j} + 14f_1 \frac{\partial f_1}{\partial x_j} \right) \left( f_0^2 + 7f_1^2 \right)^{-2} = \sum_{i=0}^{7} e_i \left( \frac{\partial f_0}{\partial x_j} + 14f_0 \frac{\partial f_1}{\partial x_j} \right) \left( 7f_1^2 - f_0^2 + 2f_0f_1(e_1 + \cdots + e_7) \right) \left( f_0^2 + 7f_1^2 \right)^{-2} = 0.
\]

An element belongs to \( \mathcal{G} \) is the exponential function:
\[
\exp(x) = e^{x_0 + x_7} \left[ \cos(x_0 \sqrt{7}) e_0 + \left( -\frac{1}{\sqrt{7}} e_1 + \cdots + e_7 \right) \sin(x_0 \sqrt{7}) \right]. \tag{3.3}
\]

The results in Theorem 3.2 also hold on octonions (no complexification), since \( \mathcal{O}^\ell \) contains \( \mathcal{O} \). If one switch the locations of \( f(x), g(x) \), and the “left” change into “right” in Theorem 3.2, then this theorem is also true, since left and right is symmetric. These principles also hold in the rest of this paper.

4. Necessary and Sufficient Conditions

If we consider the product of a left \( \mathcal{O}^\ell \)-analytic function and an \( \mathcal{O}^\ell \)-constant, we can get the necessary and sufficient conditions for the analyticity (these results obtained in this section for \( \mathcal{O} \)-analytic functions are also described in [19]).

Applying Theorem 3.2(a) and (b), if \( f(x) \) is a left \( \mathcal{O}^\ell \)-analytic function
and \( \lambda \) is a complex constant, or \( f(x) \) is a complex Stein-Weiss conjugate harmonic system and \( \lambda \) is an \( \mathbb{O}' \)-constant, then \( f(x)\lambda \) is a left \( \mathbb{O}' \)-analytic function. In what follows we will see that these conditions are also necessary in some sense.

**Theorem 4.1.** Let \( \lambda \in \mathbb{O}' \), then \( f\lambda \) is a left \( \mathbb{O}' \)-analytic function for any left \( \mathbb{O}' \)-analytic functions \( f \) if and only if \( \lambda \in \mathbb{C} \).

**Proof.** We only prove the necessity. Taking a left \( \mathbb{C} \)-analytic function \( f = x_1 e_2 - x_2 e_1 \), then

\[
D(f\lambda) = -\sum_{i,j=0}^{7} \frac{\partial f}{\partial x_i} \lambda_i [e_i, e_j, e_i] \sum_{i=0}^{7} \frac{\partial f}{\partial x_i} \lambda_i [e_i, e_j, e_i] = \lambda_i [e_i, e_j, e_i] + \lambda_i [e_i, e_j, e_i] + \lambda_i [e_i, e_j, e_i] = -2\lambda_i e_i + 2\lambda_i e_i - 2\lambda_i e_i + 2\lambda_i e_i.
\]

Thus \( \lambda_1 = \lambda_2 = \lambda_3 = 0 \). A similar technique yields \( \lambda_4 = \lambda_5 = \lambda_6 = 0 \). Hence \( \lambda \in \mathbb{C} \). \( \square \)

**Theorem 4.2.** Let \( f \in C^1(\Omega, \mathbb{O}') \). Then \( D(f\lambda) = 0 \) for any \( \lambda \in \mathbb{O}' \) if and only if \( f \) is a complex Stein-Weiss conjugate harmonic system in \( \Omega \).

Now we postpone the proof of Theorem 4.2 and consider a problem under certain conditions weaker than Theorem 4.2. In [2] the authors proposed an open problem as follows:

*Find the necessary and sufficient conditions for an \( \mathbb{O}' \)-valued function \( f \), such that the equality \( [\lambda, f(x), D] = 0 \) holds for any constant \( \lambda \in \mathbb{O}' \).*

Note that this problem is of no meaning for an associative system, but octonions is a non-associative algebra, therefore we usually encounter some difficulties while disposing some problems in octonionic analysis. In [2] the authors added the condition \( [\lambda, f(x), D] = 0 \) for the \( f(x) \) to study the Cauchy integrals on Lipschitz surfaces in octonions and then prove the analogue of Calderón’s conjecture in octonionic space.

Next we give the answer to the Open Problem as follows.

**Theorem 4.3.** Let \( f \in C^1(\Omega, \mathbb{O}') \). Then \( [D, f, \lambda] = 0 \) \( ([\lambda, f, D] = 0) \) for any \( \lambda \in \mathbb{O}' \) if and only if

\[
\frac{\partial f}{\partial x_j} = \frac{\partial f}{\partial x_i}, \quad i, j = 1, 2, \ldots, 7. \tag{4.1}
\]

**Proof.** By Proposition 2.1, we have

\[
[D, f, \lambda] = \sum_{i, j, k=0}^{7} \frac{\partial f}{\partial x_i} [e_i, e_j, e_k, \lambda] = \sum_{i, j, k=0}^{7} \frac{\partial f}{\partial x_i} [e_i, e_j, e_k, \lambda] = [e_i, e_j, e_i, \lambda] + [e_i, e_j, e_i, \lambda] + [e_i, e_j, e_i, \lambda] = -2\lambda_i e_i + 2\lambda_i e_i - 2\lambda_i e_i + 2\lambda_i e_i.
\]

If \( f \) satisfies (4.1), then \( [D, f, \lambda] = 0 \).

Inversely, let \( (\alpha, \beta, \gamma) \in \mathbb{O}' \), \( \{1, 2, \ldots, 7\} \setminus \{\alpha, \beta, \gamma\} = \{t_1, t_2, t_3, t_4\} \) and \( e_i e_j = e_j e_i = -e_i e_j, e_i e_{t_j} = e_j = -e_j e_i \).

From Propositions 2.1 and 2.2 we have \( [e_a, e_b, e_c, e_d] = 0 \) and \( [e_a, e_b, e_c] = 2(e_a e_b) e_c = 2e_c e_a \) when \( t = \alpha, \beta, \gamma \) and \( t = t_1, t_2, t_3, t_4 \), respectively.
Hence, taking $\lambda = e_i$, it follows that
\[
\left[ D, f, e_i \right] = \sum_{1 \leq i < j \leq 7} \left( \frac{\partial f_i}{\partial x_j} - \frac{\partial f_j}{\partial x_i} \right) [e_i, e_j, e_k]
\]
\[
= \left( \frac{\partial f_\beta}{\partial x_\alpha} - \frac{\partial f_\alpha}{\partial x_\beta} \right) [e_\alpha, e_\beta, e_\gamma] + \sum_{i,j,k} g_{ij} e_k
\]
\[
= 2 \left( \frac{\partial f_\beta}{\partial x_\alpha} - \frac{\partial f_\alpha}{\partial x_\beta} + \frac{\partial f_\gamma}{\partial x_\beta} - \frac{\partial f_\gamma}{\partial x_\alpha} \right) e_\beta + \sum_{i,j} h_{ij} e_i. \tag{4.2}
\]

Similarly, we take $\lambda = e_i$, then
\[
\left[ D, f, e_i \right] = 2 \left( \frac{\partial f_\beta}{\partial x_\alpha} - \frac{\partial f_\alpha}{\partial x_\beta} + \frac{\partial f_\gamma}{\partial x_\beta} - \frac{\partial f_\gamma}{\partial x_\alpha} \right) e_\beta + \sum_{i,j} h_{ij} e_i. \tag{4.3}
\]

Also we can get
\[
\left[ D, f, e_a \right] = 2 \left( \frac{\partial f_\beta}{\partial x_a} - \frac{\partial f_a}{\partial x_\beta} + \frac{\partial f_\gamma}{\partial x_\beta} - \frac{\partial f_\gamma}{\partial x_a} \right) e_\beta + \sum_{i,j} h_{ij} e_i. \tag{4.4}
\]

If we require $[D, f, \lambda] = 0$ for any constants $\lambda \in \mathbb{O}^\alpha$, from (4.2), (4.3) and (4.4) we obtain
\[
\begin{align*}
\left( \frac{\partial f_\beta}{\partial x_\alpha} - \frac{\partial f_\alpha}{\partial x_\beta} + \frac{\partial f_\gamma}{\partial x_\beta} - \frac{\partial f_\gamma}{\partial x_\alpha} & = 0, \\
\left( \frac{\partial f_\beta}{\partial x_\alpha} - \frac{\partial f_\alpha}{\partial x_\beta} + \frac{\partial f_\gamma}{\partial x_\beta} - \frac{\partial f_\gamma}{\partial x_\alpha} & = 0, \\
\left( \frac{\partial f_\beta}{\partial x_a} - \frac{\partial f_a}{\partial x_\beta} + \frac{\partial f_\gamma}{\partial x_\beta} - \frac{\partial f_\gamma}{\partial x_a} & = 0.
\end{align*}
\]

Combining above three equations with the randomicity of $(\alpha, \beta, \gamma)$ we have (4.1) holds. \( \square \)

**Proof of Theorem 4.2.** The sufficient from Theorem 3.2(b). Inversely, if we take $\lambda = 1$ in $D(f, \lambda) = 0$ it follows that $f$ is a left $\mathbb{O}^\alpha$ -analytic function. Thus for any $\lambda \in \mathbb{O}^\alpha$, we have
\[
D(f, \lambda) = (Df) \lambda - [D, f, \lambda] = -[D, f, \lambda] = 0.
\]

By Theorem 4.3 we get that $f$ satisfies (4.1). On the other hand,
\[
Df = \left( \frac{\partial}{\partial x_0} + \nabla \right) \left( f_0 + f \right) = \frac{\partial f_0}{\partial x_0} - \nabla \cdot f + \frac{\partial f}{\partial x_0} + \nabla f_0 + \nabla \times f = 0. \tag{4.5}
\]

From (4.1) it easily to get $\nabla \times f = 0$, again by (4.5) it follows that
\[
\frac{\partial f_0}{\partial x_0} - \nabla \cdot f + \frac{\partial f}{\partial x_0} + \nabla f_0 = 0,
\]

namely
\[
\frac{\partial f_0}{\partial x_0} - \nabla \cdot f = 0, \quad \frac{\partial f}{\partial x_0} + \nabla f_0 = 0.
\]
Combining this with (4.1) it shows that $\tilde{f}$ is a complex Stein-Weiss conjugate harmonic system in $\Omega$. □

5. Some Applications and Relations with the C-K Products

From Theorem A we can see that $V_{\lambda_{1\cdots l}}(x)$ are the basic components for (left) $O$-analytic functions. It is proved in [13] that the polynomials $V_{\lambda_{1\cdots l}}(x)$ are all $O$-analytic functions, therefore they are the suitable substitutions of the polynomial $z^k$ in $C$.

Again from Theorem A, since $V_{\lambda_{1\cdots l}}(x)$ is an item in the Taylor expansion of a left $O$-analytic function, $V_{\lambda_{1\cdots l}}(x)$ should be also a left $O$-analytic function. Applying Theorem 4.2, the conjugate of $V_{\lambda_{1\cdots l}}(x)$ is probably a Stein-Weiss conjugate harmonic system. The following theorem prove this is true.

**Theorem 5.1.** For any combination $(l_1,\cdots,l_k)$ of $k$ elements out of $\{1,\cdots,7\}$ repetitions being allowed, $V_{\lambda_{1\cdots l}}(x)$ is a Stein-Weiss conjugate harmonic system in $R^l$.

*Proof.* Let $s_i (i=1,\cdots,7)$ be the appearing times of $i$ in $(l_1,\cdots,l_k)$. Hence the following equality

$$V_{\lambda_{1\cdots l}}(x) = \Phi_{\lambda_{1\cdots l}}(x)$$  (5.1)

shows that $V_{\lambda_{1\cdots l}}(x)$ is a Stein-Weiss conjugate harmonic system in $R^l$, where

$$\Phi_{\lambda_{1\cdots l}}(x) = \sum_{s=0}^{[s]} \left\{ \frac{(-1)^s \kappa_1! x_0^{s+1} \prod_{j=1}^{s} (2\kappa_j+1)!}{\prod_{j=1}^{s} \kappa_j! (s_j-2\kappa_j)!} \right\}$$

is a real-valued harmonic function of order $(s_1+s_2+\cdots+s_7+1)$ with $\kappa = \sum_{j=1}^{7} \kappa_j$.

Actually, put $x_0 = 0$, the both sides of (5.1) equal to $\frac{1}{\kappa_1! \kappa_2! \cdots \kappa_7!} x_1^{\kappa_1} \cdots x_7^{\kappa_7}$. On the other hand, $V_{\lambda_{1\cdots l}}(x)$ is left $O$-analytic in $R^l$. Thus by Proposition 2.4 we have (5.1) holds. □

Combining Theorem 3.2(b) and Theorem 5.1 it really shows that all the $V_{\lambda_{1\cdots l}}(x)$ are left $O^c$-analytic functions for any $\lambda_{1\cdots l} \in O^c$. Hence the following series

$$\sum_{k=0}^{\infty} \sum_{\lambda_{1\cdots l}} V_{\lambda_{1\cdots l}}(x) \lambda_{1\cdots l}$$  (5.2)

is a left $O^c$-analytic function in some open neighborhood $\Lambda$ of the origin if $\{\lambda_{1\cdots l}\}$ satisfies certain bounded conditions.

**Theorem 5.2.** For any combination $(l_1,\cdots,l_k)$ of $k$ elements out of $\{1,\cdots,7\}$ repetitions being allowed, let $\lambda_{1\cdots l} \in O^c, k \in N$. If $\lim_{k \to \infty} \frac{1}{k!} \sup_{\lambda_{1\cdots l}} |\lambda_{1\cdots l}| = \gamma < \infty$, then the series (5.2) converges to a left $O^c$-analytic function $f(x)$ in the following region.
\[ \Lambda_\gamma = \left\{ x \in \mathbb{R}^7 : \sqrt{x_0^2 + x_i^2} < \frac{1}{\gamma}, i = 1, 2, \cdots, 7 \right\}. \]

Moreover, \( \lambda_{i-k} = \partial_{x_k} \cdots \partial_{x_i} f(0) \) \( \text{Particularly, if } \sup_{i \in \mathbb{N}} |\lambda_{i-k}| \leq C < \infty, \text{ then } f \text{ will be a left } \mathbb{C}^\gamma \text{-entire function.} \)

**Proof.** Let

\[ S_N(x) = \sum_{k=0}^{N} \sum_{k-M}^{\infty} V_{k-M} (x) \lambda_{k-M}, \quad N \in \mathbb{N}. \]

For any \( x = \sum_{i=0}^{7} x_i e_i \in \Lambda_\gamma, \text{ there exists } \gamma' > \gamma \text{ such that } \sqrt{x_0^2 + x_i^2} < \frac{1}{\gamma'}, i = 1, 2, \cdots, 7. \) Thus

\[
\sup_{x \in \Lambda_\gamma} \left| S_N(x) - S_M(x) \right| \\
\leq \sup_{x \in \Lambda_\gamma} \sum_{k=M}^{N} \sum_{k-M}^{\infty} \left| V_{k-M} (x) \right| \left| \lambda_{k-M} \right| \\
\leq \sup_{x \in \Lambda_\gamma} \frac{1}{k!} \sum_{k=0}^{N} \left| \lambda_{k-M} \right| \\
\leq \frac{7^k}{k!} \left| \lambda_{k-M} \right| \rightarrow 0 \quad (\inf (M, N) \rightarrow \infty).
\]

From Weierstrass Theorem on octonions [13] and the analyticity of \( V_{k-M} (x) \lambda_{k-M}, \text{ then there exists a left } \mathbb{O} \text{-analytic function } f \text{ in } \Lambda_\gamma \text{ such that } f(x) = \lim_{N \rightarrow \infty} S_N(x) = \sum_{k=0}^{\infty} \sum_{k-M}^{\infty} V_{k-M} (x) \lambda_{k-M}, \)

and the series uniformly converges to \( f(x) \) in each compact subset \( K \subset \Lambda_\gamma. \)

Again from the expansion of \( f(x) \) we easily get that \( \lambda_{i-k} = \partial_{x_k} \cdots \partial_{x_i} f(0). \)

If \( \sup_{i \in \mathbb{N}} \left| \lambda_{i-k} \right| \leq C < \infty, \text{ then } \Lambda_\gamma = \mathbb{R}^7, \text{ since } \lim_{k \rightarrow \infty} \frac{7^k}{k!} = 0. \) Therefore \( f \) is a left \( \mathbb{O} \text{-entire function.} \)

**Example.** Taking \( \lambda_{i-k} = 1 \) for all \( k \in \mathbb{N} \) in (5.2), then

\[ \sum_{i=0}^{\infty} \sum_{k-M}^{\infty} V_{k-M} (x) \]

is an \( \mathbb{O} \text{-entire function. In fact, (5.3) is the Taylor expansion of the exponential function } \exp(x) \text{ as in (3.3). From (3.3) we can find } \exp(x) \text{ satisfies } \exp(0) = 1, \quad \exp(x + y) = \exp(x) \cdot \exp(y) = \exp(y) \cdot \exp(x). \)

**Corollary 5.3.** For any left \( \mathbb{O} \text{-analytic function } f, \text{ if the coefficients in its Taylor series about the origin satisfy } \)

\[ \begin{cases} \partial_{x_i} f(0) \in \mathbb{C} + \epsilon \mathbb{C}, & k \in \mathbb{N}, \ i = 1, 2, \cdots, 7 \\ \partial_{x_k} \cdots \partial_{x_i} f(0) \in \mathbb{C}, & \text{otherwise.} \end{cases} \]

Then \( \bar{f} \) is a complex Stein-Weiss conjugate harmonic system.
Proof. From (5.4), we easily obtain that all the conjugates of
\[ V_{\lambda_1 \cdots \lambda_l} (x) \partial_{\lambda_1} \cdots \partial_{\lambda_l} f(0) \]
are complex Stein-Weiss conjugate harmonic systems. Hence by Weierstrass Theorem, \( f \) also is a complex Stein-Weiss conjugate harmonic system in its convergent area. \( \square \)

Combining Theorem 3.2(b), Theorems 5.1 and 5.2, by an analogous method in [6] we can define the Cauchy-Kowalewski product for any two left \( O^c \)-analytic functions \( f \) and \( g \) in \( \Omega \) which containing origin. We let their Taylor expansions be

\[ f(x) = \sum_{k=0}^{\infty} \sum_{(\lambda_1, \ldots, \lambda_l)} V_{\lambda_1 \cdots \lambda_l} (x) \partial_{\lambda_1} \cdots \partial_{\lambda_l} f(0) \]

and

\[ g(x) = \sum_{t=0}^{\infty} \sum_{(s_1, \ldots, s_i)} V_{s_1 \cdots s_i} (x) \partial_{s_1} \cdots \partial_{s_i} g(0). \]

Then the (left) Cauchy-Kowalewski product of \( f \) and \( g \) is defined by

\[ f \circ_{L} g(x) = \sum_{k,t=0}^{\infty} \sum_{(\lambda_1, \ldots, \lambda_l) \in (s_1, \ldots, s_i)} \left( \prod_{i=1}^{\lambda_l+s_i} \frac{1}{n_i! n_i!} \right) V_{\lambda_1 \cdots \lambda_l \cdots s_i} (x) \partial_{\lambda_1} \cdots \partial_{\lambda_l} \partial_{s_1} \cdots \partial_{s_i} g(0), \]

where \( n_i \) and \( n'_i \) are the appearing times of \( i \) in \( (\lambda_1, \ldots, \lambda_l) \) and \( (s_1, \ldots, s_i) \), respectively.

We have the following relation for the product and the left Cauchy-Kowalewski product between two left \( O^c \)-analytic functions.

**Theorem 5.4.** Let \( f(x), g(x) \) be two left \( O^c \)-analytic functions in \( \Omega \) which containing origin. If \( D \left( f(x)g(x) \right) = 0 \) then \( f(x)g(x) = f \circ_{L} g(x) \).

**Proof.** It is easy to see that \( f(x)g(x) = f \circ_{L} g(x) \), then by Proposition 2.4 and the analyticity of \( f(x)g(x) \) and \( f \circ_{L} g(x) \), we get \( f(x)g(x) = f \circ_{L} g(x) \). \( \square \)

**Remark.** In this paper we study the analyticity of the product of two left \( O^c \)-analytic functions. Theorem 3.2 give some sufficient conditions for the product of two left \( O^c \)-analytic functions is also a left \( O^c \)-analytic function. From Theorem 5.4 we can see that \( D \left( f(x)g(x) \right) = 0 \) for two left \( O^c \)-analytic functions \( f(x), g(x) \) if and only if this product is just equal to their left Cauchy-Kowalewski product. Since \( H \subseteq O \), our result is also true for quaternionic cases.

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