On a Subordination Result of a Subclass of Analytic Functions

Risikat Ayodeji Bello

Department of Mathematics and Statistics, College of Pure and Applied Science, Kwara State University, Malete, Nigeria
Email: reeyait26@gmail.com

Abstract

In this paper, we investigate a subordination property and the coefficient inequality for the class $M(1, b)$. The lower bound is also provided for the real part of functions belonging to the class $M(1, b)$.

Keywords

Analytic Function, Univalent Function, Hadamard Product, Subordination

1. Introduction

Let $A$ denote the class of function $f(z)$ analytic in the open unit disk $U = \{z \in \mathbb{C} : |z| < 1\}$ and let $S$ be the subclass of $A$ consisting of functions univalent in $U$ and have the form

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k,$$  \hspace{1cm} (1.1)

The class of convex functions of order $\alpha$ in $U$, denoted as $K(\alpha)$ is given by

$$K(\alpha) = \left\{ f \in S : \text{Re} \left( 1 + \frac{zf'(z)}{f'(z)} \right) > \alpha, 0 \leq \alpha < 1, z \in U \right\}.$$

Definition 1.1. The Hadamard product or convolution $f \ast g$ of the function $f(z)$ and $g(z)$, where $f(z)$ is as defined in (1.1) and the function $g(z)$ is defined as:

$$g(z) = z + \sum_{k=2}^{\infty} b_k z^k,$$

is defined as:
\[(f \ast g)(z) = z + \sum_{k=2}^{\infty} a_k b_k z^k = (g \ast f)(z), \quad (1.2)\]

**Definition 1.2.** Let \( f(z) \) and \( g(z) \) be analytic in the unit disk \( U \). Then \( f(z) \) is said to be subordination to \( g(z) \) in \( U \) and written as:

\[ f(z) \prec g(z), \quad z \in U \]

if there exist a Schwarz function \( \omega(z) \), analytic in \( U \) with \( \omega(0) = 0, |\omega(z)| < 1 \) such that

\[ f(z) = g(\omega(z)), \quad z \in U \quad (1.3) \]

In particular, if the function \( g(z) \) is univalent in \( U \), then \( f(z) \) is said to be subordinate to \( g(z) \) if

\[ f(0) = g(0), \quad f(u) \subset g(u) \quad (1.4) \]

**Definition 1.3.** The sequence \( \{c_k\}_{k=1}^{\infty} \) of complex numbers is said to be a subordinating factor sequence of the function \( f(z) \) if whenever \( f(z) \) in the form (1.1) is analytic, univalent and convex in the unit disk \( U \), the subordination is given by

\[ \sum_{k=1}^{\infty} a_k c_k z^k \prec f(z), \quad z \in U, a_k = 1 \]

We have the following theorem:

**Theorem 1.1.** (Wilf [1]) The sequence \( \{c_k\}_{k=1}^{\infty} \) is a subordinating factor sequence if and only if

\[ \text{Re}\left\{1 + 2 \sum_{k=1}^{\infty} c_k z^k \right\} > 0, \quad z \in U \quad (1.5) \]

**Definition 1.4.** A function \( P \in A \) which is normalized by \( P(0) = 1 \) is said to be in \( P(1, b) \) if

\[ |P(z) - 1| < b, \quad b > 0, \quad z \in U. \]

The class \( P(1, b) \) was studied by Janowski [2]. The family \( P(1, b) \) contains many interesting classes of functions. For example, for \( f(z) \in A \), if

\[ \left( \frac{zf'(z)}{f(z)} \right) \in P(1, 1 - \alpha), \quad 0 \leq \alpha < 1 \]

Then \( f(z) \) is starlike of order \( \alpha \) in \( U \) and if

\[ \left( 1 + \frac{zf'(z)}{f(z)} \right) \in P(1, 1 - \alpha), \quad 0 \leq \alpha < 1 \]

Then \( f(z) \) is convex of order \( \alpha \) in \( U \).

Let \( F(1, b) \) be the subclass of \( P(1, 1 - \alpha) \) consisting of functions \( P(f) \) such that

\[ P(f) = \frac{zf'(z)}{f(z)} \left( 1 + \frac{zf'(z)}{f'(z)} \right) \quad (1.6) \]

we have the following theorem
Theorem 1.2. [3] Let \( P(f) \) be given by Equation (1.6) with \( f(z) = \sum a_i z^k \). If
\[
\sum_{k=2}^{\infty} (k^2 + b - 1) |a_k| < b, b > 0
\]
then \( P(f) \in F(1,b), \ 0 < b < 0.935449 \).

It is natural to consider the class
\[
M(1,b) = \left\{ f \in A : \sum_{k=2}^{\infty} (k^2 + b - 1) |a_k| < b, b > 0 \right\}
\]
\( 0 < b < 0.935449 \)

Remark 1.1. [4] If \( b = 1 - \alpha \), then \( M(1,1 - \alpha) \) consists of starlike functions of order \( \alpha \), \( 0 \leq \alpha < 1 \) since
\[
\sum_{k=2}^{\infty} (k - \alpha) |a_k| < \sum_{k=2}^{\infty} (k^2 - \alpha) |a_k|
\]

Our main focus in this work is to provide a subordination results for functions belonging to the class \( M(1,b) \)

2. Main Results

2.1. Theorem

Let \( f(z) \in M(1,b) \), then
\[
\frac{3 + b}{2(3 + 2b)} (f * g)(z) < g(z)
\]
where \( 0 < b < 0.935449 \) and \( g(z) \) is convex function.

Proof:

Let \( f(z) \in M(1,b) \)

and suppose that
\[
g(z) = z + \sum b_k z^k \in C(\alpha)
\]
that is \( g(z) \) is a convex function of order \( \alpha \).

By definition (1.1) we have
\[
\frac{3 + b}{2(3 + 2b)} (f * g)(z)
= \frac{3 + b}{2(3 + 2b)} \left( z + \sum_{k=2}^{\infty} a_k b_k z^k \right)
= \sum_{k=2}^{\infty} \frac{3 + b}{2(3 + 2b)} a_k b_k z^k , a_i = 1, b_i = 1
\]

Hence, by Definition 1.3...to show subordination (2.1) is by establishing that
\[
\left\{ \frac{3 + b}{2(3 + 2b)} a_i \right\}_{i=1}^{\infty}
\]
is a subordinating factor sequence with \( a_i = 1 \). By Theorem 1.1, it is sufficient to show that

\[
\text{Re}\left\{1 + 2 \sum_{k=1}^{\infty} \frac{3 + b}{2(3 + 2b)} a_k z^k\right\} > 0, \quad z \in U
\] (2.4)

Now,

\[
\text{Re}\left\{1 + 2 \sum_{k=1}^{\infty} \frac{3 + b}{2(3 + 2b)} a_k z^k\right\} \\
= \text{Re}\left\{1 + \frac{3 + b}{3 + 2b} z + \sum_{k=1}^{\infty} \frac{3 + b}{3 + 2b} a_k z^k\right\} \\
> \text{Re}\left\{1 - \frac{3 + b}{3 + 2b} r - \frac{1}{3 + 2b} \sum_{k=1}^{\infty} (k^2 - b + 1) |a_k|^k\right\} \\
> \text{Re}\left\{1 - \frac{3 + b}{3 + 2b} r - \frac{br}{3 + 2b}\right\} = 1 - r > 0
\]

Since \( |z| = r < 1 \), therefore we obtain

\[
\text{Re}\left\{1 + 2 \sum_{k=1}^{\infty} \frac{3 + b}{2(3 + 2b)} a_k z^k\right\} > 0, \quad z \in U
\]

which by Theorem 1.1 shows that \( \frac{3 + b}{2(3 + 2b)} a_k \) is a subordinating factor, hence, we have established Equation (2.5).

### 2.2. Theorem

Given \( f(z) \in M(b) \), then

\[
\text{Ref}(z) > -\frac{3 + 2b}{3 + b}
\] (2.6)

The constant factor \( \frac{3 + 2b}{3 + b} \) cannot be replaced by a larger one.

**Proof:**

Let

\[
g(z) = \frac{z}{1 - z}
\]

which is a convex function, Equation (2.1) becomes

\[
\frac{3 + b}{2(3 + 2b)} f(z)^* \frac{z}{1 - z} < \frac{z}{1 - z}
\]

Since

\[
\text{Re}\left(\frac{z}{1 - z}\right) > -\frac{1}{2}, \quad |z| = r
\] (2.7)

This implies
Re\left\{ \frac{3+b}{2(3+2b)} f'\left(\frac{z}{1-z}\right) \right\} > -\frac{1}{2} \tag{2.8}

Therefore, we have

\[ \text{Re}\left( f'(z) \right) > -\frac{3+2b}{3+b} \]

which is Equation (2.6).

Now to show that sharpness of the constant factor

\[ \frac{3+b}{3+2b} \]

We consider the function

\[ f_1(z) = \frac{z(3+b)+bz^2}{3+b} \tag{2.9} \]

Applying Equation (2.1) with \( g(z) = \frac{z}{1-z} \) and \( f(z) = f_1(z) \), we have

\[ \frac{z(3+b)+bz^2}{2(3+b)} < \frac{z}{1-z} \tag{2.10} \]

Using the fact that

\[ |\text{Re}(z)| \leq |z| \tag{2.11} \]

We now show that the

\[ \min_{z \in U} \left\{ \text{Re}\left( \frac{z(3+b)+bz^2}{2(3+b)} \right) \right\} = -\frac{1}{2} \tag{2.12} \]

we have

\[ \left| \text{Re}\left( \frac{z(3+b)+bz^2}{2(3+b)} \right) \right| \leq \left| \frac{z(3+b)+bz^2}{2(3+b)} \right| \leq \left| \frac{(3+b)+bz}{2(3+b)} \right| \leq \frac{3+b}{2(3+2b)} = \frac{1}{2}, \quad (|z| = 1) \]

This implies that

\[ \left| \text{Re}\left( \frac{z(3+b)+bz^2}{2(3+b)} \right) \right| \leq \frac{1}{2} \]

and therefore

\[ -\frac{1}{2} \leq \text{Re}\left( \frac{z(3+b)+bz^2}{2(3+b)} \right) \leq \frac{1}{2} \]

Hence, we have that

\[ \min_{z \in U} \left\{ \text{Re}\left( \frac{z(3+b)+bz^2}{2(3+b)} \right) \right\} = -\frac{1}{2} \]

That is
\[
\min_{z \in D} \left\{ \Re \frac{3+b}{2(3+2b)} \left( f_i * g(z) \right) \right\} = -\frac{1}{2}
\]
which shows the Equation (2.12).

### 2.3. Theorem

Let

\[
f(z) = z + \sum_{k=2}^{\infty} a_k z^k \in M(1,b), \quad 0 < b < 0.935449
\]
then \( |a_k| \leq \frac{1}{2} \).

**Proof:**

Let

\[
f(z) = z + \sum_{k=2}^{\infty} a_k z^k \in M(1,b)
\]
then by definition of the class \( M(1,b) \),

\[
\sum_{k=2}^{\infty} (k^2 + b - 1) |a_k| \leq b, \quad 0 < b < 0.935449
\]

we have that

\[
\frac{k^2 + b - 1}{b} - k > 0
\]
which gives that

\[
\sum_{k=2}^{\infty} k |a_k| \leq \frac{k^2 + b - 1}{b} |a_k| \leq 1
\]

i.e \( \sum_{k=2}^{\infty} k |a_k| \leq 1 \)

hence

\[
2 \sum |a_k| \leq 1
\]

\[
|a_k| \leq \frac{1}{2}
\]

### References


