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Locally Defined Operators and Locally Lipschitz Composition Operators in the Space $WBV_{p(\cdot)}([a,b])$

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Abstract

We give a neccesary and sufficient condition on a function $f:\mathbb{R}\to\mathbb{R}$ such that the composition operator (Nemytskij Operator) H defined by $Hf = f\circ h$ acts in the space $WBV_{p(\cdot)}([a,b])$ and satisfies a local Lipschitz condition. And, we prove that every locally defined operator mapping the space of continuous and bounded Wiener $p(\cdot)$-variation with variable exponent functions into itself is a Nemytskij composition operator.

Keywords

Generalized Variation, $p(\cdot)$-Variation in Wiener's Sense, Variable Exponent, Convergence, Helly's Theorem, Local Operator

1. Introduction

This paper lies in the field of variable exponent function spaces, exactly we will deal with the space $WBV_{p(\cdot)}([a,b])$ of bounded $p(\cdot)$-variation in Wiener's sense with variable exponent (see [1], [2]).

Variable exponent Lebesgue spaces appeared in the literature in 1931 in the paper by Orlicz [3]. He was interested in the study of function spaces that contain all measurable functions $u:\Omega\to\mathbb{R}$ such that

$$\rho(\lambda u) = \int_\Omega \varphi(\lambda|u(x)|)\,\mathrm{d}x,$$

for some $\lambda > 0$ and φ satisfying some natural assumptions, where Ω is an open set in $\mathbb{R}^n$. This space is denotated by L^{φ} and it is now called Orlicz space. However, we point out that in [3] the case $|u(x)|^{p(x)}$ corresponding to variable exponents is not included. In the 1950's, these problems were systematically studied by Nakano [4], who developed the theory of modular function spaces. Nakano explicitly mentioned variable exponent Lebesgue spaces as an example of more general spaces he considered, see Nakano [4] p. 284. In 1991, Kováčik and Rákosník [5] established several basic properties of spaces $L^{p(x)}$ and $W^{1,p(x)}$ with variable exponents. Their results were extended by Fan and Zhao [6] in the framework of Sobolev spaces $W^{m,p(x)}$.

With the emergence of nonlinear problems in applied sciences, standard Lebesgue and Sobolev spaces demostrated their limitations in applications. The class of nonlinear problems with variable exponents growth is a new research field and it reflects a new kind of physical phenomena.

It is well known that the class of nonlinear operator equations of various types has many useful applications in describing numerous problems of the real world. A number of equations which include a given operators have arisen in many branches of science such as the theory of optimal control, economics, biological, mathematical physics and engineering. Among nonlinear operators, there is a distinguished class called composition operators. Next we define such operators.

Definition 1.1. Given a function $h:\mathbb{R}\to\mathbb{R}$, the composition operator *H*, associated to a function *f* (autonomous case) maps each function $f:[a,b]\to\mathbb{R}$ into the composition function $Hf:[a,b]\to\mathbb{R}$, given by

$$Hf(t):=h(f(t)), \quad (t\in[a,b]). \tag{1.1}$$

More generally, given $h:[a,b]\times\mathbb{R}\to\mathbb{R}$, we consider the operator *H*, defined by

$$Hf(t):=h(t,f(t)), \quad (t\in[a,b]). \tag{1.2}$$

This operator is also called *superposition operator* or *susbtitution operator* or *Nemytskij operator*. The operator in the form (1.1) is usually called the (autonomous) composition operator and the one defined by (1.2) is called non-autonomos.

A rich source of related questions are the excellent books by J. Appell and P. P. Zabrejko [7] and J. Appell, J. Banas, N. Merentes [8].

E. P. Sobolevskij in 1984 [9] proved that the autonomous composition operator associate to $h:\mathbb{R}\to\mathbb{R}$ is locally Lipschitz in the space $Lip[a,b]$ if and only if the derivative h' exists and is locally Lipschitz. In recent articles J. Appell, N. Merentes, J. L. Sánchez [10], N. Merentes, S. Rivas, J. L. Sánchez [11] and O. Mejía, N. Merentes, B. Rzepka [12], obtained several results of the Sobolevskij type. According to the authors mentioned above the importance of these results lies in the fact that in most applications to many nonlinear problems it is sufficient to impose a local Lipschitz condition, instead of a global Lipschitz condition. In fact, they proved that Sobolevskij's result is valid in the spaces $BV_{\varphi}[a,b]$, $HBV[a,b]$, $RV_{\varphi}[a,b]$, $\Phi BV[a,b]$ and $\kappa\Phi BV[a,b]$.

In this paper, we obtained two main results. The organization of this paper is as follows. Section 2, we gather some notions and preliminary facts, and necessary background about the class of functions of bounded $p(\cdot)$-variation in Wiener's sense with variable exponent, also we expose some new properties of this space. In Section 3, we establish our first main result of the Sobolevskij type which is also valid in some spaces of functions of generalized bounded variations such as $WBV_{p(\cdot)}([a,b])$. In Section 4, we enunciate and prove our second main result related to the composition operator: If a locally defined operator K maps $CWBV_{p(\cdot)}(I)$ into $C(I)$ then it is composition operator.

2. Preliminaries

Throughout this paper, we use the following notation: Let a function $p:[a,b]\to(1,\infty)$ and we will denote by $\omega_{p(x_{ts})}(f,[a,b])=\sup\left\{|f(t)-f(s)|^{p(x_{ts})}:t,s\in[a,b]\right\}$ the diameter of the image $f([a,b])$ (or the oscillation of *f* on $[a,b]$), by x_{ts} a number between $[t,s]$ and $p^+:=\sup_{x\in[a,b]}p(x)$.

In 2013 R. Castillo, N. Merentes and H. Rafeiro [1] introduced the notion of bounded variation space in the Wiener sense with variable exponent on $[a,b]$ and present a result of compactness (Helly principle) in this space.

Definition 2.1 (See [1]). Given a function $p:[a,b]\to(1,\infty)$, a partition $\pi:a=t_0<t_1<\cdots<t_n=b$ of the interval $[a,b]$ and a function $f:[a,b]\to\mathbb{R}$. The nonnegative real number

$$V_{p(\cdot)}^W(f)=V_{p(\cdot)}^W(f,[a,b]):=\sup_{\pi^*}\sum_{j=1}^{n}\left|f(t_j)-f(t_{j-1})\right|^{p(x_{j-1})}$$

is called Wiener variation with variable exponent (or $p(\cdot)$-variation in Wiener's sense) of *f* on $[a,b]$ where π^* is a tagged partition of the interval $[a,b]$, *i.e.*, a partition of the interval $[a,b]$ together with a finite sequence of numbers $x_0,\cdots,x_{n-1}$ subject to the conditions that for each *j*, $t_j\le x_j\le t_{j+1}$.

In case that $V_{p(\cdot)}^W(f,[a,b])<\infty$, we say that *f* has bounded Wiener variation with variable exponent (or bounded $p(\cdot)$-variation in Wiener's sense) on $[a,b]$. The symbol $WBV_{p(\cdot)}([a,b])$ will denote the space of functions of bounded $p(\cdot)$-variation in Wiener's sense with variable exponent on $[a,b]$.

Definition 2.2. (Norm in $WBV_{p(\cdot)}([a,b])$) The functional $\|\cdot\|_{p(\cdot)}^W:WBV_{p(\cdot)}([a,b])\to\mathbb{R}$ defined by

$$\|f\|_{p(\cdot)}^W:=|f(a)|+\mu_{p(\cdot)}(f),\quad f\in WBV_{p(\cdot)}([a,b]) \tag{2.1}$$

where $\mu_{p(\cdot)}(f):=\inf\left\{\lambda>0:V_{p(\cdot)}^W\left(\frac{f}{\lambda}\right)\le1\right\}$ is a norm on $WBV_{p(\cdot)}([a,b])$.

Theorem 2.3 (See [1]). Every sequence in $WBV_{p(\cdot)}([a,b])$ has a subsequence convergent pointwise to a function $x\in WBV_{p(\cdot)}([a,b])$.

In 2015, O. Mejía, N. Merentes and J. L. Sánchez [2] showed the following properties of elements of $WBV_{p(\cdot)}([a,b])$ that allow us to get characterizations of them.

Lemma 2.4 (General properties of the $p(\cdot)$-variation). Let $f:[a,b]\to\mathbb{R}$ be an arbitrary map. We have

(P1) minimality: if $t,s\in[a,b]$, then

$$\left|f(t)-f(s)\right|^{p(x_{ts})}\le\omega_{p(x_{ts})}(f,[a,b])\le V_{p(\cdot)}^{W}(f,[a,b]).$$

(P2) monotonicity: if $a,t,s,b\in[a,b]$ and $a\le t\le s\le b$, then $V_{p(\cdot)}^{W}(f,[a,t])\le V_{p(\cdot)}^{W}(f,[a,s])$, $V_{p(\cdot)}^{W}(f,[s,b])\le V_{p(\cdot)}^{W}(f,[t,b])$ and $V_{p(\cdot)}^{W}(f,[t,s])\le V_{p(\cdot)}^{W}(f,[a,b])$.

(P3) semi-additivity: if $t\in[a,b]$, then

$$2^{1-p^{+}}V_{p(\cdot)}^{W}(f,[a,b])\le V_{p(\cdot)}^{W}(f,[a,t])+V_{p(\cdot)}^{W}(f,[t,b])\le V_{p(\cdot)}^{W}(f,[a,b]).$$

(P4) change of a variable: if $[c,d]\subset\mathbb{R}$ and $\varphi:[c,d]\to[a,b]$ is a (not necessarily strictly) monotone function, then $V_{p(\cdot)}^{W}(f,\varphi([c,d]))=V_{p(\cdot)}^{W}(f\circ\varphi,[c,d])$.

(P5) regularity: $V_{p(\cdot)}^{W}(f,[a,b])=\sup\left\{V_{p(\cdot)}^{W}(f,[s,t]);\,s,t\in[a,b],a\le b\right\}$.

The following structural theorem is taken from [2], this gives us a characterization of the members of $WBV_{p(\cdot)}([a,b])$.

Theorem 2.5 (see [2]). The map $f:[a,b]\to\mathbb{R}$ is of bounded $p(\cdot)$-variation if and only if there exists a bounded nondecreasing function $\varphi:[a,b]\to\mathbb{R}$ a Hölderian map $g:\varphi([a,b])\to\mathbb{R}$ of exponent $\gamma=1/p(\cdot)$ and $H(g)\le1$ such that $f=g\circ\varphi$ on $[a,b]$.

Given $f\in WBV_{p(\cdot)}([a,b])$, consider the $p(\cdot)$-variation function in Wiener's sense $V_{p(\cdot),f}^{W}:[a,b]\to\mathbb{R}$ defined by

$$V_{p(\cdot),f}^{W}(x):=V_{p(\cdot)}^{W}(f;[a,x]). \tag{2.2}$$

Proposition 2.6. Suppose that $f\in WBV_{p(\cdot)}([a,b])$ is continuous at some point $y_0\in[a,b]$; then, the function $V_{p(\cdot),f}^{W}$ (2.2) is also continuous at y_0.

Proof. Let $\varepsilon>0$ and suppose that $f:[a,b]\to\mathbb{R}$ is continuous function at y_0, without loss of generality we can assume that $y_0<y<b$. Consider the difference $V_{p(\cdot),f}^{W}(y)-V_{p(\cdot),f}^{W}(y_0)$. Choose partitions $P_{y_0}=\{a=t_0,t_1,\cdots,t_s=y_0\}$ and $P_y=\{a=t_0,t_1,\cdots,t_m=y\}$ such that

$$V_{p(\cdot)}^{W}(f;[y_0,b])<V_{p(\cdot)}^{W}(f,P_{y_0};[y_0,b])+\varepsilon.$$

Afterwards, we choose δ such that $\left|f(y)-f(y_0)\right|<\varepsilon$ for $0<y-y_0<\delta$ which is possible by the continuity of *f* at y_0. By definition of $V_{p(\cdot)}^{W}(f,[a,b])$ there exist a partition $\pi:a=t_0\le t_0\le\cdots\le t_n=b$ and $\varepsilon>0$ such that

$$V_{p(\cdot)}^{W}(f,[a,b])\le\sum_{j=1}^{n}\left|f(t_j)-f(t_{j-1})\right|^{p(x_{j-1})}+\varepsilon.$$

Then for these *y*, we have

$$\begin{aligned}&\sum_{j=1}^{m}\left|f(t_j)-f(t_{j-1})\right|^{p(x_{j-1})}-\sum_{i=1}^{s}\left|f(t_i)-f(t_{i-1})\right|^{p(x_{i-1})}\\&=\sum_{j=1}^{k}\left|f(t_j)-f(t_{j-1})\right|^{p(x_{j-1})}+\sum_{j=k+1}^{m}\left|f(t_j)-f(t_{j-1})\right|^{p(x_{j-1})}-\sum_{i=1}^{s}\left|f(t_i)-f(t_{i-1})\right|^{p(x_{i-1})}\end{aligned}$$

$$
\begin{aligned}
&\le \sum_{j=1}^{k}\left|f(t_j)-f(t_{j-1})\right|^{p(x_{j-1})}+\left|f(y)-f(t_k)\right|^{p(x_{y_k y})}\\
&\quad+\sum_{j=k+1}^{m}\left|f(t_j)-f(t_{j-1})\right|^{p(x_{j-1})}-\sum_{i=1}^{s}\left|f(t_i)-f(t_{i-1})\right|^{p(x_{i-1})}\\
&\le V_{p(\cdot)}^{W}(f;P_y)+\sum_{j=k+1}^{m}\left|f(t_j)-f(t_{j-1})\right|^{p(x_{j-1})}-\sum_{i=1}^{s}\left|f(t_i)-f(t_{i-1})\right|^{p(x_{i-1})}\\
&\le \sum_{k=1}^{m+n}\left|f(t_j)-f(t_{j-1})\right|^{p(x_{j-1})}+\varepsilon-\sum_{i=1}^{s}\left|f(t_i)-f(t_{i-1})\right|^{p(x_{i-1})}\\
&\le V_{p(\cdot)}^{W}(f,[y_0,b])+\varepsilon-V_{p(\cdot)}^{W}(f,[y,b])\\
&=\left|f(y)-f(y_0)\right|^{p(x_{y_0 y})}+\sum_{j=2}^{n}\left|f(t_j)-f(t_{j-1})\right|^{p(x_{j-1})}+2\varepsilon-V_{p(\cdot)}^{W}(f,[y,b])\\
&<\varepsilon^{p(x_{y_0 y})}+2\varepsilon\le 3\max\left\{\varepsilon^{p(x_{y_0 y})},\varepsilon\right\}=\varepsilon_1.
\end{aligned}
$$

Lemma 2.7. Let $f\in WBV_{p(\cdot)}([a,b])$. Then

$$V_{p(\cdot)}^{W}\left(\frac{f}{\|f\|_{p(\cdot)}^{W}}\right)\le 1.$$

Proof. Let π^* is a tagged partition of the interval $[a,b]$, take $\lambda>\|f\|_{p(\cdot)}^{W}$. Then

$$\sum_{j=1}^{\infty}\frac{\left|f(t_j)-f(t_{j-1})\right|^{p(x_j)}}{\lambda}\le V_{p(\cdot)}^{W}\left(\frac{f}{\lambda}\right)\le 1.$$

Thus

$$V_{p(\cdot)}^{W}\left(\frac{f}{\|f\|_{p(\cdot)}^{W}}\right)=\sup_{\pi^*}\sum_{j=1}^{n}\lim_{\lambda\to\|f\|_{p(\cdot)}^{W}}\left(\frac{\left|f(t_j)-f(t_{j-1})\right|^{p(x_j)}}{\lambda}\right)\le 1.$$

Proposition 2.8. Let $\{f_n\}\in WBV_{p(\cdot)}([a,b])$ be a sequence such that f_n converges to f almost everywhere, with $f\in WBV_{p(\cdot)}([a,b])$. Then

$$\|f\|_{p(\cdot)}^{W}\le\liminf_{n\to\infty}\|f_n\|_{p(\cdot)}^{W}$$

that is, the Luxemburg norm is lower semi-continuous on $WBV_{p(\cdot)}([a,b])$.

Proof. Let $\alpha\in\mathbb{R}$ such that $\alpha<V_{p(\cdot)}^{W}\left(\frac{f}{\lambda},[a,b]\right)$ for $\lambda>0$. By the Definition 2.1, for any $\beta\in\mathbb{R}$ with $\alpha<\beta<V_{p(\cdot)}^{W}\left(\frac{f}{\lambda},[a,b]\right)$ exist a tagged partition $\pi^*=\{t_i\}_{i=0}^{n}$ of $[a,b]$ such that

$$V_{p(\cdot)}^{W}\left(\frac{f}{\lambda},\pi^*\right)=\sum_{i=1}^{m}\left|\frac{1}{\lambda}\left(f(t_i)-f(t_{i-1})\right)\right|^{p(x_{i-1})}\ge\beta.$$

By the pointwise convergence of f_n to f exist $n_0\in\mathbb{N}$ such that

$$\left|f_n(t)-f(t)\right|\le\delta:=\frac{\lambda\left(\beta^{1/p(x_{i-1})}-\alpha^{1/p(x_{i-1})}\right)}{2m^{1/p(x_{i-1})}}$$

for all $n \ge n_0$ and $t \in \pi^*$, $t_i \le x_i \le t_{i+1}, i = 0,1,\cdots,m-1$. And by the Minkowski's inequality, we get

$$\beta \le V_{p(\cdot)}^W\left(\frac{f}{\lambda},\pi^*\right) = \sum_{i=1}^m \left|\frac{1}{\lambda}\left(f(t_i) - f(t_{i-1})\right)\right|^{p(x_{i-1})}$$

$$= \sum_{i=1}^m \left[\left|\frac{1}{\lambda}\left[\left(f(t_i) - f_n(t_{i-1})\right) - \left(f_n(t_i) - f_n(t_{i-1})\right) + \left(f_n(t_i) - f(t_{i-1})\right)\right]\right|\right]^{p(x_{i-1})}$$

$$\le \sum_{i=1}^m \left[\left|\frac{1}{\lambda}\left(f(t_i) - f_n(t_{i-1})\right)\right| + \left|\frac{1}{\lambda}\left(f_n(t_i) - f_n(t_{i-1})\right)\right| + \left|\frac{1}{\lambda}\left(f_n(t_i) - f(t_{i-1})\right)\right|\right]^{p(x_{i-1})}$$

$$\le \left[\left(\sum_{i=1}^m \left|\frac{1}{\lambda}\left(f(t_i) - f_n(t_{i-1})\right)\right|^{p(x_{i-1})}\right)^{\frac{1}{p(x_{i-1})}} + \left(\sum_{i=1}^m \left|\frac{1}{\lambda}\left(f_n(t_i) - f_n(t_{i-1})\right)\right|^{p(x_{i-1})}\right)^{\frac{1}{p(x_{i-1})}}\right.$$

$$\left. + \left(\sum_{i=1}^m \left|\frac{1}{\lambda}\left(f_n(t_{i-1}) - f(t_{i-1})\right)\right|^{p(x_{i-1})}\right)^{\frac{1}{p(x_{i-1})}}\right]^{p(x_{i-1})}$$

$$\le \left[\left(\sum_{i=1}^m \left|\frac{1}{\lambda}\left(\frac{\lambda\left(\beta^{1/p(x_{i-1})} - \alpha^{1/p(x_{i-1})}\right)}{2m^{1/p(x_{i-1})}}\right)\right|^{p(x_{i-1})}\right)^{\frac{1}{p(x_{i-1})}} + \left(\sum_{i=1}^m \left|\frac{1}{\lambda}\left(f_n(t_i) - f_n(t_{i-1})\right)\right|^{p(x_{i-1})}\right)^{\frac{1}{p(x_{i-1})}}\right.$$

$$\left. + \left(\sum_{i=1}^m \left|\frac{1}{\lambda}\left(\frac{\lambda\left(\beta^{1/p(x_{i-1})} - \alpha^{1/p(x_{i-1})}\right)}{2m^{1/p(x_{i-1})}}\right)\right|^{p(x_{i-1})}\right)^{\frac{1}{p(x_{i-1})}}\right]^{p(x_{i-1})}$$

$$\le \left[\beta^{\frac{1}{p(x_{i-1})}} - \alpha^{\frac{1}{p(x_{i-1})}} + \left(V_{p(\cdot)}^W\left(\frac{f_n}{\lambda},\pi^*\right)\right)^{\frac{1}{p(x_{i-1})}}\right]^{p(x_{i-1})}, \quad \text{for all } n \ge N_0$$

therefore

$$\alpha \le V_{p(\cdot)}^W\left(\frac{f_n}{\lambda},\pi^*\right) \le V_{p(\cdot)}^W\left(\frac{f_n}{\lambda},I\right), \quad \text{for all } n \ge N_0$$

hence

$$\alpha \le \inf_{n \ge N_0} V_{p(\cdot)}^W\left(\frac{f_n}{\lambda},I\right),$$

that is,

$$\alpha \le \liminf_{n\to\infty} V_{p(\cdot)}^W\left(\frac{f_n}{\lambda},I\right).$$

Passing the limit as α tends $V_{p(\cdot)}^W\left(\frac{f_n}{\lambda},I\right)$, we get that $V_{p(\cdot)}^W$ is sequentially lower semicontinuous, *i.e.*,

$$V_{p(\cdot)}^W\left(\frac{f}{\lambda}\right) \le \liminf_{n\to\infty} V_{p(\cdot)}^W\left(\frac{f_n}{\lambda}\right),$$

if $f_n \in \mathbb{R}^I, n \in \mathbb{N}$ and $\lim_{n\to+\infty} f_n(x) = f(x)$ for all $x \in I$. By the Definition 2.1 it follows that

$$\|f\|_{p(\cdot)}^{W} \le \liminf_{n\to\infty} \|f_n\|_{p(\cdot)}^{W}$$

Lemma 2.9 (Invariance Principle). Let $h:\mathbb{R}\to\mathbb{R}$ be a function. Then, the composition operator (1.1) maps the space $WBV_{p(\cdot)}([a,b])$ into itself if and only if it maps, for any other choice of $c<d$, the space $WBV_{p(\cdot)}([a,b])$ into itself.

Proof. The function $v:[c,d]\to[a,b]$ defined by

$$v(t) = \frac{b-a}{d-c}(t-c)+a$$

is an affine homeomorphism with inverse the function $v^{-1}:[a,b]\to[c,d]$ defined by

$$v^{-1}(s) = \frac{d-c}{b-a}(s-a)+c,$$

such that: $v(c)=a$ and $v(d)=b$. Thus, $v:\mathcal{P}([c,d])\to\mathcal{P}([a,b])$ defined by

$$\begin{aligned} v(\pi) &= v(\{t_0,t_1,\cdots,t_m\}) = \{v(t_0),v(t_1),\cdots,v(t_m)\} \\ &= \{v(t_i)\}_{i=1}^{m} \in \mathcal{P}([a,b]), \end{aligned}$$

defines a 1-1 correspondence between all partitions $\mathcal{P}([c,d])$ of $[c,d]$ and all partitions $\mathcal{P}([a,b])$ of $[a,b]$ since v is strictly increasing. Consequently, for $u \in WBV_{p(\cdot)}([a,b])$, we obtain

$$\begin{aligned} v_{p(\cdot)}^{W}(u,[a,b]) &= \sup_{\mathcal{P}([a,b])} \sum_{i=1}^{m} \left| u(v(t_i)) - u(v(t_{i-1})) \right|^{p(x_{i-1})} \\ &= \sup_{\mathcal{P}([c,d])} \sum_{i=1}^{m} \left| (u\circ v)(t_i) - (u\circ v)(t_{i-1}) \right|^{p(x_{i-1})} \\ &= v_{p(\cdot)}^{W}(u\circ v,[c,d]). \end{aligned}$$

3. Locally Lipschitz Composition Operators

In this section, we expose one of the main results of this paper. We demonstrate that a result of the Sobolevskij type is also valid in the space $WBV_{p(\cdot)}([a,b])$ of bounded $p(\cdot)$-variation in the Wiener's sense with variable exponent.

Theorem 3.1. Let $h:\mathbb{R}\to\mathbb{R}$ be a function. If the composition operator H generated by h maps the space $WBV_{p(\cdot)}([a,b])$ into itself then H is locally Lipschitz if and only if h' exist and is locally Lipschitz in $\mathbb{R}$.

Proof. First let us assume that h' is locally Lipschitz in $\mathbb{R}$. For $r>0$ we denote by $K_1(r)$ the minimal Lipschitz constant of h' and by $K_2(r)$ the supremum of $|h'|$ on the bounded set

$$B_r := \bigcup_{a\le t\le b} \left\{ f(t) : \|f\|_{p(\cdot)}^{W} \le r \right\} \subset \mathbb{R}.$$

The finiteness of $K_2(r)$ implies that H satisfies a local Lipschitz condition in the norm $\|\cdot\|_\infty$ (norm of supremum), so we only have to prove a local Lipschitz condition for H with respect to the $p(\cdot)$-norm (2.1). We do this by applying twice the mean value theorem.

Fix $f,g \in WBV_{p(\cdot)}([a,b])$ with $\|f\|_{p(\cdot)}^{W}, \|g\|_{p(\cdot)}^{W} \le r$. Given a partition $\pi = \{t_0, t_1, \cdots, t_m\}$ of $[a,b]$, we split the index set {1, …, m} into a union $I \cup J$ of disjoint sets *I* and *J* by defining the following:

$j \in I$ if

$$\left|f(t_j)-g(t_j)\right|+\left|f(t_{j-1})-g(t_{j-1})\right| \le \left|f(t_j)-f(t_{j-1})\right|+\left|g(t_j)-g(t_{j-1})\right|,$$

$j \in J$ if

$$\left|f(t_j)-g(t_j)\right|+\left|f(t_{j-1})-g(t_{j-1})\right| > \left|f(t_j)-f(t_{j-1})\right|+\left|g(t_j)-g(t_{j-1})\right|.$$

By the classical mean value theorem we find α_j between $g(t_j)$ and $f(t_j)$ such that

$$Hf(t_j)-Hg(t_j)=h'(\alpha_j)\left[f(t_j)-g(t_j)\right],\ (j=1,\cdots,m).$$

Now, by definition of *I* we have

$$\left|\alpha_j-\alpha_{j-1}\right| \le 2\left|f(t_j)-f(t_{j-1})\right|+2\left|g(t_j)-g(t_{j-1})\right|,\ (j \in I).$$

Making a simple calculation

$$\begin{aligned}
&\left|Hf(t_j)-Hg(t_j)-Hf(t_{j-1})+Hg(t_{j-1})\right| \\
&=\left|h'(\alpha_j)\left[f(t_j)-g(t_j)\right]-h'(\alpha_{j-1})\left[f(t_{j-1})-g(t_{j-1})\right]\right| \\
&=\left|\left(h'(\alpha_j)-h'(\alpha_{j-1})\right)\left[f(t_j)-g(t_j)\right]+h'(\alpha_{j-1})\left[f(t_j)-g(t_j)-f(t_{j-1})+g(t_{j-1})\right]\right| \\
&\le K_1(r)\left|\alpha_j-\alpha_{j-1}\right|\|f-g\|_\infty+K_2(r)\left|f(t_j)-g(t_j)-f(t_{j-1})+g(t_{j-1})\right| \\
&\le\left[2K_1(r)\|f-g\|_\infty+K_2(r)\right]\left|f(t_j)-g(t_j)-f(t_{j-1})+g(t_{j-1})\right| \\
&=K_3(r)\left|f(t_j)-g(t_j)-f(t_{j-1})+g(t_{j-1})\right|.
\end{aligned}$$

Since $p:[a,b]\to(1,\infty)$ and adding on $j \in I$ we get that

$$\begin{aligned}
&\sum_{j\in I}\left|Hf(t_j)-Hg(t_j)-Hf(t_{j-1})+Hg(t_{j-1})\right|^{p(x_{j-1})} \\
&\le \sum_{j\in I}K_3(r)^{p(x_{j-1})}\left|f(t_j)-g(t_j)-f(t_{j-1})+g(t_{j-1})\right|^{p(x_{j-1})} \\
&\le K_4(r)\sum_{j\in I}\left|(f-g)(t_j)-(f-g)(t_{j-1})\right|^{p(x_{j-1})} \\
&\le K_4(r)V_{p(\cdot)}^{W}(f-g) \\
&\le K_4(r)\|f-g\|_{p(\cdot)}^{W}.
\end{aligned}$$

Again by the mean value theorem we find β_j between $f(t_j)$ and $f(t_{j-1})$ and γ_j between $g(t_j)$ and $g(t_{j-1})$ such that

$$Hf(t_j)-Hf(t_j)=h'(\beta_j)\left[f(t_j)-f(t_j)\right]\quad(j=1,2,\cdots,m)$$

and

$$Hg\left(t_j\right)-Hg\left(t_j\right)=h'\left(\gamma_j\right)\left[g\left(t_j\right)-g\left(t_j\right)\right]\quad\left(j=1,2,\cdots,m\right).$$

By definition of J we have

$$\left|\beta_j-\gamma_j\right|\le 2\left|f\left(t_j\right)-g\left(t_j\right)\right|+2\left|f\left(t_{j-1}\right)-g\left(t_{j-1}\right)\right|.$$

Again a simple calculation shows that

$$\begin{aligned}
&\left|Hf\left(t_j\right)-Hg\left(t_j\right)-Hf\left(t_j\right)+Hg\left(t_j\right)\right|\\
&=\left|h'\left(\beta_j\right)\left[f\left(t_j\right)-f\left(t_{j-1}\right)\right]-h'\left(\gamma_j\right)\left[g\left(t_j\right)-g\left(t_{j-1}\right)\right]\right|\\
&=\left|\left(h'\left(\beta_j\right)-h'\left(\gamma_j\right)\right)\left[f\left(t_j\right)-f\left(t_{j-1}\right)\right]-h'\left(\alpha_{j-1}\right)\left[f\left(t_j\right)-f\left(t_{j-1}\right)-g\left(t_j\right)+g\left(t_{j-1}\right)\right]\right|\\
&\le K_5\left(r\right)\left|\beta_j-\gamma_j\right|\left|f\left(t_j\right)-f\left(t_{j-1}\right)\right|+K_6\left(r\right)\left|f\left(t_j\right)-g\left(t_j\right)-f\left(t_{j-1}\right)+g\left(t_{j-1}\right)\right|\\
&\le\left[4K_5\left(r\right)\|f\|_\infty+K_6\left(r\right)\right]\left|f\left(t_j\right)-g\left(t_j\right)-f\left(t_{j-1}\right)+g\left(t_{j-1}\right)\right|\\
&=K_7\left(r\right)\left|f\left(t_j\right)-g\left(t_j\right)-f\left(t_{j-1}\right)+g\left(t_{j-1}\right)\right|.
\end{aligned}$$

Since $p:\left[a,b\right]\to\left(1,\infty\right)$ and adding on $j\in J$ we get that

$$\begin{aligned}
&\sum_{j\in I}\left|Hf\left(t_j\right)-Hg\left(t_j\right)-Hf\left(t_j\right)+Hg\left(t_j\right)\right|^{p\left(x_{j-1}\right)}\\
&=\sum_{j\in I}K_7\left(r\right)^{p\left(x_{j-1}\right)}\left|f\left(t_j\right)-g\left(t_j\right)-f\left(t_{j-1}\right)+g\left(t_{j-1}\right)\right|^{p\left(x_{j-1}\right)}\\
&=K_8\left(r\right)\sum_{j\in I}\left|f\left(t_j\right)-g\left(t_j\right)-f\left(t_{j-1}\right)+g\left(t_{j-1}\right)\right|^{p\left(x_{j-1}\right)}\\
&=K_8\left(r\right)V_{p(\cdot)}^W\left(f-g\right)\\
&=K_8\left(r\right)\|f-g\|_{p(\cdot)}^W.
\end{aligned}$$

Summing up both partial sums and observing that $K_4\left(r\right)$ and $K_8\left(r\right)$ do not depend on the partition π we conclude that

$$V_{p(\cdot)}^W\left(\frac{Hu-Hv}{\left(K_4\left(r\right)+K_8\left(r\right)\right)\|u-v\|_{p(\cdot)}^W}\right)\le 1$$

which proves the assertion.

Conversely, suppose that H satisfies a Lipschitz condition. By assumption, the constant

$$K\left(r\right):=\sup\left\{\frac{\|Hu-Hv\|_{p(\cdot)}^W}{\|u-v\|_{p(\cdot)}^W}:u,v\in WBV_{p(\cdot)}\left(\left[a,b\right]\right),\|u\|_{p(\cdot)}^W,\|v\|_{p(\cdot)}^W\le r,u\ne v\right\}\quad(3.1)$$

is finite for each $r>0$. Considering, in particular, both functions u and v in (3.1) constant, we see that

$$\left|h\left(u\right)-h\left(v\right)\right|\le K\left(r\right)\left|u-v\right|\quad\left(u,v\in\mathbb{R},\left|u\right|,\left|v\right|\le r\right).$$

This shows that h is locally Lipschitz, and so the derivative h' exists almost everywhere in $\mathbb{R}$. It remains to prove that h' exists everywhere in $\mathbb{R}$ and is locally Lipschitz. For the proof of the first claim we show that h' exists in any closed interval

$I=[a,b]$.

Given $r>0$, consider $z\in WBV_{p(\cdot)}([a,b])$ with $\|z\|_{p(\cdot)}^{W}\le\frac{r}{2}$. Let $\{\alpha_n\}_{n=1}^{\infty}$ be a decreasing sequence of positive real numbers converging to 0; without loss of generality, we may assume that $\alpha_n\le\frac{r}{2}$ for all $n\in\mathbb{N}$. Define a sequence of functions $h_{\alpha_n,z}:[a,b]\to\mathbb{R}$ by

$$h_{\alpha_n,z}(t)=\frac{h(z(t)+\alpha_n)-h(z(t))}{\alpha_n}\qquad (t\in[a,b]). \tag{3.2}$$

Since the composition operator H associate to h acts in the space $WBV_{p(\cdot)}([a,b])$, by assumption, the functions $h_{\alpha_n,z}$ given by (3.2) belong to $WBV_{p(\cdot)}([a,b])$.

Now, we show that the sequences $\{h_{\alpha_n,z}\}_{n=1}^{\infty}$ have uniformly bounded $p(\cdot)$-variation in Wiener's sense for all $z\in WBV_{p(\cdot)}([a,b])$ with $\|z\|_{p(\cdot)}^{W}\le\frac{r}{2}$. In fact, let $\pi=\{t_0,t_1,\cdots,t_m\}$ be a partition of the interval of $[a,b]$. For each $n\in\mathbb{N}$ define functions u_n and v by

$$u_n(t)=z(t)+\alpha_n,\quad v(t)=z(t)\quad (t\in[a,b]). \tag{3.3}$$

Then, $\|u_n\|_{p(\cdot)}^{W}\le r$ and $\|v\|_{p(\cdot)}^{W}\le r$. Furthermore, from Lemma 2.7, (3.2) and (3.3), we obtain the estimates

$$\begin{aligned}
&\sum_{j=1}^{m}\frac{\left|\alpha_n\left[h_{\alpha_n,z}(t_j)-h_{\alpha_n,z}(t_{j-1})\right]\right|^{p(x_j)}}{\|H_{u_n}-H_v\|_{p(\cdot)}^{W}}\\
&=\sum_{j=1}^{m}\frac{\left|h(z(t_j)+\alpha_n)-h(z(t_j))-h(z(t_{j-1})+\alpha_n)+h(z(t_{j-1}))\right|^{p(x_j)}}{\|H_{u_n}-H_v\|_{p(\cdot)}^{W}}\\
&=\sum_{j=1}^{m}\frac{\left|h(u_n(t_j))-h(v(t_j))-h(u_n(t_{j-1}))+h(v(t_{j-1}))\right|^{p(x_j)}}{\|H_{u_n}-H_v\|_{p(\cdot)}^{W}}\\
&=\sum_{j=1}^{m}\frac{|Hu_n-Hv|}{\|H_{u_n}-H_v\|_{p(\cdot)}^{W}}\le V_{p(\cdot)}^{W}\left(\frac{Hu_n-Hv}{\|H_{u_n}-H_v\|_{p(\cdot)}^{W}};[a,b]\right)\le 1.
\end{aligned}$$

Since the partition $\pi=\{t_0,t_1,\cdots,t_m\}$ was arbitrary, the inequality

$$V_{p(\cdot)}^{W}\left(\frac{\alpha_n h_{\alpha_n,z}}{\|H_{u_n}-H_v\|_{p(\cdot)}^{W}};[a,b]\right)\le 1$$

holds for every $n\in\mathbb{N}$ and each $z\in WBV_{p(\cdot)}([a,b])$ with $\|z\|_{p(\cdot)}^{W}\le\frac{r}{2}$. From Lemma 2.7, the definition of the function $h_{\alpha_n,z}$ in (3.2), and the definition of the functions u_n and v in (3.3), we further get

$$\left\|\alpha_n h_{\alpha_n,z}\right\|_{p(\cdot)}^W = \left\|h(z+\alpha_n)-h(z)\right\|_{p(\cdot)}^W = \left\|h(u_n)-h(v)\right\|_{p(\cdot)}^W$$
$$\le K(r)\left\|u_n - v\right\|_{p(\cdot)}^W = K(r)\alpha_n$$

hence $\left\|h_{\alpha_n,z}\right\|_{p(\cdot)}^W \le K(r)$. By Lemma 2.7, we conclude that

$$V_{p(\cdot)}^W\left(h_{\alpha_n,z}\right) \le K(r), \tag{3.4}$$

which shows that the sequence $\left\{h_{\alpha_n,z}\right\}_{n=1}^{\infty}$ satisfies the hypotheses of Theorem 2.3.

Theorem 2.3 ensures the existence of a pointwise convergent subsequence of $\left\{h_{\alpha_n,z}\right\}_{n=1}^{\infty}$; without loss of generality we assume that the whole sequence $\left\{h_{\alpha_n,z}\right\}_{n=1}^{\infty}$ converges pointwise on $[a,b]$ to some function $f \in WBV_{p(\cdot)}([a,b])$.

Now setting $z(t) := \lambda t$, where $\lambda > 0$ small enough such that $\|z\|_{p(\cdot)}^W \le \frac{r}{2}$. By (3.3) we note that

$$\begin{aligned} f(t) &= \lim_{n\to\infty} \frac{h(z(t)+\alpha_n)-h(z(t))}{\alpha_n} \\ &= \lim_{n\to\infty} \frac{h(\lambda t+\alpha_n)-h(\lambda t)}{\alpha_n} = \lambda h'(\lambda t) \end{aligned} \tag{3.5}$$

for almost all $t \in [a,b]$. Since the primitive of *f* and the function $t \mapsto h(\lambda t)$ are both absolutely continuous and have the same derivative on $[a,b]$, we conclude that they differ only by some constant on $[a,b]$, and so h' exists everywhere on $[a,b]$. From the invariance principle (Lemma 2.9), we deduce that the derivative h' of *h* exists on any interval, and so everywhere in $\mathbb{R}$.

It remains to prove that h' satisfies a local Lipschitz condition. Denoting by *F* the composition operator associate to the function f from (3.5), we claim that, for $z \in WBV_{p(\cdot)}([a,b])$ with $\|z\|_{p(\cdot)}^W \le \frac{r}{2}$, we have

$$\|Fz\|_{p(\cdot)}^W \le K(r), \tag{3.6}$$

where $K(r)$ is the Lipschitz constant from (3.1). In fact, by Theorem 2.3 we conclude that

$$\|f\|_{p(\cdot)}^W \le \liminf_{n\to\infty} \|h_n\|_{p(\cdot)}^W,$$

whenever the sequence $\{h_n\}_{n=1}^{\infty}$ of functions $h_n \in WBV_{p(\cdot)}([a,b])$ converges pointwise on $[a,b]$ to some function *f*. Combining this with (3.4) and the observation that $h_{\alpha_n,z}(a) \to g(a)$ as $n \to \infty$ we obtain (3.6). We conclude that the composition operator *F* maps the space $WBV_{p(\cdot)}([a,b])$ into itself, and so the corresponding function f is locally Lipschitz on $\mathbb{R}$. By (3.5), the same is true for the function h'.

4. Locally Defined Operators

In this section, we present our second main result, which is related to the notion of locally defined operator. We prove that every locally defined operator mapping the space of continuous and bounded $p(\cdot)$-variation in Wiener's sense functions into itself is a composition operator (Nemytskij operator).

Definition 4.1. Let $I=[a,b]$ be a closed interval of the real line $\mathbb{R}$, $a,b\in\mathbb{R}, a<b$ and let $\mathbb{X}=\mathbb{X}(I)$, $\mathbb{Y}=\mathbb{Y}(I)$ be function spaces $\varphi: I\to\mathbb{R}$. An operator $K:\mathbb{X}\to\mathbb{Y}$ is called a locally defined, or $(\mathbb{X},\mathbb{Y})$-local operator, briefly, a local operator, if for every open interval $J\subset\mathbb{R}$ and for all functions $f,g\in\mathbb{X}$, the implication

$$f|_{J\cap I}=g|_{J\cap I}\Rightarrow K(f)|_{J\cap I}=K(g)|_{J\cap I}$$

holds true.

Remark 4.1. For some pairs $(\mathbb{X},\mathbb{Y})$ of function spaces the forms of local operators $K:\mathbb{X}\to\mathbb{Y}$ (or their representation theorems) have been established. For instance in [13] it was done is the case when $\mathbb{X}=C^n(I)$ and $\mathbb{Y}=C(I)$ or $\mathbb{Y}=C^1(I)$, in [14]-[16] in the case when $\mathbb{X}$ and $\mathbb{Y}$ are the spaces of n-times (k-times, respectively) Whitney differentiable functions, in [17], [18] in the case when $\mathbb{X}$ is the space of Hölder functions and $\mathbb{Y}=C(I)$, in [19] for continuous and monotone functions, in [20] in the case when $\mathbb{X}=CW_\varphi(I)$ for functions of bounded φ-variation in the sense of Wiener and $\mathbb{Y}=C(I)$ and in [21] in the case when $\mathbb{X}=RV_p(I)$ for functions of bounded Riesz-variation and $\mathbb{Y}=C(I)$.

Definition 4.2. (See [13]) An operator $K:\mathbb{X}\to\mathbb{Y}$ is said to be

1) left-hand defined, if and only if for every $s_0\in I$ and for every two functions $f,g\in\mathbb{X}$,

$$f|_{(-\infty,s_0)\cap I}=g|_{(-\infty,s_0)\cap I}\Rightarrow K(f)|_{(-\infty,s_0)\cap I}=K(g)|_{(-\infty,s_0)\cap I}.$$

2) right-hand defined, if and only if for every $s_0\in I$ and for every two functions $f,g\in\mathbb{X}$,

$$f|_{I\cap(-\infty,s_0)}=g|_{I\cap(-\infty,s_0)}\Rightarrow K(f)|_{I\cap(-\infty,s_0)}=K(g)|_{I\cap(-\infty,s_0)}.$$

From now on, let $CWBV_{p(\cdot)}(I)=WBV_{p(\cdot)}(I)\cap C(I)$, where $C(I)$ stands for the space of continuous functions defined on *I*. We begin this section with some definitions.

Theorem 4.3. (See [13]) The operator $K:\mathbb{X}\to\mathbb{Y}$ is locally defined if and only if it is left and right defined operator.

The locally defined operators have been the subject of intensive research and many applications of then can be found in the literature (See, for instance [22], [23] and the references therein).

Theorem 4.4. Let $p:[a,b]\to(1,\infty)$. If a locally defined operator K maps $CWBV_{p(\cdot)}(I)$ into $C(I)$ then there exist a unique function $h:I\times\mathbb{R}\to\mathbb{R}$ such that, for all $f\in CWBV_{p(\cdot)}(I)$,

$$K(f)(t)=h(t,f(t)),\quad t\in I.$$

Proof. We begin by showing that for every $f,g\in CWBV_{p(\cdot)}(I)$ and for every $s_0\in\operatorname{int}(I)$ the condition

$$f(s_0)=g(s_0) \tag{4.1}$$

implies that

$$K(f)(s_0) = K(g)(s_0).$$

To this end choose arbitrary $s_0 \in \operatorname{int}(I)$ and take an arbitrary pair of functions $f, g \in CWBV_{p(\cdot)}(I)$ which fulfil (4.1). The function $\gamma : I \to \mathbb{R}$ defined by

$$\gamma(t) = \begin{cases} f(t) & \text{for } t \in [a, x_0]; \\ g(t) & \text{for } t \in (x_0, b] \end{cases}$$

belongs to $CWBV_{p(\cdot)}(I)$. Indeed, define the functions $f_1, g_1 : I \to \mathbb{R}$ by

$$f_1(t) = \begin{cases} f(t) - f(x_0) & \text{for } t \in [a, x_0]; \\ 0 & \text{for } t \in (x_0, b] \end{cases}$$

and

$$g_1(t) = \begin{cases} 0 & \text{for } t \in [a, x_0]; \\ g(t) - g(x_0) & \text{for } t \in (x_0, b] \end{cases}$$

Since $f, g \in CWBV_{p(\cdot)}(I)$, f, g are continuous in $I, V_{p(\cdot)}^W(f) < \infty$ and $V_{p(\cdot)}^W(g) < \infty$. Let $\pi = \{t_i\}_{i=0}^m$ be a partition of I such that $t_{\ell-1} \le s_0 < t_\ell$ for some $1 \le \ell \le m$. Then

$$\sum_{i=1}^m \left| f_1(t_i) - f_1(t_{i-1}) \right|^{p(x_{i-1})} = \sum_{i=1}^{\ell-1} \left| f(t_i) - f(t_{i-1}) \right|^{p(x_{i-1})} + \left| f(s_0) - f(t_{\ell-1}) \right|^{p(x_k)} \le V_{p(\cdot)}^W(f).$$

Hence $V_{p(\cdot)}^W(f_1) < \infty$. By a similar reasoning, we have $V_{p(\cdot)}^W(g_1) < \infty$. Finally $f_1 + g_1 \in CWBV_{p(\cdot)}(I)$, as $CWBV_{p(\cdot)}(I)$ is a linear space. Thus

$$V_p^R(f_1 + g_1) < \infty. \tag{4.2}$$

Since, for all $t, t' \in I$

$$(f_1 + g_1)(t) - (f_1 + g_1)(t') = \gamma(t) - \gamma(t'),$$

the condition (4.2) implies that $\gamma \in CWBV_{p(\cdot)}(I)$. As

$$f\big|_{(-\infty, s_0) \cap I} = \gamma\big|_{(-\infty, s_0) \cap I} \quad \text{and} \quad g\big|_{(s_0, \infty) \cap I} = \gamma\big|_{(s_0, \infty) \cap I}$$

according to Definition 4.2, we get

$$K(f)\big|_{(-\infty, s_0) \cap I} = K(\gamma)\big|_{(-\infty, s_0) \cap I} \quad \text{and} \quad K(g)\big|_{(s_0, \infty) \cap I} = K(\gamma)\big|_{(s_0, \infty) \cap I}.$$

Therefore, by the continuity of $K(f), K(g)$ and $K(\gamma)$ en s_0, we obtain

$$K(f)(s_0) = K(\gamma)(s_0) = K(g)(s_0).$$

Suppose now that s_0 is the left endpoint of the interval I (*i.e.*, $s_0 = a$). By the continuity of f and g at s_0, there exist a sequence $(t_n)_{n \in \mathbb{N}}$ such that: $s_0 < t_{n+1} < t_n, |t_n - s_0| < (b - s_0)/n, n \in \mathbb{N}$ and

$$\left| f(t_n) - f(s_0) \right| < \frac{1}{n^2}, \quad \left| g(t_n) - g(s_0) \right| < \frac{1}{n^2}, \; n \in \mathbb{N}. \tag{4.3}$$

The sequence of functions $\gamma_n : I \to \mathbb{R}, n \in \mathbb{N}$, defined by

$$\gamma_{2k}(t)=\begin{cases}\dfrac{f(s_{2k})-f(s_0)}{s_{2k}-s_0}(t-s_0)+f(s_0) & \text{for } t\in[s_0,s_{2k}];\\ \dfrac{g(s_{2i-1})-f(s_{2i})}{s_{2i-1}-s_{2i}}(t-s_{2i})+f(s_{2i}) & \text{for } t\in(s_{2i},s_{2i-1}], i\in\{1,\cdots,k\};\\ \dfrac{f(s_{2i})-g(s_{2i+1})}{s_{2i}-s_{2i+1}}(t-s_{2i+1})+g(s_{2i+1}) & \text{for } t\in(s_{2i+1},s_{2i}], i\in\{1,\cdots,k-1\};\\ g(s_1) & \text{for } t\in(s_1,b]\end{cases}$$

$$\gamma_{2k-1}(t)=\begin{cases}\dfrac{g(s_{2k-1})-g(s_0)}{s_{2k-1}-s_0}(t-s_0)+g(s_0) & \text{for } t\in[s_0,s_{2k-1}];\\ \dfrac{f(s_{2i-2})-g(s_{2i-1})}{s_{2i-2}-s_{2i-1}}(t-s_{2i-1})+g(s_{2i-1}) & \text{for } t\in(s_{2i-1},s_{2i-2}], i\in\{2,\cdots,k\};\\ \dfrac{g(s_{2i-3})-f(s_{2i-2})}{s_{2i-3}-s_{2i-2}}(t-s_{2i-2})+f(s_{2i-2}) & \text{for } t\in(s_{2i-2},s_{2i-3}], i\in\{2,\cdots,k-1\};\\ g(s_1) & \text{for } t\in(s_1,b]\end{cases}$$

for all $k\in\mathbb{N}$, belong to the space $WBV_{p(\cdot)}(I)$. Indeed, by the definition of $\gamma_{2k}, k\in\mathbb{N}$, the triangle inequality, (4.1) and (4.3), we have

$$\left|\gamma_{2k}(t_i)-\gamma_{2k}(s_0)\right|^{p(x_{i-1})}\le\left(\frac{2}{i^2}\right)^{p(x_{i-1})}$$

and

$$\left|\gamma_{2k}(t_i)-\gamma_{2k}(t_j)\right|^{p(x_{i-1})}\le\left|\gamma_{2k}(t_i)-\gamma_{2k}(s_0)\right|^{p(x_{i-1})}+\left|\gamma_{2k}(t_j)-\gamma_{2k}(s_0)\right|^{p(x_{i-1})}<\left(\frac{2}{i^2}\right)^{p(x_{i-1})}$$

for all $i,j\in\{1,\cdots,2k\}, i<j, k\in\mathbb{N}$. Therefore

$$\begin{aligned}&\sum_{i=1}^{2k}\left|\gamma_{2k}(t_j)-\gamma_{2k}(s_0)\right|^{p(x_{i-1})}\\ &\le\sum_{i=1}^{2k}\left(\frac{2}{i^2}\right)^{p(x_{i-1})}\le M\sum_{i=1}^{2k}\frac{1}{i^{2p(x_{i-1})}},\quad\left(1<M=2^{p(x_{i-1})}<+\infty\right),\end{aligned}$$

so

$$V_{p(\cdot)}^{W}(\gamma_{2k},I)\le M\sum_{i=1}^{2k}\frac{1}{i^{2p(x_{i-1})}},\ k\in\mathbb{N}.\tag{4.4}$$

Similar reasoning shows, that

$$V_{p(\cdot)}^{W}(\gamma_{2k-1},I)\le M\sum_{i=1}^{2k-1}\frac{1}{i^{2p(x_{i-1})}},\ k\in\mathbb{N}.\tag{4.5}$$

From (4.4) and (4.5), we obtain that $\gamma_n\in WBV_{p(\cdot)}(I)$ and

$$V_{p(\cdot)}^{W}(\gamma_n,I)\le M\sum_{i=1}^{n}\frac{1}{i^{2p(x_{i-1})}},\ k\in\mathbb{N}.\tag{4.6}$$

Let us observe that

$$\gamma_{2k-1}(s_0)=\gamma_{2k}(s_0)=f(s_0)=g(s_0),\ k\in\mathbb{N},\tag{4.7}$$

and for all $k,i\in\mathbb{N}$,

$$\gamma_{2k}(t_{2k})=f(t_{2k})=\gamma_{2k+i}(t_{2k}),\quad \gamma_{2k-1}(t_{2k-1})=g(t_{2k-1})=\gamma_{2k-1+i}(t_{2k-1}), \tag{4.8}$$

and for every $t\in I\setminus\{t_k:k\in\mathbb{N}\}$ there exist $n_0\in\mathbb{N}$ such that

$$\gamma_n(t)=\gamma_{n_0}(t),\quad n\ge n_0,\ n\in\mathbb{N}. \tag{4.9}$$

Put

$$\gamma(t):=\lim_{n\to+\infty}\gamma_n(t),\ \ t\in I.$$

From (4.7), (4.8) and (4.9) the function γ is well defined and

$$\begin{aligned}&\left|\gamma(t)-\gamma(t_{2k})\right|\le\left|g(t_{2k+1})-f(t_{2k})\right|\\&\le\left|g(t_{2k+1})-g(s_0)\right|+\left|f(t_{2k})-f(s_0)\right|,\ \text{ for all } t\in[t_{2k+1},t_{2k})\end{aligned} \tag{4.10}$$

and

$$\begin{aligned}&\left|\gamma(t)-\gamma(t_{2k})\right|\le\left|g(t_{2k-1})-f(t_{2k})\right|\\&\le\left|g(t_{2k-1})-g(s_0)\right|+\left|f(t_{2k})-f(s_0)\right|,\ \text{ for all } t\in[t_{2k},t_{2k-1}).\end{aligned} \tag{4.11}$$

To show that γ is continuous at γ, fix an $\epsilon>0$. By the continuity of *f* and *g* at s_0, there exist $n_0\in\mathbb{N}$ such that

$$\left|g(t_n)-g(s_0)\right|<\epsilon/3,\quad \left|f(t_n)-f(s_0)\right|<\epsilon/3,\quad n\in\mathbb{N},\ n\ge n_0. \tag{4.12}$$

Take an arbitrary $t\in\left(s_0,s_{n_0}\right)$. There exist $k\in\mathbb{N}$ such that $2k-1>n_0$ and either $t\in[t_{2k+1},t_{2k})$ or $t\in[t_{2k},t_{2k-1})$. Since, by triangle inequality and (4.7)

$$\begin{aligned}\left|\gamma(t)-\gamma(s_0)\right|&\le\left|\gamma(t)-\gamma(t_{2k})\right|+\left|\gamma(t_{2k})-\gamma(s_0)\right|\\&\le\left|\gamma(t)-\gamma(t_{2k})\right|+\left|f(t_{2k})-f(s_0)\right|,\end{aligned}$$

therefore, by (4.10) and (4.12)

$$\left|\gamma(t)-\gamma(s_0)\right|\le\left|g(t_{2k+1})-g(s_0)\right|+2\left|f(t_{2k})-f(s_0)\right|<\epsilon$$

in the case when $t\in[t_{2k+1},t_{2k})$, and by (4.11) and (4.12)

$$\left|\gamma(t)-\gamma(s_0)\right|\le\left|g(t_{2k-1})-g(s_0)\right|+2\left|f(t_{2k})-f(s_0)\right|<\epsilon$$

in the case when $t\in[t_{2k},t_{2k-1})$. As the continuity of γ at the remaining points is obvious, γ is continuous.

By the lower semicontinuity of $V^W_{p(\cdot)}$ (Proposition 2.8) and (4.6)

$$V^W_{p(\cdot)}(\gamma,I)\le\liminf_{n\to+\infty}M\sum_{i=1}^{n}\frac{1}{i^{2p(x_{i-1})}},$$

and the convergence of series $\sum_{i=1}^{\infty}\frac{1}{i^{2p(x_{i-1})}}$ implies that $\gamma\in WBV_{p(\cdot)}(I)$.

Thus there exist a function $\gamma\in CWBV_{p(\cdot)}(I)$ and sequence $(t_k)_{k\in\mathbb{N}}$ such that

$$\gamma(t_{2k-1})=g(t_{2k-1}),\quad \gamma(t_{2k})=f(t_{2k}),\ t_k\in I,\ k\in\mathbb{N}.$$

According to the first part of the proof, we have

$$K(\gamma)(t_{2k-1})=K(g)(t_{2k-1})\ \text{ and }\ K(\gamma)(t_{2k})=K(f)(t_{2k}),\ k\in\mathbb{N}.$$

Hence, by continuity of $K(\gamma), K(f)$ and $K(g)$ at s_0, letting $k \to \infty$, we get

$$K(f)(s_0) = K(\gamma)(s_0) = K(g)(s_0).$$

When s_0 is the right endpoint of *I*, the argument is similar.

To define the function $h: I \times \mathbb{R} \to \mathbb{R}$, fix arbitrarily an $y_0 \in \mathbb{R}$, let us define a function $P_{y_0}: I \to \mathbb{R}$ by

$$P_{y_0}(t) := y_0, \quad t \in I. \tag{4.13}$$

Of course P_{y_0}, as a constant function, belongs to $CWBV_{p(\cdot)}(I)$. For $s_0 \in I, y_0 \in \mathbb{R}$, put

$$h(s_0, y_0) := K(P_{y_0})(s_0).$$

Since, by (4.13), for all functions *f*,

$$f(s_0) = P_{f(s_0)}(s_0),$$

according to what has already been proved, we have

$$K(f)(s_0) = K(P_{f(s_0)})(s_0) = h(s_0, f(s_0)). \tag{4.14}$$

To prove the uniqueness of *h*, assume that $\bar{h}: I \times \mathbb{R} \to \mathbb{R}$ is such that

$$K(f)(t) = \bar{h}(t, f(t))$$

for all $f \in CWBV_{p(\cdot)}(I)$ and $t \in I$. To show that $h = \bar{h}$ let us fix arbitrarily $t \in I, y \in \mathbb{R}$ and take $f \in CWBV_{p(\cdot)}(I)$ with $f(t) = y$. From (4.14), we have

$$\bar{h}(t, y) = \bar{h}(t, f(t)) = K(f)(t) = h(t, f(t)) = h(t, y),$$

which proves the uniqueness of *h*.

5. Conclusion

In this paper, we get two important results. In Theorem 3.1, we show that the result of the Sobolevkij type is valid for the space of functions of bounded $p(\cdot)$-variation in Wiener's sense ($WBV_{p(\cdot)}([a,b])$) on $[a,b]$. And the Theorem 4.4, we show that if a locally defined operator K maps $CWBV_{p(\cdot)}(I)$ into $C(I)$ then it is composition operator.

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