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Functions of Bounded $(p(\cdot),2)$-Variation in De la Vallée Poussin-Wiener's Sense with Variable Exponent

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Abstract

In this paper we establish the notion of the space of bounded $(p(\cdot),2)$- variation in De la Vallée Poussin-Wiener's sense with variable exponent. We show some properties of this space $BV^{W}_{(p(\cdot),2)}[a,b]$ and we show that any uniformly bounded composition operator that maps this space into itself necessarily satisfies the so-called Matkowski's conditions.

Keywords

Generalized Variation, De la Vallée Poussin, $(p(\cdot),2)$-Variation in Wiener's Sense, Variable Exponent, Composition Operator, Matkowski's Condition

1. Introduction

In 1881, C. Jordan gave the notion of variation of a function in [1], and from this moment, many generalizations and extensions have been given. Consequently, the study of notions of generalized bounded variation forms an important direction in the field of mathematical analysis. Another important generalization of the space of bounded variation in the Jordan's sense is the notion of the space of functions of second bounded variation studied by Ch. J. de la Vallée Poussin in 1908 in [2]. It is defined as follows:

Definition 1 *Let* π *be a partition of the interval* $[a,b]$ *of the form* $\pi=\{a=t_0<t_1<\cdots<t_n=b\}$, *and* f *be a function* $f:[a,b]\to\mathbb{R}$. *The nonnegative real number*

$$V^{(2)}(f)=V^{(2)}(f;[a,b]):=\sup_{\pi}\sum_{j=1}^{n-1}\left|\frac{f(t_{j+1})-f(t_j)}{t_{j+1}-t_j}-\frac{f(t_j)-f(t_{j-1})}{t_j-t_{j-1}}\right|,$$

is called the second variation of f on $[a,b]$, where the supremum is taken over all partitions π of $[a,b]$. In the case that $V^{(2)}(f)<\infty$, we say that f has bounded second variation on $[a,b]$ and we denote it by $f\in BV^{(2)}[a,b]$.

A well-known generalization of the functions of bounded variation was done by N. Wiener in 1924 in [3]. The *p*-variation of a function f is the supremum of the sums of the p^{th} powers of absolute increments of f over non overlapping intervals. Wiener mainly focused on the case $p=2$, the 2-variation.

Definition 2 *Let* π *be a partition of the interval* $[a,b]$ *of the form* $\pi=\{a=t_0<t_1<\cdots<t_n=b\}$, f *be a function* $f:[a,b]\to\mathbb{R}$ *and* $1<p<\infty$. *The nonnegative real number*

$$V_p(f)=V_p(f;[a,b]):=\sup_{\pi}\sum_{j=1}^{n-1}\left|f(t_j)-f(t_{j-1})\right|^p,$$

is called the Wiener *p*-variation of f on $[a,b]$ where the supremum is taken over all partitions π of $[a,b]$. In the case that $V_p(f)<\infty$, we say that f has bounded Wiener *p*-variation on $[a,b]$ and we denote it by $f\in BV_p^W[a,b]$.

The p^{th}-variations were reconsidered in a probabilistic context by R. Dudley in [4] and [5], in 1994 and 1997, respectively. Many basic properties of the variation in the sense of Wiener and a number of important applications of the concept can be found in [6] [7]. The paper by V. V. Chistyakov and O. E. Galkin in [8] in 1998 is very important in the context of *p*-variation.

The class of nonlinear problems with exponent growth is a new research field and it reflects a new kind of physical phenomena. In 2000 the field began to expand even further. Motivated by problems in the study of electrorheological fluids, Diening [9] raised the question of when the Hardy-Littlewood maximal operator and other classical operators in harmonic analysis are bounded on variable Lebesgue spaces. These and related problems are the subject of active research to this day. These problems are interesting in applications (see [10] [11] [12] [13]) and gave rise to a revival of the interest in Lebesgue and Sobolev spaces with variable exponent, the origins of which can be traced back to the work of Orlicz [14] in the 1930's. In the 1950's, this study was carried on by Nakano [15] [16] who made the first systematic study of spaces with variable exponent. Later, Polish and Czechoslovak mathematicians investigated the modular function spaces (see for example Musielak [17] [18], Kovacik and Rakosnik [19] and Kozlowski [20]). We refer to the book [13] for detailed information on the theoretical approach for the Lebesgue and Sobolev spaces with variable exponents. Recently, in [21] Castillo, Merentes and Rafeiro studied a new space of functions of generalized bounded variation. They introduced the notion of bounded variation in the Wiener sense with variable exponent $p(\cdot)$ on $[a,b]$ and study some of its properties.

Definition 3 *Given a function* $p:[a,b]\to(1,\infty)$, *a partition*

$\pi = \{a = t_0 < t_1 < \cdots < t_n = b\}$ *of the interval* $[a,b]$*, and a function* $f:[a,b] \to \mathbb{R}$*, the nonnegative real number*

$$V_{p(\cdot)}^W(f) = V_{p(\cdot)}^W(f,[a,b]) := \sup_{\pi^*} \sum_{=1}^{n} \left| f(t_j) - f(t_{j-1}) \right|^{p(x_{j-1})}, \tag{1.1}$$

is called the Wiener variation with variable exponent (or $p(\cdot)$-variation in Wiener's sense) of f on $[a,b]$ where π^* is a tagged partition of the interval $[a,b]$, *i.e.*, a partition of the interval $[a,b]$ together with a finite sequence of numbers $x_0,\cdots,x_{n-1}$ subject to the conditions that for each *j*, $t_j \le x_j \le t_{j+1}$.

In case that $V_{p(\cdot)}^W(f;[a,b]) < \infty$, we say that f has bounded Wiener variation with variable exponent (or bounded $p(\cdot)$-variation in Wiener's sense) on $[a,b]$. The symbol $WBV_{p(\cdot)}[a,b] = BV_{p(\cdot)}^W[a,b]$ will denote the space of functions of bounded $p(\cdot)$-variation in Wiener's sense with variable exponent on $[a,b]$.

The aim of this paper is to provide a description of the new class formed by the functions of bounded $(p(\cdot),2)$-variation in the sense of Wiener as an extension to the double case of the previous concept. Also, we prove structural properties for mappings of bounded $(p(\cdot),2)$-variation in the Wiener's sense. Finally, we show that any uniformly bounded composition operator that maps the space $BV_{(p(\cdot),2)}^W[a,b]$ into itself necessarily satisfies the so-called Matkowski's conditions.

2. Preliminaries

In this section we present some definitions and propositions that will be used through out this paper.

Definition 4 *Let* $1 < p < \infty$*,* π *be a partition* $\pi = \{a = t_0 < t_1 < \cdots < t_n = b\}$ *of the interval* $[a,b]$*, and* $f:[a,b] \to \mathbb{R}$ *be a function. The nonnegative real number*

$$V_{(p,2)}^W(f) = V_{(p,2)}^W(f;[a,b]) := \sup_{\pi} \sum_{j=1}^{n-1} \left| \frac{f(t_{j+1}) - f(t_j)}{t_{j+1} - t_j} - \frac{f(t_j) - f(t_{j-1})}{t_j - t_{j-1}} \right|^p,$$

is called the De La Vallée Poussin-Wiener variation (or $(p,2)$-variation in Wiener's sense) of f on $[a,b]$ where the supremum is taken over all partitions π of $[a,b]$. In the case that $V_{(p,2)}^W(f) < \infty$, we say that f has bounded $(p,2)$-variation on $[a,b]$ and we denote by $f \in BV_{(p,2)}^W[a,b]$.

For the interested readers can see some of the properties in [2] [7] and other related problems in [22].

Proposition 1 *Let* $f:[a,b] \to \mathbb{R}$ *be a function with* $a,b > 0$ *and consider* $1 < p < \infty$*. Then*

1) $V_{(p,2)}^W(f;[a,b]) = 0$ if and only if f is a liner function.

2) If $V_{(p,2)}^W(f;[a,b]) < \infty$, then f is bounded in $[a,b]$.

3) $V_{(p,2)}^W(\cdot;[a,b])$ is a convex function.

Proof. 1) Suppose first that f is a linear function. If $f(t)=\alpha t+\beta$ for all $t\in[a,b]$, with $\alpha,\beta\in\mathbb{R}$, then by Definition 4, it follows easily that $V_{(p,2)}^{W}(f;[a,b])=0$.

Now, if $V_{(p,2)}^{W}(f;[a,b])=0$, then by Definition 4 we have

$$\begin{aligned}0&=V_{(p,2)}^{W}(f;[a,b])\\&=\sup_{\pi}\sum_{j=1}^{n-1}\left|\frac{f(t_{j+1})-f(t_j)}{t_{j+1}-t_j}-\frac{f(t_j)-f(t_{j-1})}{t_j-t_{j-1}}\right|^p.\end{aligned}$$

Hence, for any partition $\pi=\{a=t_0<t_1<\cdots<t_n=b\}$ of the interval $[a,b]$, we should have that

$$\sum_{j=1}^{n-1}\left|\frac{f(t_{j+1})-f(t_j)}{t_{j+1}-t_j}-\frac{f(t_j)-f(t_{j-1})}{t_j-t_{j-1}}\right|^p=0.$$

Then, any term in the sum should be zero. Since the function $t\to t^p$ vanishes only at zero, it follows that

$$\frac{f(t_{j+1})-f(t_j)}{t_{j+1}-t_j}=\frac{f(t_j)-f(t_{j-1})}{t_j-t_{j-1}}\quad\text{for all } j=1,2,\cdots,n-1.$$

Therefore, f is equal to a linear function.

2) Suppose that $f\in BV_{(p,2)}^{W}[a,b]$ and f is not bounded, then there exists a sequence $\{t_n\}_{n\ge1}$, $t_n\in(a,b)$, $n\ge1$ such that $|f(t_n)|\to\infty$ when $n\to\infty$. Let $\{t_m\}_{m\ge1}$ be a subsequence of $\{t_n\}_{n\ge1}$ such that $\{t_m\}_{m\ge1}$ converge to $x\in[a,b]$. Then, as $\{f(t_m)\}_{m\ge1}$ is a subsequence of $\{f(t_n)\}_{n\ge1}$, so

$$|f(t_m)|\to\infty\quad\text{when } n\to\infty.$$

We have that

$$\left|\frac{f(t_{n+1})-f(t_n)}{t_{n+1}-t_n}-\frac{f(t_n)-f(t_{n-1})}{t_n-t_{n-1}}\right|^p\le V_{(p,2)}^{W}(f;[a,b]),\quad n\ge1.$$

Moreover for $\pi=\{a\le t\le t_m\le\cdots\le b\}$ we get

$$\left|\frac{f(t_m)-f(t)}{t_m-t}-\frac{f(t)-f(a)}{t-a}\right|^p\le V_{(p,2)}^{W}(f,[a,t_m])\le V_{(p,2)}^{W}(f,[a,b]).$$

In consequence, $V_{(p,2)}^{W}(f;[a,b])=\infty$, since

$$\left|\frac{f(t_m)-f(t)}{t_m-t}-\frac{f(t)-f(a)}{t-a}\right|^p\to\infty,$$

as $m\to\infty$, which is a contradiction with $f\in BV_{(p,2)}^{W}[a,b]$. Therefore f is bounded.

3) Let $f,g:[a,b]\to\mathbb{R}$ be two functions, $\alpha,\beta\in[0,1]$ such that $\alpha+\beta=1$ and $\pi=\{a=t_0<t_1<\cdots<t_n=b\}$ be a partition of $[a,b]$. Since t^p is convex and nondecreasing, we have that

$$
\begin{aligned}
&\alpha V_{(p,2)}^{W}\left(f;[a,b]\right)+\beta V_{(p,2)}^{W}\left(g;[a,b]\right)\\
&=\alpha\sup_{\pi}\sum_{j=1}^{n-1}\left|\frac{f(t_{j+1})-f(t_j)}{t_{j+1}-t_j}-\frac{f(t_j)-f(t_{j-1})}{t_j-t_{j-1}}\right|^p\\
&\quad+\beta\sup_{\pi}\sum_{j=1}^{n-1}\left|\frac{g(t_{j+1})-g(t_j)}{t_{j+1}-t_j}-\frac{g(t_j)-g(t_{j-1})}{t_j-t_{j-1}}\right|^p\\
&\geq\sup_{\pi}\sum_{j=1}^{n-1}\left|\alpha\left[\frac{f(t_{j+1})-f(t_j)}{t_{j+1}-t_j}-\frac{f(t_j)-f(t_{j-1})}{t_j-t_{j-1}}\right]\right.\\
&\quad\left.+\beta\left[\frac{g(t_{j+1})-g(t_j)}{t_{j+1}-t_j}-\frac{g(t_j)-g(t_{j-1})}{t_j-t_{j-1}}\right]\right|^p\\
&=\sup_{\pi}\sum_{j=1}^{n-1}\left|\frac{(\alpha f+\beta g)(t_{j+1})-(\alpha f+\beta g)(t_j)}{t_{j+1}-t_j}\right.\\
&\quad\left.-\frac{(\alpha f+\beta g)(t_j)-(\alpha f+\beta g)(t_{j-1})}{t_j-t_{j-1}}\right|^p\\
&=V_{(p,2)}^{W}\left(\alpha f+\beta g;[a,b]\right).
\end{aligned}
$$

Then, $V_{(p,2)}^{W}\left(\cdot;[a,b]\right)$ is a convex function.

Definition 5 *(Norm in* $BV_{(p,2)}^{W}[a,b]$*) The functional* $\|\cdot\|_{(p,2)}^{W}:BV_{(p,2)}^{W}[a,b]\to\mathbb{R}$ *defined by*

$$
\|f\|_{(p,2)}^{W}:=|f(a)|+|f'(a)|+V_{(p,2)}^{W}\left(f;[a,b]\right)^{\frac{1}{p}} \tag{2.1}
$$

is a norm.

In [7], the authors have shown that the linear space $BV_{(p,2)}^{W}[a,b]$ with the norm (2.1) is a Banach space and $BV_{(p,2)}^{W}[a,b]\subset BV_{p}^{W}[a,b]$.

3. Main Results

In [23] the authors present and study the space of functions of bounded $p(\cdot)$-variation as an extension of the space $BV_{p}^{W}[a,b]$. In this section, our goal is to study the corresponding space of functions of bounded second $p(\cdot)$-variation, with $p(\cdot)$ be a variable exponent, as an extension of $BV_{(p,2)}^{W}[a,b]$.

Definition 6 *Let* p *be a function* $p:[a,b]\to(1,\infty)$, π *be a partition* $\pi=\{a=t_0<t_1<\cdots<t_n=b\}$ *of the interval* $[a,b]$ *and* $f:[a,b]\to\mathbb{R}$ *be a function. The nonnegative real number*

$$
V_{(p(\cdot),2)}^{W}(f)=V_{(p(\cdot),2)}^{W}\left(f;[a,b]\right):=\sup_{\pi^*}\sum_{j=1}^{n-1}\left|\frac{f(t_{j+1})-f(t_j)}{t_{j+1}-t_j}-\frac{f(t_j)-f(t_{j-1})}{t_j-t_{j-1}}\right|^{p(x_{j-1})},
$$

is called the De La Vallée Poussin-Wiener variation with variable exponent (or $(p(\cdot),2)$-variation in De La Vallée Poussin-Wiener's sense) of f on $[a,b]$, where π^* is a tagged partition of the interval $[a,b]$, *i.e.*, a partition of the interval $[a,b]$ together with a finite sequence of numbers $x_0,\cdots,x_{n-2}$ subject to the conditions $t_j \le x_j \le t_{j+1}$ for each j. It is worth to note that by definition (we take supremum over all partitions), the number $V^W_{(p(\cdot),2)}(f)$ does not depend on the election of the argument of the exponent. In the case that $V^W_{(p(\cdot),2)}(f)<\infty$, we say that f has bounded $(p(\cdot),2)$-variation on $[a,b]$.

We will denote by $BV^W_{(p(\cdot),2)}[a,b]=V^W_{(p(\cdot),2)}[a,b]$ the space of functions of bounded $(p(\cdot),2)$-variation in Wiener's sense with variable exponent in $[a,b]$. It is endowed with the functional:

$$\|f\|_{BV^W_{(p(\cdot),2)}[a,b]}=|f(a)|+|f'_+(a)|+\inf\left\{\lambda>0;V^W_{(p(\cdot),2)}\left(\frac{f}{\lambda};[a,b]\right)\le 1\right\}. \tag{3.1}$$

Then,

$$\left(BV^W_{(p(\cdot),2)}[a,b],\|\cdot\|_{BV^W_{(p(\cdot),2)}[a,b]}\right):=\left\{f:[a,b]\to\mathbb{R};\|f\|_{BV^W_{(p(\cdot),2)}[a,b]}<\infty\right\}.$$

Remark 3.1 *Given a function* $p:[a,b]\to[1,\infty)$.

1) If $p(x)=1$ for all $x\in[a,b]$, then $BV^W_{(p(\cdot),2)}[a,b]=BV^2[a,b]$.

2) If $p(x)=p$ for all $x\in[a,b]$ and $1<p<\infty$ then $BV^W_{(p(\cdot),2)}[a,b]=BV^W_{(p,2)}[a,b]$, *i.e.*, the space of bounded $(p(\cdot),2)$-variation in De la Vallée Poisson-Wiener's sense with variable exponent is exactly the space of bounded $(p,2)$-variation in De la Vallée Poisson-Wiener's sense.

Given a function $p:[a,b]\to(1,\infty)$, that is, a variable exponent function, let us define as in the literature,

$$p^-:=\operatorname{essinf}_{x\in[a,b]}p(x)=\sup\left\{\beta\in\mathbb{R}:\left|\left\{x\in[a,b];p(x)<\beta\right\}\right|=0\right\},$$

and

$$p^+:=\operatorname{esssup}_{x\in[a,b]}p(x)=\inf\left\{\alpha\in\mathbb{R}:\left|\left\{x\in[a,b];p(x)>\alpha\right\}\right|=0\right\}.$$

It is said that the exponent p is admissible if the range of p is in $(1,\infty)$ and p^+ is finite.

Let us recall a classical concept in the theory of function spaces. Let X be a vector space over $\mathbb{R}$. A convex and left-continuous function $\rho:X\to[0,\infty]$ is called a convex pseudo-modular on X if for arbitrary x and y, there holds:

1) $\rho(0x)=0$,

2) $\rho(\alpha x)=\rho(x)$ for every $\alpha\in\mathbb{R}$ such that $|\alpha|=1$,

3) $\rho(\alpha x+(1-\alpha)y)\le\alpha\rho(x)+(1-\alpha)\rho(y)$ for every $\alpha\in[0,1]$.

It is possible to see that for p be an admissible function, the functional $V^W_{(p(\cdot),2)}(\cdot;[a,b])$ is a convex pseudo-modular.

Proposition 2 *Let* p *be an admissible function. Then* $V^W_{(p(\cdot),2)}(\cdot;[a,b])$ *is a convex pseudo-modular.*

Proof. We have that for any $f\in BV^W_{(p(\cdot),2)}[a,b]$, $V^W_{(p(\cdot),2)}(0f;[a,b])=V^W_{(p(\cdot),2)}(0;[a,b])=0$. Moreover, the fact that for any

$f \in BV^W_{(p(\cdot),2)}[a,b]$, $V^W_{(p(\cdot),2)}(\alpha f;[a,b]) = V^W_{(p(\cdot),2)}(f;[a,b])$ whenever $|\alpha| = 1$ follows immediately from the definition.

Finally, with the same kind of argument than in Proposition 1(c) it follows that for $\alpha \in [0,1]$ and $f, g \in BV^W_{(p(\cdot),2)}[a,b]$ we have that

$$V^W_{(p(\cdot),2)}(\alpha f + (1-\alpha)g;[a,b]) \le \alpha V^W_{(p(\cdot),2)}(f;[a,b]) + (1-\alpha)V^W_{(p(\cdot),2)}(g;[a,b]).$$

Definition 7 *A convex and left-continuous function* $\rho : X \to [0,\infty]$ *is called semimodular on* X *if*

1) $\rho(0) = 0$,

2) $\rho(-x) = \rho(x)$ for every $x \in X$, and

3) if $\rho(\lambda x) = 0$ for every $\lambda \in \mathbb{R}$, then $x = 0$.

For p be an admissible function, the functional $V^W_{(p(\cdot),2)}(\cdot,[a,b])$ is a semimodular on X.

Proposition 3 *Let* p *be an admissible function. Then* $V^W_{(p(\cdot),2)}(\cdot,[a,b])$ *is a semimodular.*

Proof. Let $f \in BV^W_{(p(\cdot),2)}[a,b]$ and π^* be a tagged partition of $[a,b]$, then

$$\begin{aligned} V^W_{(p(\cdot),2)}(-f) &= \sup_{\pi^*} \sum_{j=1}^{n-1} \left| \frac{-(f(t_{j+1}) - f(t_j))}{t_{j+1} - t_j} - \frac{-(f(t_j) - f(t_{j-1}))}{t_j - t_{j-1}} \right|^{p(x_{j-1})} \\ &= \sup_{\pi^*} \sum_{j=1}^{n-1} \left(|-1| \left| \frac{f(t_{j+1}) - f(t_j)}{t_{j+1} - t_j} - \frac{f(t_j) - f(t_{j-1})}{t_j - t_{j-1}} \right| \right)^{p(x_{j-1})} \\ &= \sup_{\pi^*} \sum_{j=1}^{n-1} \left| \frac{f(t_{j+1}) - f(t_j)}{t_{j+1} - t_j} - \frac{f(t_j) - f(t_{j-1})}{t_j - t_{j-1}} \right|^{p(x_{j-1})} \\ &= V^W_{(p(\cdot),2)}(f). \end{aligned}$$

On the other hand, if

$$\begin{aligned} V^W_{(p(\cdot),2)}(\lambda f) &= \sup_{\pi^*} \sum_{j=1}^{n-1} \left| \frac{\lambda f(t_{j+1}) - \lambda f(t_j)}{t_{j+1} - t_j} - \frac{\lambda f(t_j) - \lambda f(t_{j-1})}{t_j - t_{j-1}} \right|^{p(x_{j-1})} \\ &= \sup_{\pi^*} \sum_{j=1}^{n-1} \left(|\lambda| \left| \frac{f(t_{j+1}) - f(t_j)}{t_{j+1} - t_j} - \frac{f(t_j) - f(t_{j-1})}{t_j - t_{j-1}} \right| \right)^{p(x_{j-1})} \\ &= \sup_{\pi^*} \sum_{j=1}^{n-1} |\lambda|^{p(x_{j-1})} \left| \frac{f(t_{j+1}) - f(t_j)}{t_{j+1} - t_j} - \frac{f(t_j) - f(t_{j-1})}{t_j - t_{j-1}} \right|^{p(x_{j-1})} = 0, \end{aligned}$$

for every λ, necessarily it follows that $f = 0$.

Proposition 4 *Let* X *be a vector space,* ρ *be a semimodular on* X *and* $f \in X$*. Then*

1) $\rho(f) \le 1$ if and only if $\|f\|_\rho \le 1$,

2) if $\|f\|_\rho \le 1$, then $\rho(f) \le \|f\|_\rho$,

3) if $\|f\|_\rho > 1$, then $\rho(f) \ge \|f\|_\rho$,

4) for every $f \in X$, $\|f\|_\rho \le \rho(f)+1$.

Theorem 1 *Let* $f:[a,b]\to\mathbb{R}$ *be a function and* p *be an admissible function, then* $BV^2[a,b] \subset BV^W_{(p(\cdot),2)}[a,b]$.

Proof. Let p be an admissible function, π^* be a tagged partition of the interval $[a,b]$, $f \in BV^2[a,b]$ and

$$\sigma = \left\{ j \in \pi^* : \left| \frac{f(t_{j+1}) - f(t_j)}{t_{j+1}-t_j} - \frac{f(t_j)-f(t_{j-1})}{t_j - t_{j-1}} \right| \le 1 \right\}.$$

$$\sum_{j=1}^{n-1} \left| \frac{f(t_{j+1}) - f(t_j)}{t_{j+1}-t_j} - \frac{f(t_j)-f(t_{j-1})}{t_j - t_{j-1}} \right|^{p(x_{j-1})}$$

$$= \sum_{j\in\sigma} \left| \frac{f(t_{j+1}) - f(t_j)}{t_{j+1}-t_j} - \frac{f(t_j)-f(t_{j-1})}{t_j - t_{j-1}} \right|^{p(x_{j-1})} + \sum_{j\notin\sigma} \left| \frac{f(t_{j+1}) - f(t_j)}{t_{j+1}-t_j} - \frac{f(t_j)-f(t_{j-1})}{t_j - t_{j-1}} \right|^{p(x_{j-1})}$$

$$\le \sum_{j\in\sigma} \left| \frac{f(t_{j+1}) - f(t_j)}{t_{j+1}-t_j} - \frac{f(t_j)-f(t_{j-1})}{t_j - t_{j-1}} \right| + \sum_{j\notin\sigma} \left| \frac{f(t_{j+1}) - f(t_j)}{t_{j+1}-t_j} - \frac{f(t_j)-f(t_{j-1})}{t_j - t_{j-1}} \right|^{p(x_{j-1})}$$

$$\le \sum_{j=1}^{n-1} \left| \frac{f(t_{j+1}) - f(t_j)}{t_{j+1}-t_j} - \frac{f(t_j)-f(t_{j-1})}{t_j - t_{j-1}} \right| + \sum_{j\notin\sigma} \left| \frac{f(t_{j+1}) - f(t_j)}{t_{j+1}-t_j} - \frac{f(t_j)-f(t_{j-1})}{t_j - t_{j-1}} \right|^{p(x_{j-1})}$$

$$\le V^{(2)}(f;[a,b]) + \sum_{j\notin\sigma} \left| \frac{f(t_{j+1}) - f(t_j)}{t_{j+1}-t_j} - \frac{f(t_j)-f(t_{j-1})}{t_j - t_{j-1}} \right|^{p(x_{j-1})}.$$

Then,

$$V^W_{(p(\cdot),2)}(f) := \sup_{\pi^*} \sum_{j=1}^{n-1} \left| \frac{f(t_{j+1}) - f(t_j)}{t_{j+1}-t_j} - \frac{f(t_j)-f(t_{j-1})}{t_j - t_{j-1}} \right|^{p(x_{j-1})}$$

$$\le V^{(2)}(f;[a,b]) + \sup_{\pi^*} \sum_{j\notin\sigma} \left| \frac{f(t_{j+1}) - f(t_j)}{t_{j+1}-t_j} - \frac{f(t_j)-f(t_{j-1})}{t_j - t_{j-1}} \right|^{p(x_{j-1})}.$$

The proof of the fact that $\sup_{\pi^*} \sum_{j\notin\sigma} \left| \frac{f(t_{j+1}) - f(t_j)}{t_{j+1}-t_j} - \frac{f(t_j)-f(t_{j-1})}{t_j - t_{j-1}} \right|^{p(x_{j-1})} < \infty$ will be by contradiction. That is, we assume that $\sup_{\pi^*} \sum_{j\notin\sigma} \left| \frac{f(t_{j+1}) - f(t_j)}{t_{j+1}-t_j} - \frac{f(t_j)-f(t_{j-1})}{t_j - t_{j-1}} \right|^{p(x_{j-1})} = \infty$. Therefore, there exists a tagged partition π^* such that

$$\sum_{j\notin\sigma} \left| \frac{f(t_{j+1}) - f(t_j)}{t_{j+1}-t_j} - \frac{f(t_j)-f(t_{j-1})}{t_j - t_{j-1}} \right|^{p(x_{j-1})} = \infty.$$

Since $j \notin \sigma$ and $p(\cdot) > 1$ we get

$$\left|\frac{f(t_{j+1})-f(t_j)}{t_{j+1}-t_j}-\frac{f(t_j)-f(t_{j-1})}{t_j-t_{j-1}}\right|>1.$$

But this is satisfied only for a finite number of terms, because in opposite case we would get

$$V^{(2)}(f;[a,b])\geq\sum_{j\notin\sigma}\left|\frac{f(t_{j+1})-f(t_j)}{t_{j+1}-t_j}-\frac{f(t_j)-f(t_{j-1})}{t_j-t_{j-1}}\right|>\sum_{j\notin\sigma}1\to\infty,$$

which is a contradiction as $f\in BV^2[a,b]$. Then, taking supremum we get

$$V^W_{(p(\cdot),2)}(f;[a,b])=\sup_{\pi^*}\sum_{j=1}^{n-1}\left|\frac{f(t_{j+1})-f(t_j)}{t_{j+1}-t_j}-\frac{f(t_j)-f(t_{j-1})}{t_j-t_{j-1}}\right|^{p(x_{j-1})}<\infty.$$

Theorem 2 *Let* p *be an admissible function. If* $f\in BV^W_{(p(\cdot),2)}[a,b]$, *then it follows that for any* $c\in(a,b)$

$$V^W_{(p(\cdot),2)}(f;[a,c])+V^W_{(p(\cdot),2)}(f;[c,b])\leq V^W_{(p(\cdot),2)}(f;[a,b]). \tag{3.2}$$

Proof. By the definition of $V^W_{(p(\cdot),2)}(f;[a,c])$ and $V^W_{(p(\cdot),2)}(f;[c,b])$ we have that, for each $\epsilon>0$, there are partitions $\pi_{(a,c)}$ and $\pi_{(c,b)}$ with $\pi_{(a,c)}:=\{a=\overline{t}_0,\cdots,\overline{t}_m=c\}$ and $\pi_{(c,b)}:=\{c=t_0,\cdots,t_r=b\}$, and sequences of points $\{x_j\}_{j=0}^{m-2}$ and $\{y_j\}_{j=0}^{r-2}$ such that $\overline{t}_j\leq x_j\leq\overline{t}_{j+1}$ for $j=0,\cdots,m-2$ and $t_j\leq y_j\leq t_{j+1}$ for $j=0,\cdots,r-2$ that satisfies

$$\sum_{j=1}^{m-1}\left|\frac{f(\overline{t}_{j+1})-\overline{t}(t_j)}{\overline{t}_{j+1}-\overline{t}_j}-\frac{f(\overline{t}_j)-f(\overline{t}_{j-1})}{\overline{t}_j-\overline{t}_{j-1}}\right|^{p(x_{j-1})}>V^W_{(p(\cdot),2)}(f;[a,c])-\frac{\epsilon}{2},$$

and

$$\sum_{j=1}^{r-1}\left|\frac{f(t_{j+1})-f(t_j)}{t_{j+1}-t_j}-\frac{f(t_j)-f(t_{j-1})}{t_j-t_{j-1}}\right|^{p(y_{j-1})}>V^W_{(p(\cdot),2)}(f;[c,b])-\frac{\epsilon}{2}.$$

Taking $\pi=\pi_{(a,c)}\cup\pi_{(c,b)}=\{a=u_0,\cdots,u_{r+m-1}=b\}$ and the points $\{z_j\}_j:=\{x_j\}_{j=0}^{m-2}\cup\{y_j\}_{j=0}^{r-2}$, we get a partition of $[a,b]$ such that

$$\begin{aligned}&\sum_{j=1}^{m+r-2}\left|\frac{f(u_{j+1})-f(u_j)}{u_{j+1}-u_j}-\frac{f(u_j)-f(u_{j-1})}{u_j-u_{j-1}}\right|^{p(z_{j-1})}\\&=\sum_{j=1}^{r-1}\left|\frac{f(t_{j+1})-f(t_j)}{t_{j+1}-t_j}-\frac{f(t_j)-f(t_{j-1})}{t_j-t_{j-1}}\right|^{p(y_{j-1})}\\&+\sum_{j=1}^{m-1}\left|\frac{f(\overline{t}_{j+1})-\overline{t}(t_j)}{\overline{t}_{j+1}-\overline{t}_j}-\frac{f(\overline{t}_j)-f(\overline{t}_{j-1})}{\overline{t}_j-\overline{t}_{j-1}}\right|^{p(x_{j-1})},\end{aligned}$$

which implies that

$$\sum_{j=1}^{m+r-2}\left|\frac{f(u_{j+1})-f(u_j)}{u_{j+1}-u_j}-\frac{f(u_j)-f(u_{j-1})}{u_j-u_{j-1}}\right|^{p(z_{j-1})} > V^W_{(p(\cdot),2)}(f;[c,b])-\frac{\epsilon}{2}+V^W_{(p(\cdot),2)}(f;[a,c])-\frac{\epsilon}{2}. \tag{3.3}$$

Letting $\epsilon\to 0$ first, and then taking the corresponding supremum in the left-hand side of (3.3), it follows (3.2).

Define
$$\omega_{p(x_{ts\sigma})}(f;[a,b]) := \sup_{t,s,\sigma\in[a,b]}\left\{\left|\frac{f(t)-f(s)}{t-s}-\frac{f(s)-f(\sigma)}{s-\sigma}\right|^{p(x_{ts\sigma})}\right\}.$$

Lemma 1 ***Basic properties of the*** $(p(\cdot),2)$ ***-variation in De La Vallée Poussin-Wiener's sense*** *Let* $f:[a,b]\to\mathbb{R}$ *be an arbitrary map. We have the following properties:*

(P1) For any $t,s,\sigma\in[a,b]$, we have that

$$\left|\frac{f(t)-f(s)}{t-s}-\frac{f(s)-f(\sigma)}{s-\sigma}\right|^{p(x_{ts\sigma})} \le \omega_{p(x_{ts\sigma})}(f;[a,b]) \le V^W_{(p(\cdot),2)}(f;[a,b]).$$

(P2) Monotonicity: If $t,s\in[a,b]$ and $a\le t\le s\le b$, then $V^W_{(p(\cdot),2)}(f;[a,t])\le V^W_{(p(\cdot),2)}(f;[a,s])$, $V^W_{(p(\cdot),2)}(f;[s,b])\le V^W_{(p(\cdot),2)}(f;[t,b])$, and $V^W_{(p(\cdot),2)}(f;[t,s])\le V^W_{(p(\cdot),2)}(f;[a,b])$.

(P3) Semi-additivity: If $t\in(a,b)$, then

$$V^W_{(p(\cdot),2)}(f;[a,t])+V^W_{(p(\cdot),2)}(f;[t,b])\le V^W_{(p(\cdot),2)}(f;[a,b]).$$

(P4) Change of variable: If $\varphi:[c,d]\to[a,b]$ is a monotone function, then

$$V^W_{(p(\cdot),2)}(f;\varphi[c,d]) = V^W_{(p(\cdot),2)}(f\circ\varphi;[c,d]). \tag{3.4}$$

(P5) Regularity: $V^W_{(p(\cdot),2)}(f;[a,b])=\sup\left\{V^W_{(p(\cdot),2)}(f;[s,t]);s,t\in[a,b]\right\}$.

Proof. (P1) We have that for any $t,s,\sigma\in[a,b]$,

$$\begin{aligned}&\left|\frac{f(t)-f(s)}{t-s}-\frac{f(s)-f(\sigma)}{s-\sigma}\right|^{p(x_{ts\sigma})}\\ &\le \sup\left\{\left|\frac{f(t)-f(s)}{t-s}-\frac{f(s)-f(\sigma)}{s-\sigma}\right|^{p(x_{ts\sigma})};t,s,\sigma\in[a,b]\right\}\\ &:=\omega_{p(x_{ts\sigma})}(f;[a,b])\\ &\le \sup_{\pi}\sum_{j=1}^{m-1}\left|\frac{f(t_{j+1})-f(t_j)}{t_{j+1}-t_j}-\frac{f(t_j)-f(t_{j-1})}{t_j-t_{j-1}}\right|^{p(x_{j-1})}\\ &=V^W_{(p(\cdot),2)}(f;[a,b]).\end{aligned}$$

(P2) Let $a\le t\le s\le b$ and the partition $\pi:=\{a=t_0<t_1<\cdots<t_{m1}=t<\cdots<t_{m2}=s<\cdots<t_n=b\}$. Then

$$
\begin{aligned}
V^W_{(p(\cdot),2)}(f;[a,t]) &= \sup_{\pi^*}\sum_{j=1}^{m1}\left|\frac{f(t_{j+1})-f(t_j)}{t_{j+1}-t_j}-\frac{f(t_j)-f(t_{j-1})}{t_j-t_{j-1}}\right|^{p(x_{j-1})}\\
&\le \sup_{\pi^*}\sum_{j=1}^{m1}\left|\frac{f(t_{j+1})-f(t_j)}{t_{j+1}-t_j}-\frac{f(t_j)-f(t_{j-1})}{t_j-t_{j-1}}\right|^{p(x_{j-1})}\\
&\quad+\sup_{\pi^*}\sum_{j=m1+1}^{m2}\left|\frac{f(t_{j+1})-f(t_j)}{t_{j+1}-t_j}-\frac{f(t_j)-f(t_{j-1})}{t_j-t_{j-1}}\right|^{p(x_{j-1})}\\
&\le \sup_{\pi^*}\sum_{j=1}^{m2}\left|\frac{f(t_{j+1})-f(t_j)}{t_{j+1}-t_j}-\frac{f(t_j)-f(t_{j-1})}{t_j-t_{j-1}}\right|^{p(x_{j-1})}\\
&= V^W_{(p(\cdot),2)}(f;[a,s]).
\end{aligned}
$$

The other cases follow in a similar way.

(P3) Semi-additivity: It is obtained in Theorem 2.

(P4) It follows as in ([23], Lemma 2 (P4)). Indeed, let $[c,d]\subset\mathbb{R}$, $\varphi:[c,d]\to[a,b]$ be a (not necessarily strictly) monotone function, π_0 be a tagged partition of the interval $[c,d]$, $T_1=\{\tau_j\}_{j=0}^m\in\pi_0$ and $T=\{t_j\}_{j=0}^m$ with $t_j=\varphi(\tau_j)$, then

$$
\begin{aligned}
&V^W_{(p(\cdot),2)}(f\circ\varphi,T_1)\\
&=\sup_{T_1}\sum_{j=1}^{m}\left|\frac{f(\varphi(\tau_{j+1}))-f(\varphi(\tau_j))}{\tau_{j+1}-\tau_j}-\frac{f(\varphi(\tau_j))-f(\varphi(\tau_{j-1}))}{\tau_j-\tau_{j-1}}\right|^{p(x_{j-1})}\\
&=\sup_{T}\sum_{j=1}^{m}\left|\frac{f(t_{j+1})-f(t_j)}{t_{j+1}-t_j}-\frac{f(t_j)-f(t_{j-1})}{t_j-t_{j-1}}\right|^{p(x_{j-1})}\\
&=V^W_{(p(\cdot),2)}(f,T)\le V^W_{(p(\cdot),2)}(f,\varphi([c,d])).
\end{aligned}
$$

On the other hand, if a partition $T=\{t_j\}_{j=0}^m$ of $\varphi([c,d])$ is such that $t_{j-1}<t_j$ for $j=1,\cdots,m$ then there exists $\tau_j\in[c,d]$ such that $t_j=\varphi(\tau_j)$ and, again by the monotonicity of φ

$$V^W_{(p(\cdot),2)}(f,T)=V^W_{(p(\cdot),2)}(f\circ\varphi,T_1)\le V^W_{(p(\cdot),2)}(f,\varphi([c,d])).$$

(P5) By monotonicity $V^W_{(p(\cdot),2)}(f;[a,b])\ge\sup\{V^W_{(p(\cdot),2)}(f;[s,t]);s,t\in[a,b]\}$. On the other hand, for any $\alpha<V^W_{(p(\cdot),2)}(f;[a,b])$ such that there exists a tagged partition $\Pi=\{t_i\}_{i=0}^n$ of $[a,b]$ with $V^W_{(p(\cdot),2)}(f;\Pi)\ge\alpha$. We define $\bar{\pi}$ a partition of the interval $[t_0,t_m]$ then $\Pi\in\bar{\pi}$ and $V^W_{(p(\cdot),2)}(f;\bar{\pi})\ge V^W_{(p(\cdot),2)}(f;\Pi)\ge\alpha$, *i.e.*,

$$V^W_{(p(\cdot),2)}(f;[a,b])\le\sup\{V^W_{(p(\cdot),2)}(f;[s,t]);s,t\in[a,b]\}.$$

Lemma 2 *If* $\beta_1 > \beta_2$, *then* $V^W_{(p(\cdot),2)}\left(\frac{f}{\beta_1};[a,b]\right) \le V^W_{(p(\cdot),2)}\left(\frac{f}{\beta_2};[a,b]\right)$ *for all* $f \in BV^W_{(p(\cdot),2)}[a,b]$.

Proof. Let β_1, β_2 such that $\beta_1 > \beta_2$. Then, consider any partition π of $[a,b]$, $\pi = \{a = t_0, \cdots, t_n = b\}$ and any finite sequence of numbers $x_0, \cdots, x_{n-2}$ subject to the conditions $t_j \le x_j \le t_{j+1}$ for each $j \le n-2$. It follows that

$$\left|\frac{\left(\frac{f}{\beta_1}\right)(t_{i+1}) - \left(\frac{f}{\beta_1}\right)(t_i)}{t_{i+1}-t_i} - \frac{\left(\frac{f}{\beta_1}\right)(t_i) - \left(\frac{f}{\beta_1}\right)(t_{i-1})}{t_i - t_{i-1}}\right|^{p(x_{i-1})}$$

$$= \left|\frac{1}{\beta_1}\left[\frac{f(t_{i+1}) - f(t_i)}{t_{i+1}-t_i} - \frac{f(t_i) - f(t_{i-1})}{t_i - t_{i-1}}\right]\right|^{p(x_{i-1})}$$

$$\le \left|\frac{1}{\beta_2}\left[\frac{f(t_{i+1}) - f(t_i)}{t_{i+1}-t_i} - \frac{f(t_i) - f(t_{i-1})}{t_i - t_{i-1}}\right]\right|^{p(x_{i-1})}$$

$$= \left|\frac{\left(\frac{f}{\beta_2}\right)(t_{i+1}) - \left(\frac{f}{\beta_2}\right)(t_i)}{t_{i+1}-t_i} - \frac{\left(\frac{f}{\beta_2}\right)(t_i) - \left(\frac{f}{\beta_2}\right)(t_{i-1})}{t_i - t_{i-1}}\right|^{p(x_{i-1})}$$

as $\frac{1}{\beta_2} \ge \frac{1}{\beta_1}$. Then, as this inequality follows for all terms in the sum

$$\sum_{i=1}^{n-1}\left|\frac{\left(\frac{f}{\beta_1}\right)(t_{i+1}) - \left(\frac{f}{\beta_1}\right)(t_i)}{t_{i+1}-t_i} - \frac{\left(\frac{f}{\beta_1}\right)(t_i) - \left(\frac{f}{\beta_1}\right)(t_{i-1})}{t_i - t_{i-1}}\right|^{p(x_{i-1})}$$

$$\le \sum_{i=1}^{n-1}\left|\frac{\left(\frac{f}{\beta_2}\right)(t_{i+1}) - \left(\frac{f}{\beta_2}\right)(t_i)}{t_{i+1}-t_i} - \frac{\left(\frac{f}{\beta_2}\right)(t_i) - \left(\frac{f}{\beta_2}\right)(t_{i-1})}{t_i - t_{i-1}}\right|^{p(x_{i-1})}$$

Taking supremum in any partition, it follows that

$$V^W_{(p(\cdot),2)}\left(\frac{f}{\beta_1};[a,b]\right) \le V^W_{(p(\cdot),2)}\left(\frac{f}{\beta_2};[a,b]\right).$$

Proposition 5 *Let* p *be an admissible function. The space* $BV^W_{(p(\cdot),2)}[a,b]$ *is a vectorial space.*

Proof. Let $f, g \in BV^W_{(p(\cdot),2)}[a,b]$ and consider any partition $\pi = \{a = t_0, \cdots, t_n = b\}$ and any finite sequence of numbers $x_0, \cdots, x_{n-2}$ subject to the conditions $t_j \le x_j \le t_{j+1}$ and $\alpha, \beta \in \mathbb{R}$. By definition, there exists β_1, β_2 such that

$$V^{W}_{(p(\cdot),2)}\left(\frac{f}{\beta_1};[a,b]\right)\le 1<\infty \quad\text{and}\quad V^{W}_{(p(\cdot),2)}\left(\frac{g}{\beta_2};[a,b]\right)\le 1<\infty.$$

Let $\hat{\beta}:=\max\{\beta_1,\beta_2\}>0$. By Lemma 2, it follows that

$$V^{W}_{(p(\cdot),2)}\left(\frac{f}{\hat{\beta}};[a,b]\right)<V^{W}_{(p(\cdot),2)}\left(\frac{f}{\beta_1};[a,b]\right)<\infty$$

$$V^{W}_{(p(\cdot),2)}\left(\frac{g}{\hat{\beta}};[a,b]\right)<V^{W}_{(p(\cdot),2)}\left(\frac{g}{\beta_2};[a,b]\right)<\infty.$$

The rest of the proof follows analyzing the possible cases.

1) If $\alpha=\beta=0$, then $\alpha f+\beta g\in BV^{W}_{(p(\cdot),2)}[a,b]$.

2) If $\alpha\neq 0$ and/or $\beta\neq 0$. Let $\mu=(|\alpha|+|\beta|)\hat{\beta}>0$, and consider any tagged partition π^* of $[a,b]$, $\pi^*=\{a=t_0\le\cdots\le t_n=b\}$ which is any partition π of $[a,b]$ and any finite sequence of numbers $x_0,\cdots,x_{n-2}$ subject to the conditions $t_j\le x_j\le t_{j+1}$ for each $j\le n-2$. Then, by convexity of t^p, when $1<p<\infty$, it follows that

$$\sum_{j=1}^{n-1}\left|\frac{\left(\frac{\alpha f+\beta g}{\mu}\right)(t_{j+1})-\left(\frac{\alpha f+\beta g}{\mu}\right)(t_j)}{t_{j+1}-t_j}-\frac{\left(\frac{\alpha f+\beta g}{\mu}\right)(t_j)-\left(\frac{\alpha f+\beta g}{\mu}\right)(t_{j-1})}{t_j-t_{j-1}}\right|^{p(x_{j-1})}$$

$$=\sum_{j=1}^{n-1}\left|\frac{1}{\mu}\frac{\left[\alpha\left(f(t_{j+1})-f(t_j)\right)+\beta\left(g(t_{j+1})-g(t_j)\right)\right]}{t_{j+1}-t_j}-\frac{1}{\mu}\frac{\left[\alpha\left(f(t_j)-f(t_{j-1})\right)+\beta\left(g(t_j)-g(t_{j-1})\right)\right]}{t_j-t_{j-1}}\right|^{p(x_{j-1})}$$

$$\le\sum_{j=1}^{n-1}\left(\frac{|\alpha|}{\mu}\left|\frac{f(t_{j+1})-f(t_j)}{t_{j+1}-t_j}-\frac{f(t_j)-f(t_{j-1})}{t_j-t_{j-1}}\right|+\frac{|\beta|}{\mu}\left|\frac{g(t_{j+1})-g(t_j)}{t_j-t_{j-1}}-\frac{g(t_j)-g(t_{j-1})}{t_j-t_{j-1}}\right|\right)^{p(x_{j-1})}$$

$$\le\sum_{j=1}^{n-1}\left(\frac{|\alpha|}{|\alpha|+|\beta|}\frac{1}{\hat{\beta}}\left|\frac{f(t_{j+1})-f(t_j)}{t_{j+1}-t_j}-\frac{f(t_j)-f(t_{j-1})}{t_j-t_{j-1}}\right|\right.$$
$$\left.+\frac{|\beta|}{|\alpha|+|\beta|}\frac{1}{\hat{\beta}}\left|\frac{g(t_{j+1})-g(t_j)}{t_{j+1}-t_j}-\frac{g(t_j)-g(t_{j-1})}{t_j-t_{j-1}}\right|\right)^{p(x_{j-1})}$$

$$\le\sum_{j=1}^{n-1}\left[\frac{|\alpha|}{|\alpha|+|\beta|}\left(\frac{1}{\hat{\beta}}\left|\frac{f(t_{j+1})-f(t_j)}{t_{j+1}-t_j}-\frac{f(t_j)-f(t_{j-1})}{t_j-t_{j-1}}\right|\right)^{p(x_{j-1})}\right.$$
$$\left.+\frac{|\beta|}{|\alpha|+|\beta|}\left(\frac{1}{\hat{\beta}}\left|\frac{g(t_{j+1})-g(t_j)}{t_{j+1}-t_j}-\frac{g(t_j)-g(t_{j-1})}{t_j-t_{j-1}}\right|\right)^{p(x_{j-1})}\right]$$

$$\le\frac{|\alpha|}{|\alpha|+|\beta|}\sum_{j=1}^{n-1}\left(\frac{1}{\hat{\beta}}\left|\frac{f(t_{j+1})-f(t_j)}{t_{j+1}-t_j}-\frac{f(t_j)-f(t_{j-1})}{t_j-t_{j-1}}\right|\right)^{p(x_{j-1})}$$
$$+\frac{|\beta|}{|\alpha|+|\beta|}\sum_{j=1}^{n-1}\left(\frac{1}{\hat{\beta}}\left|\frac{g(t_{j+1})-g(t_j)}{t_{j+1}-t_j}-\frac{g(t_j)-g(t_{j-1})}{t_j-t_{j-1}}\right|\right)^{p(x_{j-1})}.$$

Therefore,

$$\sum_{j=1}^{n-1}\left|\frac{\left(\frac{\alpha f+\beta g}{\mu}\right)(t_{j+1})-\left(\frac{\alpha f+\beta g}{\mu}\right)(t_j)}{t_{j+1}-t_j}-\frac{\left(\frac{\alpha f+\beta g}{\mu}\right)(t_j)-\left(\frac{\alpha f+\beta g}{\mu}\right)(t_{j-1})}{t_j-t_{j-1}}\right|^{p(x_{j-1})}$$

$$\leq\frac{|\alpha|}{|\alpha|+|\beta|}\sum_{j=1}^{n-1}\left(\frac{1}{\hat{\beta}}\left|\frac{f(t_{j+1})-f(t_j)}{t_{j+1}-t_j}-\frac{f(t_j)-f(t_{j-1})}{t_j-t_{j-1}}\right|\right)^{p(x_{j-1})}$$

$$+\frac{|\beta|}{|\alpha|+|\beta|}\sum_{j=1}^{n-1}\left(\frac{1}{\hat{\beta}}\left|\frac{g(t_{j+1})-g(t_j)}{t_{j+1}-t_j}-\frac{g(t_j)-g(t_{j-1})}{t_j-t_{j-1}}\right|\right)^{p(x_{j-1})}$$

$$\leq\frac{|\alpha|}{|\alpha|+|\beta|}V_{(p(\cdot),2)}^{W}\left(\frac{f}{\hat{\beta}};[a,b]\right)+\frac{|\beta|}{|\alpha|+|\beta|}V_{(p(\cdot),2)}^{W}\left(\frac{g}{\hat{\beta}};[a,b]\right)<\infty.$$

Then, taking supremum over all partitions, we get that

$$V_{(p(\cdot),2)}^{W}\left(\frac{\alpha f+\beta g}{\mu};[a,b]\right)$$
$$\leq\frac{|\alpha|}{|\alpha|+|\beta|}V_{(p(\cdot),2)}^{W}\left(\frac{f}{\hat{\beta}};[a,b]\right)+\frac{|\beta|}{|\alpha|+|\beta|}V_{(p(\cdot),2)}^{W}\left(\frac{g}{\hat{\beta}};[a,b]\right)$$
$$\leq\frac{|\alpha|}{|\alpha|+|\beta|}+\frac{|\beta|}{|\alpha|+|\beta|}=1<\infty.$$

Therefore $\alpha f+\beta g\in BV_{(p(\cdot),2)}^{W}[a,b]$.

The other properties of a vectorial space follow similarly.

Theorem 3 *Let* p *be an admissible function. The space* $BV_{(p(\cdot),2)}^{W}[a,b]$ *is a normed space.*

Proof. Let p be an admissible function. Let us analyze all the properties of a norm.

1) By definition of $\|\cdot\|_{BV_{(p(\cdot),2)}^{W}[a,b]}$, we have that $\|f\|_{BV_{(p(\cdot),2)}^{W}[a,b]}\geq 0$ for all $f\in BV_{(p(\cdot),2)}^{W}[a,b]$

2) To prove that $\|\alpha f\|_{BV_{(p(\cdot),2)}^{W}[a,b]}=|\alpha|\|f\|_{BV_{(p(\cdot),2)}^{W}[a,b]}$ for any $\alpha\in\mathbb{R}$, we consider the possible cases:

- If $\alpha=0$, then

$$\|\alpha f\|_{BV_{(p(\cdot),2)}^{W}[a,b]}=\|0\|_{BV_{(p(\cdot),2)}^{W}[a,b]}=0=0\|f\|_{BV_{(p(\cdot),2)}^{W}[a,b]}=\alpha\|f\|_{BV_{(p(\cdot),2)}^{W}[a,b]}$$

for any $f\in BV_{(p(\cdot),2)}^{W}[a,b]$.

- If $\alpha\neq 0$, then

$$\|\alpha f\|_{BV_{(p(\cdot),2)}^{W}[a,b]}=|\alpha f(a)|+|\alpha f_{+}'(a)|+\inf\left\{\lambda>0;V_{(p(\cdot),2)}^{W}\left(\frac{\alpha f}{\lambda};[a,b]\right)\leq 1\right\}$$
$$=|\alpha||f(a)|+|\alpha||f_{+}'(a)|+\inf\left\{\lambda>0;V_{(p(\cdot),2)}^{W}\left(\frac{f}{\frac{\lambda}{\alpha}};[a,b]\right)\leq 1\right\}$$

$$=|\alpha||f(a)|+|\alpha||f'_+(a)|+\inf\left\{\alpha\frac{\lambda}{\alpha}>0;V^W_{(p(\cdot),2)}\left(\frac{f}{\frac{\lambda}{\alpha}};[a,b]\right)\le 1\right\}$$

$$=|\alpha||f(a)|+|\alpha||f'_+(a)|+\alpha\inf\left\{\frac{\lambda}{\alpha}>0;V^W_{(p(\cdot),2)}\left(\frac{f}{\frac{\lambda}{\alpha}};[a,b]\right)\le 1\right\}$$

$$=|\alpha||f(a)|+|\alpha||f'_+(a)|+\alpha\inf\left\{\beta>0;V^W_{(p(\cdot),2)}\left(\frac{f}{\beta};[a,b]\right)\le 1\right\}$$

$$=|\alpha|\|f\|_{BV^W_{(p(\cdot),2)}[a,b]},$$

3) Property $\|f+g\|_{BV^W_{(p(\cdot),2)}[a,b]}\le\|f\|_{BV^W_{(p(\cdot),2)}[a,b]}+\|g\|_{BV^W_{(p(\cdot),2)}[a,b]}$ is satisfied by using that $|f+g|\le|f|+|g|$, $\left|(f+g)'_+\right|=|f'_++g'_+|\le|f'_+|+|g'_+|$ and the previous proposition.

4) Let us see that $\|f\|_{BV^W_{(p(\cdot),2)}[a,b]}=0$ if and only if $f=0$.

- If $\|f\|_{BV^W_{(p(\cdot),2)}[a,b]}=0$, then by definition of the norm, $f(a)=0$ and $f'_+(a)=0$, and

$$\inf\left\{\lambda>0;V^W_{(p(\cdot),2)}\left(\frac{f}{\lambda};[a,b]\right)\le 1\right\}=0.$$

Hence, we have by Proposition 3 and Proposition 4 (2) that

$$V^W_{(p(\cdot),2)}(f;[a,b])\le\|f\|_{BV^W_{(p(\cdot),2)}[a,b]}.$$

Therefore, $V^W_{(p(\cdot),2)}(f;[a,b])=0$, and hence,

$$\sup_{\pi^*}\sum_{j=1}^{n-1}\left|\frac{f(t_{j+1})-f(t_j)}{t_{j+1}-t_j}-\frac{f(t_j)-f(t_{j-1})}{t_j-t_{j-1}}\right|^{p(x_{j-1})}=0.$$

Therefore, for any tagged partition π^* of the interval $[a,b]$, that is a partition $\pi=\{a=t_0<\cdots<t_n=b\}$ together with a finite sequence of numbers $x_0,\cdots,x_n$ subject to the conditions $t_j\le x_j\le t_{j+1}$ for each j, we have that

$$\left|\frac{f(t_{j+1})-f(t_j)}{t_{j+1}-t_j}-\frac{f(t_j)-f(t_{j-1})}{t_j-t_{j-1}}\right|^{p(x_{j-1})}=0,\quad\forall j\in\{1,\cdots,n-1\}.$$

So that

$$\frac{f(t_{j+1})-f(t_j)}{t_{j+1}-t_j}=\frac{f(t_j)-f(t_{j-1})}{t_j-t_{j-1}},\quad\forall j\in\{1,\cdots,n-1\}.$$

Consider the partition $\pi=\{a\le t_1<t_2=c<t\le b\}$. We get that

$$\lim_{c\to a+}\frac{f(t)-f(c)}{t-c}=\lim_{c\to a+}\frac{f(c)-f(a)}{c-a}=f'_+(a)=0.$$

Then

$$\frac{f(t)-f(a)}{t-a}=0.$$

As $f(a)=0$ is obtained that $f(t)=0$ for all $t\in[a,b]$.

- In other hand, if $f=0$, then $f(t)=0$ for all $t\in[a,b]$. Hence, $f'_+(a)=0$ and $V^W_{(p(\cdot),2)}(f;[a,b])=V^W_{(p(\cdot),2)}(0;[a,b])=0$. Therefore, by definition, $\|f\|_{BV^W_{(p(\cdot),2)}[a,b]}=0.$

Theorem 4 *Let* p *be an admissible function. The space* $BV^W_{(p(\cdot),2)}[a,b]$ *is a Banach space endowed with the norm in* (3.1).

Let $\{f_n\}_{n\in\mathbb{N}}$ be a Cauchy sequence in $BV^W_{(p(\cdot),2)}[a,b]$. Then, for all $\epsilon>0$, there exists $N(\epsilon)$ such that

$$\|f_m-f_n\|_{BV^W_{(p(\cdot),2)}[a,b]}<\epsilon,\quad \forall m,n>N(\epsilon).$$

Therefore, by definition it follows that

$$\inf\left\{\lambda>0;V^W_{(p(\cdot),2)}\left(\frac{f_m-f_n}{\lambda};[a,b]\right)\le 1\right\}<\epsilon,\quad \forall m,n>N(\epsilon),\tag{3.5}$$

$$|(f_m-f_n)(a)|<\epsilon,\quad \forall m,n>N(\epsilon),\tag{3.6}$$

and

$$\left|(f_m-f_n)'_+(a)\right|<\epsilon,\quad \forall m,n>N(\epsilon).\tag{3.7}$$

Then, by (3.5) and Proposition 4 (2) we have that

$$V^W_{(p(\cdot),2)}(f_m-f_n;[a,b])<\epsilon.$$

It implies that for fixed t, $\{f_n(t)\}_{n\in\mathbb{N}}$ is a Cauchy sequence in $\mathbb{R}$. Indeed,

$$V^W_{(p(\cdot),2)}\left(\frac{f_m-f_n}{\epsilon}\right)\le 1,\quad \forall m,n>N(\epsilon)$$

then for all $x,y,z\in[a,b]$, $f=f_m-f_n$ we get

$$\left|\frac{1}{\epsilon}\left(\frac{f(z)-f(y)}{z-y}-\frac{f(y)-f(x)}{y-x}\right)\right|^{p(y)}\le V^W_{(p(\cdot),2)}\left(\frac{f_m-f_n}{\epsilon}\right)\le 1$$

so

$$\left|\frac{f(z)-f(y)}{z-y}-\frac{f(y)-f(x)}{y-x}\right|^{p(y)}\le \epsilon^{p(y)}.$$

As

$$\begin{aligned}\left|\frac{f(z)-f(y)}{z-y}\right|^{p(y)}&\le\left|\left|\frac{f(z)-f(y)}{z-y}\right|-\left|\frac{f(y)-f(x)}{y-x}\right|\right|^{p(y)}\\&\le\left|\frac{f(z)-f(y)}{z-y}-\frac{f(y)-f(x)}{y-x}\right|^{p(y)}\end{aligned}$$

thus

$$\left|\frac{f(z)-f(y)}{z-y}\right|^{p(y)} \le \epsilon^{p(y)}.$$

Therefore

$$\left|f(z)-f(y)\right|^{p(y)} \le \left(\epsilon|z-y|\right)^{p(y)}$$

and by property of log

$$p(y)\log\left|f(z)-f(y)\right| \le p(y)\log\left(\epsilon|z-y|\right).$$

Then

$$\log\left|f(z)-f(y)\right| \le \log\left(\epsilon|z-y|\right)$$

and hence

$$\left|f(z)-f(y)\right| \le \epsilon' = \exp^{\log(\epsilon|z-y|)} = \epsilon|z-y|.$$

i.e.

$$\left|(f_m-f_n)(z)-(f_m-f_n)(y)\right| \le \epsilon', \quad \forall m,n > N(\epsilon).$$

Let $f(t) := \lim_{n\to\infty} f_n(t)$ for any $t\in[a,b]$ and let π be any partition $\pi := \{a=t_0,\cdots,t_k=b\}$ of $[a,b]$ and a sequence $x_0,\cdots,x_{k-1}$ such that $t_j \le x_j \le t_{j+1}$ for any $1\le j<k-1$. It follows that for all $m,n>N(\epsilon)$

$$\sum_{j=1}^{k}\left|\frac{(f_m-f_n)(t_{j+1})-(f_m-f_n)(t_j)}{t_{j+1}-t_j}-\frac{(f_m-f_n)(t_j)-(f_m-f_n)(t_{j-1})}{t_j-t_{j-1}}\right|^{p(x_{j-1})} < \epsilon.$$

Then, letting $n\to\infty$, for any $m>N(\epsilon)$ it follows that

$$\sum_{j=1}^{k}\left|\frac{(f_m-f)(t_{j+1})-(f_m-f)(t_j)}{t_{j+1}-t_j}-\frac{(f_m-f)(t_j)-(f_m-f)(t_{j-1})}{t_j-t_{j-1}}\right|^{p(x_{j-1})} < \epsilon. \tag{3.8}$$

Therefore, as (3.8) follows for any tagged partition π^* of $[a,b]$, taking supremum over all tagged partitions it follows that

$$V^{W}_{(p(\cdot),2)}\left(f_m-f;[a,b]\right) < \epsilon, \quad \forall m>N(\epsilon). \tag{3.9}$$

Moreover, by (3.6) and (3.7), we have that

$$\left|(f_m-f_n)(a)\right| < \epsilon, \quad \left|(f_m-f_n)'_+(a)\right| < \epsilon, \quad \forall m,n>N(\epsilon).$$

Then, letting $n\to\infty$, we have that

$$\left|(f_m-f)(a)\right| < \epsilon, \quad \left|(f_m-f)'_+(a)\right| < \epsilon, \quad \forall m>N(\epsilon). \tag{3.10}$$

Then, (3.9) and (3.10) imply that for m sufficiently large

$$\|f_m-f\|_{BV^{W}_{(p(\cdot),2)}[a,b]} < 3\epsilon.$$

Hence, as

$$\|f\|_{BV^W_{(p(\cdot),2)}[a,b]} \le \|f_m - f\|_{BV^W_{(p(\cdot),2)}[a,b]} + \|f_m\|_{BV^W_{(p(\cdot),2)}[a,b]} < \infty,$$

we obtain that $f \in BV^W_{(p(\cdot),2)}[a,b]$.

Theorem 5 *Let* p *be an admissible function. Then, we have:*

1) If $f \in BV^W_{(p(\cdot),2)}[a,b]$, then f is bounded in all the interval $[a,b]$.

2) $BV^W_{(p(\cdot),2)}[a,b] \hookrightarrow BV^W_{(q(\cdot),2)}[a,b]$ for functions $q(x) \ge p(x)$.

Let us proof (a). Suppose that $f \in BV^W_{(p(\cdot),2)}[a,b]$ and f is not bounded. Then, there exists a sequence $\{t_n\}_{n\ge1}$, $t_n \in (a,b)$, $n \ge 1$ such that $|f(t_n)| \to \infty$ when $n \to \infty$. Let $\{t_m\}_{m\ge1}$ be a subsequence of $\{t_n\}_{n\ge1}$ such that $\{t_m\}_{m\ge1}$ converge to $x \in [a,b]$. As $\{f(t_m)\}_{m\ge1}$ is a subsequence of $\{f(t_n)\}_{n\ge1}$, so

$$|f(t_m)| \to \infty \quad \text{when } n \to \infty.$$

Case 1: Suppose that $x = a$ and let t such that $a \le t_m < t < b$ for some $t_m \in \{t_m\}_{m\ge1}$, then

$$\left|\frac{f(b)-f(t)}{b-t} - \frac{f(t)-f(t_m)}{t-t_m}\right|^{p(x_t)} \le V^W_{(p(\cdot),2)}(f;[a,b])$$

and since $u \to u^s$ is continuous

$$\left|\frac{f(b)-f(t)}{b-t} - \frac{\lim_{m\to\infty} f(t)-f(t_m)}{t-x}\right|^{p(x_t)}$$

$$= \lim_{m\to\infty}\left|\frac{f(b)-f(t)}{b-t} - \frac{f(t)-f(t_m)}{t-t_m}\right|^{p(x_t)} \le V^W_{(p(\cdot),2)}(f;[a,b]).$$

On the other hand $|f(t)-f(t_m)|$ tend to infinity as $m \to \infty$. Then

$$\lim_{m\to\infty}\left|\frac{f(b)-f(t)}{b-t} - \frac{f(t)-f(t_m)}{t-t_m}\right|^{p(x_t)} = \infty,$$

and hence $V^W_{(p(\cdot),2)}(f;[a,b]) = \infty$, which is a contradiction.

Case 2: Suppose that $x \ne a$ and let t such that $a < t < t_m < b$ for some $t_m \in \{t_m\}_{m\ge1}$, then

$$\left|\frac{f(t_m)-f(t)}{t_m-t} - \frac{f(t)-f(a)}{t-a}\right|^{p(x_t)} \le V^W_{(p(\cdot),2)}(f;[a,b]).$$

Since $u \to u^s$ is continuous

$$\left|\frac{\lim_{m\to\infty} f(t_m)-f(t)}{x-t} - \frac{f(t)-f(a)}{t-a}\right|^{p(x_t)}$$

$$= \lim_{m\to\infty}\left|\frac{f(t_m)-f(t)}{t_m-t} - \frac{f(t)-f(a)}{t-a}\right|^{p(x_t)} \le V^W_{(p(\cdot),2)}(f;[a,b]).$$

On the other hand $|f(t_m)-f(t)|$ tend to infinity as $m \to \infty$ then

$$\left|\frac{\lim_{m\to\infty} f(t_m)-f(t)}{x-t}-\frac{f(t)-f(a)}{t-a}\right|^{p(x_t)} \to \infty \quad \text{as } n\to\infty,$$

and then $V^W_{(p(\cdot),2)}(f)=\infty$, which is a contradiction.

Let us proof (b). Taking $\|f\|_{BV^W_{(p(\cdot),2)}[a,b]}=1$, since $V^W_{(p(\cdot),2)}(f;[a,b])\le 1$, it follows that

$$\sum_{j=1}^{n-1}\left|\frac{f(t_{j+1})-f(t_j)}{t_{j+1}-t_j}-\frac{f(t_j)-f(t_{j-1})}{t_j-t_{j-1}}\right|^{p(x_{j-1})}\le 1,$$

for any tagged partition $\pi:=\{a=t_0<\cdots<t_n=b\}$ and any sequence of points x_j such that $t_j\le x_j\le t_{j+1}$ for $j=0,\cdots,n-2$. Therefore,

$$\sum_{j=1}^{n-1}\left|\frac{f(t_{j+1})-f(t_j)}{t_{j+1}-t_j}-\frac{f(t_j)-f(t_{j-1})}{t_j-t_{j-1}}\right|^{q(x_{j-1})}$$
$$\le\sum_{j=1}^{n-1}\left|\frac{f(t_{j+1})-f(t_j)}{t_{j+1}-t_j}-\frac{f(t_j)-f(t_{j-1})}{t_j-t_{j-1}}\right|^{p(x_{j-1})}\le 1,$$

since in particular $\left|\frac{f(t_{j+1})-f(t_j)}{t_{j+1}-t_j}-\frac{f(t_j)-f(t_{j-1})}{t_j-t_{j-1}}\right|^{p(x_{j-1})}\le 1$ for any $1\le j\le n-1$. Taking supremum to both sides, we obtain that $V^W_{(q(\cdot),2)}(f;[a,b])\le V^W_{(p(\cdot),2)}(f;[a,b])$. Then, by definition it follows that

$$\|f\|_{BV^W_{(q(\cdot),2)}[a,b]}\le\|f\|_{BV^W_{(p(\cdot),2)}[a,b]},$$

and the general case follows from the homogeneity of the norm. □

4. Functions in $BV^W_{(p(\cdot),2)}[a,b]$ and Hölder Continuous Functions

In this section we prove also that if a function is the composition of a bounded monotone function with a $(\gamma(\cdot)+1)$-Hölder continuous function with $\gamma(\cdot)=1/p(\cdot)$, then the function is in $BV^W_{(p(\cdot),2)}[a,b]$.

Definition 8 *A function* $g:[a,b]\to\mathbb{R}$ *is Hölder continuous of exponent* γ, *where* $\gamma(\cdot)$ *is a positive function such that* $0\le\gamma(x)\le 1$, *if*

$$|g(t_i)-g(t_{i-1})|\le C|t_i-t_{i-1}|^{\gamma(x_{i-1})}$$

for all $x_{i-1}\in[a,b]$. The least number C satisfying the above inequality is called the Hölder constant of g.

Proposition 6 *Let* p *be an admissible function and* $f:[a,b]\to\mathbb{R}$ *such that* $f=g\circ\varphi$, *where* $\varphi:[a,b]\to\mathbb{R}$ *is a bounded monotone function and* $g:\varphi[a,b]\to\mathbb{R}$ *is* $(\gamma(\cdot)+1)$*-Hölder continuous with* $\gamma(\cdot)=\frac{1}{p(\cdot)}$. *Then*

$f \in BV^W_{(p(\cdot),2)}[a,b]$.

Proof. Assume that φ is nondecreasing. Since $\varphi([a,b]) = [\varphi(a),\varphi(b)]$, by virtue of the change of variable

$$V^W_{(p(\cdot),2)}(f;[a,b]) = V^W_{(p(\cdot),2)}(g\circ\varphi;[a,b]) = V^W_{(p(\cdot),2)}(g;[\varphi(a),\varphi(b)]). \tag{4.1}$$

If $T = \{t_i\}_{i=0}^n$ is a partition of $[\varphi(a),\varphi(b)]$ and $\{x_j\}$ is a sequence of points $x_j \in (t_j,t_{j+1})$ for $j = 0,\cdots,n-2$ then

$$\begin{aligned}
&\sum_{i=1}^{n-1}\left|\frac{g(t_{i+1})-g(t_i)}{t_{i+1}-t_i} - \frac{g(t_i)-g(t_{i-1})}{t_i-t_{i-1}}\right|^{p(x_{i-1})}\\
&\le \sum_{i=1}^{n-1}\left(\left|\frac{g(t_{i+1})-g(t_i)}{t_{i+1}-t_i}\right| + \left|\frac{g(t_i)-g(t_{i-1})}{t_i-t_{i-1}}\right|\right)^{p(x_{i-1})}\\
&\le \sum_{i=1}^{n-1}\left(\frac{C|t_{i+1}-t_i|^{\gamma(x_{i-1})+1}}{|t_{i+1}-t_i|} + \frac{C|t_i-t_{i-1}|^{\gamma(x_{i-1})+1}}{|t_i-t_{i-1}|}\right)^{p(x_{i-1})}\\
&\le \sum_{i=1}^{n-1}\left(C|t_{i+1}-t_i|^{\gamma(x_{i-1})} + C|t_i-t_{i-1}|^{\gamma(x_{i-1})}\right)^{p(x_{i-1})}\\
&\le \sum_{i=1}^{n-1}2^{p(x_{i-1})}\left(C^{p(x_{i-1})}|t_{i+1}-t_i|^{(\gamma(x_{i-1}))p(x_{i-1})} + C^{p(x_{i-1})}|t_i-t_{i-1}|^{(\gamma(x_{i-1}))p(x_{i-1})}\right)\\
&\le \sum_{i=1}^{n-1}2^{p^+}\left(C^{p^+}|t_{i+1}-t_i| + C^{p^+}|t_i-t_{i-1}|\right) \le 2^{p^+ +1}C^{p^+}|\varphi(b)-\varphi(a)|.
\end{aligned}$$

Therefore, by taking supremum over any tagged partition, it follows that

$$V^W_{(p(\cdot),2)}(g;[\varphi(a),\varphi(b)]) \le 2^{p^+ +1}C^{p^+}|\varphi(b)-\varphi(a)| < \infty$$

by the boundedness of φ. Hence, by (4.1)

$$V^W_{(p(\cdot),2)}(f;[a,b]) = V^W_{(p(\cdot),2)}(g;[\varphi(a),\varphi(b)]) < \infty.$$

5. The Matkowski's Condition

Let us show as an application that, any uniformly bounded composition operator that maps the space $BV^W_{(p(\cdot),2)}[a,b]$ into itself satisfies the Matkowski's condition.

Theorem 6 *Suppose that the composition operator* H *generated by* h *maps* $BV^W_{(p(\cdot),2)}[a,b]$ *into itself and satisfies the following inequality*

$$\|Hf_1 - Hf_2\|^W_{(p(\cdot),2)} \le \gamma\left(\|f_1 - f_2\|^W_{(p(\cdot),2)}\right), \quad \left(f_1, f_2 \in BV^W_{(p(\cdot),2)}[a,b]\right), \tag{5.1}$$

for any function $\gamma:[0,\infty)\to[0,\infty)$. Then, there exist functions $\alpha,\beta \in BV^W_{(p(\cdot),2)}[a,b]$ such that

$$h(t,x) = \alpha(t)x + \beta(t), \quad t\in[a,b], x\in\mathbb{R}. \tag{5.2}$$

Proof. By hypothesis, for $x\in\mathbb{R}$ fixed, the constant function $f(t) = x$, $t\in[a,b]$ belongs to $BV^W_{(p(\cdot),2)}[a,b]$. Since H maps $BV^W_{(p(\cdot),2)}[a,b]$ into itself, we have that $(Hf)(t) = h(t, f(t)) \in BV^W_{(p(\cdot),2)}[a,b]$.

From inequality (5.1) and definition of the norm $\|\cdot\|^W_{(p(\cdot),2)}$, we have for $f_1, f_2 \in BV^W_{(p(\cdot),2)}[a,b]$,

$$\begin{aligned}&\inf\left\{\lambda > 0; V^W_{(p(\cdot),2)}\left(\frac{Hf_1 - Hf_2}{\lambda};[a,b]\right) \le 1\right\}\\ &\le \|Hf_1 - Hf_2\|^W_{(p(\cdot),2)} \le \gamma\left(\|f_1 - f_2\|^W_{(p(\cdot),2)}\right),\end{aligned}$$

and then

$$V^W_{(p(\cdot),2)}\left(\frac{Hf_1 - Hf_2}{\gamma\left(\|f_1 - f_2\|^W_{(p(\cdot),2)}\right)};[a,b]\right) \le 1. \tag{5.3}$$

Consider $a \le s < t \le b$ and let $\pi_m := \{t_0, t_1, \cdots, t_{2m}\} \in \pi$ be the equidistant partition defined by

$$t_0 = s,\ t_j - t_{j-1} = \frac{t-s}{2m}, \quad (j = 1, 2, \cdots, 2m).$$

Given $u, v \in \mathbb{R}$ with $u \ne v$, define $f_1, f_2 : [a,b] \to \mathbb{R}$ by

$$f_1(x) := \begin{cases} v, & \text{if } x = t_j \text{ for some even } j, \\ \frac{u+v}{2}, & \text{if } x = t_j \text{ for some odd } j, \\ \text{linear}, & \text{otherwise} \end{cases}$$

and

$$f_2(x) := \begin{cases} \frac{u+v}{2}, & \text{if } x = t_j \text{ for some even } j, \\ u, & \text{if } x = t_j \text{ for some odd } j, \\ \text{linear}, & \text{otherwise.} \end{cases}$$

Then, the difference $f_1 - f_2$ satisfies that $|f_1(x) - f_2(x)| = \frac{|u-v|}{2}$ for all $x \in [a,b]$. Therefore, by the inequality (5.1)

$$\begin{aligned}\|Hf_1 - Hf_2\|^W_{(p(\cdot),2)} &\le \gamma\left(\|f_1 - f_2\|^W_{(p(\cdot),2)}\right)\\ &\le \gamma\left(\frac{|u-v|}{2}\right),\end{aligned}$$

and hence, by definition

$$V^W_{(p(\cdot),2)}\left(\frac{Hf_1 - Hf_2}{\gamma\left(\frac{|u-v|}{2}\right)};[a,b]\right) \le 1. \tag{5.4}$$

From the inequality (5.4), and the definition, it follows that for any partition $\{t_0, t_2, t_4, \cdots, t_{2(m-1)}\}$ of $[a,b]$

$$\sum_{j=1}^{m-1}\left|\frac{h(f_1)(t_{2j})-h(f_2)(t_{2j})-h(f_1)(t_{2j-1})+h(f_2)(t_{2j-1})}{|t_{2j}-t_{2j-1}|\gamma\left(\frac{|u-v|}{2}\right)}\right.$$

$$\left.-\frac{h(f_1)(t_{2j-1})-h(f_2)(t_{2j-1})-h(f_1)(t_{2j-2})+h(f_2)(t_{2j-2})}{|t_{2j-1}-t_{2j-2}|\gamma\left(\frac{|u-v|}{2}\right)}\right|^{p(x_{j-1})} \leq 1.$$

However, by the definition of f_1 and f_2, we have that

$$\sum_{j=1}^{m-1}\left|\frac{h(f_1)(t_{2j})-h(f_2)(t_{2j})-h(f_1)(t_{2j-1})+h(f_2)(t_{2j-1})}{|t_{2j}-t_{2j-1}|\gamma\left(\frac{|u-v|}{2}\right)}\right.$$

$$\left.-\frac{h(f_1)(t_{2j-1})-h(f_2)(t_{2j-1})-h(f_1)(t_{2j-2})+h(f_2)(t_{2j-2})}{|t_{2j-1}-t_{2j-2}|\gamma\left(\frac{|u-v|}{2}\right)}\right|^{p(x_{j-1})}$$

$$=\sum_{j=1}^{m-1}\left(\frac{2\left|h(v)+h(u)-2h\left(\frac{u+v}{2}\right)\right|}{\frac{t-s}{2m}\gamma\left(\frac{|u-v|}{2}\right)}\right)^{p(x_{j-1})}$$

$$=\sum_{j=1}^{m-1}\left(\frac{4m}{t-s}\frac{\left|h(v)+h(u)-2h\left(\frac{u+v}{2}\right)\right|}{\gamma\left(\frac{|u-v|}{2}\right)}\right)^{p(x_{j-1})} \leq 1.$$

Then, since $1<p(x_{j-1})<\infty$ and $j=1,2,\cdots,2m$, it follows that

$$\sum_{j=1}^{m-1}\left(\frac{4}{t-s}\frac{\left|h(v)+h(u)-2h\left(\frac{u+v}{2}\right)\right|}{\gamma\left(\frac{|u-v|}{2}\right)}\right)^{p(x_{j-1})}$$

$$\leq\sum_{j=1}^{m-1}\left(\frac{4m}{t-s}\frac{\left|h(v)+h(u)-2h\left(\frac{u+v}{2}\right)\right|}{\gamma\left(\frac{|u-v|}{2}\right)}\right)^{p(x_{j-1})} \leq 1.$$

Hence, necessarily

$$h(v)+h(u)-2h\left(\frac{u+v}{2}\right)=0.$$

So that, we conclude that $h(s,\cdot)$ satisfies the Jensen equation in $\mathbb{R}$. The continuity of h with respect to the second variable implies that for every $t\in[a,b]$ there exists $\alpha,\beta:[a,b]\to\mathbb{R}$ such that

$$h(t,x)=\alpha(t)x+\beta(t),\quad (t\in[a,b],x\in\mathbb{R}).$$

Since $\beta(t)=h(t,0)$, $t\in[a,b]$, $\alpha(t)=h(t,1)-\beta(t)$ and $h(\cdot,x)\in BV^W_{(p(\cdot),2)}[a,b]$ for each $x\in\mathbb{R}$ we obtain that $\alpha,\beta\in BV^W_{(p(\cdot),2)}[a,b]$.

Now we will give the definition of uniformly bounded mapping introduced by J. Matkowski in [24].

Definition 9 *Let X and Y be two metric (or normed) spaces. A mapping $H:X\to Y$ is uniformly bounded if, for any $t>0$ there exists a nonnegative real number $\gamma(t)$ such that for any nonempty set $B\subset X$ we have*

$$diam(B)\le t\to diamH(B)\le\gamma(t).$$

With the same kind of argument than in ([23], Theorem 7), we can see that any uniformly bounded composition operator acting between general Lipschitz function normed space must be of the form (5.2):

Theorem 7 *Let $h:[a,b]\times\mathbb{R}\to\mathbb{R}$ and H the composition operator associated to h. Suppose that H maps $BV^W_{(p(\cdot),2)}[a,b]$ into itself and it is uniformly continuous, then there exists functions $\alpha,\beta\in BV^W_{(p(\cdot),2)}[a,b]$, such that*

$$h(t,x)=\alpha(t)x+\beta(t),\quad (t\in[a,b],x\in\mathbb{R}).$$

Proof. It follows as ([23], Theorem 7) by Theorem 6.

6. Absolutely Continuous Functions

We now define the analog of absolute *p*-continuous functions of order two in the framework of variable space.

Definition 10 *Given a function $p:[a,b]\to(1,\infty)$, by modulus of $p(\cdot)$-continuity of order two of a function $f:[a,b]\to\mathbb{R}$, we mean*

$$\omega_\delta^{(p(\cdot),2)}(f):=\sup_{\|\pi^*\|\le\delta}\sup_{\pi^*}\sum_{j=1}^{n-1}\left|\frac{f(t_{j+1})-f(t_j)}{t_{j+1}-t_j}-\frac{f(t_j)-f(t_{j-1})}{t_j-t_{j-1}}\right|^{p(x_{j-1})},$$

where the supremum is taken over all tagged partitions $\pi^*=\{a=t_0<t_1<\cdots<t_n=b\}$ of the interval $[a,b]$ together with a finite sequence of numbers $x_0,\cdots,x_{n-2}$ subject to the conditions $t_j\le x_j\le t_{j+1}$ for each j such that the norm of π^* is at most δ.

Lemma 3 *Let p be an admissible function. The modulus of $p(\cdot)$-continuity of order two is a sub-additive function.*

Proof. Let $f,g:[a,b]\to\mathbb{R}$.

$$\begin{aligned}
&\omega_\delta^{(p(\cdot),2)}(f+g)\\
&=\sup_{\|\pi\|\le\delta}\sup_{\pi}\sum_{j=1}^{n-1}\left|\frac{(f+g)(t_{j+1})-(f+g)(t_j)}{t_{j+1}-t_j}-\frac{(f+g)(t_j)-(f+g)(t_{j-1})}{t_j-t_{j-1}}\right|^{p(x_{j-1})}\\
&\le 2^{p^+-1}\sup_{\|\pi\|\le\delta}\sup_{\pi}\sum_{j=1}^{n-1}\left(\left|\frac{f(t_{j+1})-f(t_j)}{t_{j+1}-t_j}-\frac{f(t_j)-f(t_{j-1})}{t_j-t_{j-1}}\right|^{p(x_{j-1})}\right.\\
&\quad\left.+\left|\frac{g(t_{j+1})-g(t_j)}{t_{j+1}-t_j}-\frac{g(t_j)-g(t_{j-1})}{t_j-t_{j-1}}\right|^{p(x_{j-1})}\right)\\
&=2^{p^+-1}\left(\omega_\delta^{(p(\cdot),2)}(f)+\omega_\delta^{(p(\cdot),2)}(g)\right).
\end{aligned}$$

If $f\in BV^W_{(p(\cdot),2)}[a,b]$ and $\lim_{\delta\to0}\omega_\delta^{(p(\cdot),2)}(f)=0$, we say that f is absolutely $p(\cdot)$-continuous of order two, that is, $f\in C^{(p(\cdot),2)}[a,b]$.

Theorem 8 *Let* p *be an admissible function. Then* $C^{(p(\cdot),2)}[a,b]$ *is a closed subspace of* $BV^W_{(p(\cdot),2)}[a,b]$.

Proof. We take a sequence $\{f_n\}_{n\in\mathbb{N}}$ of functions in $C^{(p(\cdot),2)}[a,b]$ such that

$$s-\lim_{n\to\infty}f_n=f\in BV^W_{(p(\cdot),2)}[a,b]. \tag{6.1}$$

By the sub-additivity of $\omega_\delta^{(p(\cdot),2)}(f)$ we have that

$$\omega_\delta^{(p(\cdot),2)}(f)\le\omega_\delta^{(p(\cdot),2)}(f-f_n)+\omega_\delta^{(p(\cdot),2)}(f_n).$$

Moreover, since $V^W_{(p(\cdot),2)}(f)\ge\omega_\delta^{(p(\cdot),2)}(f)$ and $V^W_{(p(\cdot),2)}(2f)\precsim V^W_{(p(\cdot),2)}(f)$, using Proposition 2.3. in [21] and the strong limit (6.1) we have that, for each fixed δ, $\omega_\delta^{(p(\cdot),2)}(f-f_n)\to0$ when $n\to\infty$. Since $\omega_\delta^{(p(\cdot),2)}(f_n)\to0$ when $\delta\to0$ by hypothesis, we obtain that $\omega_\delta^{(p(\cdot),2)}(f)\to0$ when $\delta\to0$.

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