
A. A. Durmagambetov

L. N. Gumilyov Eurasian National University
Email: aset.durmagambet@gmail.com

The original online version of this article (Durmagambetov, A. A. (2016) The Riemann Hypothesis-Millennium Prize Problem. Advances in Pure Mathematics, 6, 915-920. 10.4236/apm.2016.612069) unfortunately contains a mistake. The author wishes to correct the errors in Theorem 2 of the result part.

2. Results

These are the well-known Abel’s results.

Theorem 1. Let the function $\phi(x)$ be limited on every finite interval, and $\frac{d\phi}{dx}(x)$ is continuous and limited on every finite interval then

$$\sum_{a \leq x \leq b} \phi(x) = \int_a^b \{x-[x]-1/2\} \frac{d\phi}{dx} + (a-[a]-1/2)\phi(a) - (b-[b]-1/2)\phi(b)$$

Corollary 1. Let the function $s > 1$, $\phi(x) = x^{-s}$, $a, b \in N$ then

$$\sum_{a \leq x \leq b} n^{-s} = \frac{b^{1-s} - a^{1-s}}{1-s} - \int_a^b \frac{(x-[x]-1/2)}{x^{s+1}} dx + \frac{1}{2} (b^{-s} - a^{-s})$$

Our goal is to use this theorem on the analogs of zeta functions. We are interested in the analytical properties of the following generalizations of zeta functions:

$$P(s) = \sum_{p \leq x} \frac{1}{p^s}, Q(s) = \sum_{p \leq x} \frac{1}{p(p-1)}$$

$$P_m(s) = \sum_{p \leq x} \frac{1}{p^s}, Q_m(s) = \sum_{p \leq x} \frac{1}{p(p-1)}$$
Let \( N \) be the set of all natural numbers and \( N^m_p = \{ n \in N, n \geq m, n \text{ prime number} \} \).

Below we will always let \( m > 3 \), this limitation is introduced only to simplify the calculations. Considering all the information above let us rewrite

\[ \zeta^m_p(s) = \sum_{n \in N^m_p} \frac{1}{n^s}. \]

For the function \( \zeta^m_p(s) = \zeta(s) - P^m(s) \), let us apply the results obtained by Muntz for the zeta function representation. With the help of the given definitions we formulate the analog of Muntz theorem.

**Lemma 1.** Let the function

\[ \delta(s) = P^m(s) - Q^m(s), \]

then

\[ \delta(s) = -sP^m(s + 1) + s^2O(P^m(s + 2)). \]

**Proof:** According to the theorem conditions we have

\[ \delta(s) = \sum_{p \in N^m_p} \left[ \frac{1}{p^s} - \frac{1}{(p-1)^s} \right] = \sum_{p \in N^m_p} \frac{1}{p^s} \left[ 1 - \frac{1}{(-1/p + 1)^s} \right] \]

\[ = -s \sum_{p \in N^m_p} \frac{1}{p^{s+1}} + s^2O(P^m(s + 2)). \]

**Lemma 2.** Let the function

\[ \gamma_1(s) = \sum_{p \in N^m_p} \int_{p-1}^{p} \frac{x}{x^s + 1} \, dx, \gamma_2(s) = -\sum_{p \in N^m_p} \int_{p-1}^{p} \frac{x}{x^s + 1} \, dx, \gamma_3(s) = -\sum_{p \in N^m_p} \int_{p-1}^{p} \frac{1/2}{x^s + 1} \, dx, \]

then

\[ \gamma_1(s) = \frac{1}{1-s} \sum_{p \in N^m_p} \left[ \frac{1}{p^{s-1}} - \frac{1}{(p-1)^{s-1}} \right] = \frac{\delta(s-1)}{1-s} \]

\[ \gamma_2(s) = -\frac{1}{s} \sum_{p \in N^m_p} \left[ \frac{p-1}{p^s} - \frac{p-1}{(p-1)^s} \right] = -\frac{\delta(s-1)}{s} \cdot \frac{P^m(s)}{s} \]

\[ \gamma_3(s) = -\frac{1}{2s} \sum_{p \in N^m_p} \left[ \frac{1}{p^s} - \frac{1}{(p-1)^s} \right] = -\frac{\delta(s)}{2s}. \]

\[ s[\gamma_1(s) + \gamma_2(s) + \gamma_3(s)] = s \left[ \frac{\delta(s-1)}{s-1} - \frac{\delta(s-1)}{s} + \frac{P^m(s)}{s} \cdot \frac{\delta(s)}{s} \right] \]

**Proof:** Follows from computing of integrals.

**Lemma 3.** Let the function
\[ \phi(x) = x^{-s}, s > 1, \ a, b, m - \text{prime numbers} \]
\[ (a, b) \cap \mathbb{N}_p^m = \emptyset, \ \{a, a \geq m\} = \mathbb{N}_p^m \text{ then} \]
\[ -\delta(s) - m^{-s} = \sum_{a, b \in \mathbb{N}_p^m} [(b-1)^{-s} - a^{-s}] \]  \hfill (16)
\[ \sum_{a, b, c \in \mathbb{N}_p^m} s \frac{(x - \lfloor x \rfloor - 1/2)}{x^{1+1}} dx = s \int_{m}^{\varphi(x)} \frac{1}{x^{1+1}} dx - s[\gamma_1(s) + \gamma_2(s) + \gamma_3(s)] \]  \hfill (17)

**PROOF:** Computing the sums, we have
\[ \sum_{a, b, c \in \mathbb{N}_p^m} [(b-1)^{-s} - a^{-s}] = -m^{-s} + \sum_{p \in \mathbb{N}_p} [(p-1)^{-s} - p^{-s}] = -m^{-s} - \delta(s-1) \]  \hfill (18)

**Theorem 2.** Let the function
\[ \phi(x) = x^{-s}, s > 1, \ a, b, m - \text{prime numbers} \]
\[ (a, b) \cap \mathbb{N}_p^m = \emptyset, \ \{a, a \geq m\} = \mathbb{N}_p^m \text{ then} \]
\[ s P^m(s) = \zeta(s) - \left[ -m^{-s} - s \int_{m}^{\varphi(x)} \frac{1}{x^{1+1}} dx - \delta(s) - m^{-s} + O\left(P^m(s+1)\right) \right] \]  \hfill (19)

**PROOF:** Using Corollary 1, we have
\[ \zeta_p^m(s) = \sum_{a, b, c \in \mathbb{N}_p^m, a < c < b} n^{-s} \]
\[ = \sum_{a, b \in \mathbb{N}_p^m} \frac{(b-1)^{-s} - a^{-s}}{1-s} - \sum_{a, b \in \mathbb{N}_p^m} s \int_{a}^{b} \frac{1}{x^{1+1}} dx \]
\[ + \frac{1}{2} \sum_{a, b, c \in \mathbb{N}_p^m} [(b-1)^{-s} - a^{-s}] \]  \hfill (20)
\[ \zeta_p^m(s) = -\delta(s-1) - m^{-s} - s \int_{m}^{\varphi(x)} \frac{1}{x^{1+1}} dx \]
\[ + s\left[\gamma_1(s) + \gamma_2(s) + \gamma_3(s)\right] - \delta(s) - m^{-s} \]
\[ = -\delta(s-1) - m^{-s} - s \int_{m}^{\varphi(x)} \frac{1}{x^{1+1}} dx \]
\[ + s\left[\frac{\delta(s-1)}{s-1} + \frac{P^m(s)}{s} - \frac{\delta(s)}{s} - m^{-s}\right] - \delta(s) - m^{-s} \]  \hfill (21)
\[ \zeta_p^m(s) = \delta(s-1) + P^m(s) - m^{-s} - s \int_{m}^{\varphi(x)} \frac{1}{x^{1+1}} dx - \delta(s) - m^{-s} \]  \hfill (22)
\[ \zeta(s) - P^m = (s-2)P^m(s) + P^m(s) - m^{-s} - s \int_{m}^{\varphi(x)} \frac{1}{x^{1+1}} dx \]
\[ - \delta(s) - m^{-s} + O\left(P^m(s+1)\right). \]  \hfill (23)
\[ \zeta(s) = s P^m(s) - m^{-s} - s \int_{m}^{\varphi(x)} \frac{1}{x^{1+1}} dx - \delta(s) - m^{-s} + O\left(P^m(s+1)\right). \]  \hfill (24)
\[ s^m P^m(s) = \zeta(s) - \left[ -m^{s-\frac{1}{2}} - s \sum_{n=1}^\infty \frac{\left( x - \left\lfloor x \right\rfloor - \frac{1}{2} \right)}{x^{s+1}} \right] \int \delta(x) - \left( -m^{s-\frac{1}{2}} + O\left( P^m(s+1) \right) \right) \] (25)

From the last equation we obtain the regularity of the function \( \zeta^m_p(s), P^m(s) \) as \( s \) satisfied \( 1/2 < \text{Re}(s) < 1 \).

**Theorem 3.** The Riemann’s function has nontrivial zeros only on the line \( \text{Re}(s) = 1/2 \).

**PROOF:** For \( R^2(s) = \sum_{m=2}^\infty P(ms)/m \), we have

\[ \left| R^2(s) \right| = \sum_{m=2}^\infty P(ms)/m \leq \sum_{m=2}^\infty \left| P(ms)/m \right| \leq C_\delta \sum_{m=2}^\infty \left| 2^{\delta m}/m \right| < C_\delta < \infty \] (26)

Applying the formula from the theorem 2

\[ \ln \left( \zeta(s) \right) = P(s) + \sum_{m=2}^\infty P(ms)/m = P(s) + R^2(s) = \zeta(s) - \zeta^m_p(s) - P_m(s) + R^2(s) \] (27)

estimating by the module

\[ \left| \ln(\zeta(s)) \right| \leq \left| \zeta(s) \right| + \left| \zeta^m_p(s) \right| + \left| R^2(s) \right| + \left| P_m(s) \right| \] (28)

Estimating the zeta function, potentiating, we obtain

\[ \left| \zeta(s) \right| \geq \exp \left[ -\left| \zeta(s) \right| - \left| \zeta^m_p(s) \right| - \left| R^2(s) \right| - \left| P_m(s) \right| \right] \] (29)

According to the theorem 1 \( \zeta(s) \) limited for \( z \) from the following multitude

\[ (s,|s| < R, |s| > 1 + \delta, \delta > 0) \] (30)

similarly, applying the theorem 2 for \( \zeta^m_p(s) \) we obtain its limitation in the same multitude. For the function \( \left| R^2(s) \right| \) we have a limitation for all \( z \), belonging to the half-plane \( \text{Re}(s) > 1/2 + 1/R \). similarly, applying the theorem 2 for \( \left| \zeta^m_p(s) \right| \) we obtain its limitation in the same multitude and finally we obtain:

\[ \left| \zeta(s) \right| \geq \exp \left[ -C_\gamma \right], \quad \text{Re}(s) > 1/2 + 1/R, |s| < R, |s| > 1 + \delta, \delta > 0 \] (31)

These estimations for \( \left| P(s) \right|, \left| R^2(s) \right|, \left| P_m(s) \right| \) prove that zeta function does not have zeros on the half-plane \( \text{Re}(s) > 1/2 + 1/R \) due to the integral representation (3) these results are projected on the half-plane \( \text{Re}(s) < 1/2 \) for the case of nontrivial zeros.

The Riemann’s hypothesis is proved.