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Sums of Squares of Polygonal Numbers

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Abstract

Polygonal numbers and sums of squares of primes are distinct fields of number theory. Here we consider sums of squares of consecutive (of order and rank) polygonal numbers. We try to express sums of squares of polygonal numbers of consecutive orders in matrix form. We also try to find the solution of a Diophantine equation $x^2 + y^2 = 2z^2 + \omega^2$ in terms of polygonal numbers.

Keywords

Polygonal Numbers, Sums of Squares, Triangular Numbers

1. Introduction [1]

Polygonal numbers have been meticulously studied since their very beginnings in ancient Greece. Numerous discoveries stemmed from these peculiar numbers can be seen in the basic fundamental group work of number theory today with finding such as pascal's triangle and Fermat triangular number theorem. It becomes a popular field of research for mathematicians. The concept of polygonal numbers was first defined by the Greek Mathematical hypsicles in the year 170 BC. If the polygonal numbers are divided successively into triangles it will ultimately end up with right triangle. The right triangles immediately remind us of Pythagorean property. This leads to the idea of finding sums of squares of consecutive polygonal numbers. In this paper we calculate sums of squares polygonal numbers of consecutive orders. We also calculate the sums of squares of m-gonal numbers of consecutive ranks. We analyze some properties of the above.

2. Polygonal Number

2.1. Definition

For $m = 3, 4, \cdots$

$$P(m,n) = (m-2)\frac{n(n-1)}{2} + n \quad \text{with} \quad n \in \mathbb{Z}$$

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are called generalized m-gonal numbers.

Also $P(m,n)=(m-2)T_{n-1}+n$

where

$T_{n-1}=\dfrac{n(n-1)}{2}$ with $n\in\mathbb{Z}$, a triangular number of rank $n-1$

Sums of Squares of Polygonal numbers of Consecutive Orders of Same Rank

2.2. Proposition

$$P^2(m,n)+P^2(m+1,n)$$
$$=\frac{n^2}{4}\Big\{\big[m(m+1)+(m-4)^2+(m-11)\big]n^2-\big[\big[m(m-1)+(m-5)^2+(m-12)\big]-2\big]2n$$
$$+\big[(m-2)(m-1)+(m-6)^2+(m-13)\big]\Big\}$$

Proof

$$P^2(m,n)+P^2(m+1,n)=\left[\frac{(m-2)n(n-1)}{2}+n\right]^2+\left[\frac{(m-1)n(n-1)}{2}+n\right]^2$$
$$=\frac{n^2(n-1)^2}{4}\left[(m-2)^2+(m-1)^2\right]+2n^2+\left[(m-2)+(m-1)\right]n^2(n-1)$$
$$=\frac{n^2(n-1)^2}{4}\left[m^2+(m^2-6m+5)\right]+2n^2+[2m-3]n^2(n-1)$$
$$=\frac{n^2}{4}\Big\{\big[m(m+1)+(m-4)^2+(m-11)\big]n^2-\big[\big[m(m-1)+(m-5)^2+(m-12)\big]-2\big]2n$$
$$+\big[(m-2)(m-1)+(m-6)^2+(m-13)\big]\Big\}$$

Sums of squares of Polygonal numbers of Consecutive Orders in Matrix Form [2]

Expressing the coefficients of n^4,n^3 and n^2 for 3 consecutive sums of squares in a 3×3 matrix the coefficients of sums of squares of any three consecutive terms of higher order can be obtained.

The coefficient matrix A_m of sums of squares polygonal numbers is

$$A_m=$$
$$\begin{bmatrix} \big[m(m+1)+(m-4)^2+(m-11)\big] & 2\big[\big[m(m-1)+(m-5)^2+(m-12)\big]-2\big] & \big[(m-2)(m-1)+(m-6)^2+(m-13)\big] \\ \big[(m+1)(m+2)+(m-3)^2+(m-10)\big] & 2\big[\big[m(m+1)+(m-4)^2+(m-11)\big]-2\big] & \big[(m-1)m+(m-5)^2+(m-12)\big] \\ \big[(m+2)(m+3)+(m-2)^2+(m-9)\big] & 2\big[\big[(m+1)(m+2)+(m-3)^2+(m-10)\big]-2\big] & \big[m(m+1)+(m-4)^2+(m-11)\big] \end{bmatrix}$$

for $m\geq 4$.

In general,

$$A_m=\begin{bmatrix} t_{m+2} & 2[t_{m+1}-2] & t_m \\ t_{m+3} & 2[t_{m+2}-2] & t_{m+1} \\ t_{m+4} & 2[t_{m+3}-2] & t_{m+2} \end{bmatrix}$$

where $t_m=\big[(m-2)(m-1)+(m-6)^2+(m-13)\big]$.

Recursive matrix form

Consider the initial matrix as the coefficients of n^4,n^3, and n^2 in $P^2(m,n)+P^2(m+1,n)$ for $m=4,5,6$

$$A_1 = \begin{bmatrix} 13 & -6 & 1 \\ 25 & -22 & 5 \\ 41 & -46 & 13 \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

The elements of next order A_2 depends on the previous order elements in A_1 except the elements of first row.

$$A_2 = \begin{bmatrix} 25 & -22 & 5 \\ 41 & -46 & 13 \\ 61 & -78 & 25 \end{bmatrix} = \begin{bmatrix} a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \\ 2a_{31} - a_{21} + 4 = a_{41} & 2\left(a_{31} - 2\right) = a_{42} & a_{21} = a_{43} \end{bmatrix}$$

The first two rows elements of A_3 are already occurred in A_2 and the third row elements are depend on the elements of first two row elements.

$$A_3 = \begin{bmatrix} a_{31} & a_{32} & a_{33} \\ a_{41} & a_{42} & a_{43} \\ 2a_{41} - a_{31} + 4 = a_{51} & 2\left(a_{41} - 2\right) = a_{52} & a_{31} = a_{53} \end{bmatrix}$$

In general, the matrix of order A_k depends on the previous order matrix elements A_{k-1}.

$$A_k = \begin{bmatrix} a_{(k-1)1} & a_{(k-1)2} & a_{(k-1)3} \\ a_{k1} & a_{k2} & a_{k3} \\ 2a_{k1} - a_{(k-1)1} + 4 & 2\left[a_{k1} - 2\right] & a_{(k-1)1} \end{bmatrix}$$

Sums of squares of Polygonal Numbers with Consecutive ranks *n*, *n*+1.

2.3. Proposition [3]

$$P^2(m,n) + P^2(m,n+1) = m^2 T_{n^2} - 2mT_{n-1} 16MAG_{n^2} + 2T_{n-1}^2 + 1$$

Proof

$$P^2(m,n) + P^2(m,n+1) = \left[\frac{(m-2)n(n-1)}{2} + n\right]^2 + \left[\frac{(m-2)(n+1)n}{2} + (n+1)\right]^2$$

$$= \frac{(m-2)^2 n^2}{4}\left[(n-1)^2 + (n+1)^2\right] + \left[n^2 + (n+1)^2\right] + (m-2)n\left[n(n-1) + (n+1)^2\right]$$

$$= \frac{(m-2)^2 n^2}{2}\left(n^2+1\right) + \left(2n^2 + 2n + 1\right) + (m-2)n\left(2n^2 + n + 1\right)$$

$$= m^2 T_{n^2} - mn(n-1)\left(2n^2+1\right) + 2n^3(n-1) - 2n^2(n-1) + 1$$

$$= m^2 T_{n^2} - mn(n-1)\left(2n^2+1\right) + 2n^2(n-1)^2 + 1$$

$$P^2(m,n) + P^2(m,n+1) = m^2 T_{n^2} - 2mT_{n-1} 16MAG_{n^2} + 2T_{n-1}^2 + 1.$$

2.4. Proposition

The Triple $P(m,n), P(m+k,n), P(m+2k,n)$ form the solution of the Diophantine equation $x^2 + y^2 = 2z^2 + \omega^2$, ω is a constant.

Proof

Consider the Diophantine equation $x^2 + y^2 = 2z^2 + \omega^2$

We try for the solution in polygonal numbers.
Take $x = P(m,n)$ and $y = P(m+2k,n)$

$$x^2 + y^2 = P^2(m,n) + P^2(m+2k,n)$$

$$= \left[(m-2)T_{n-1} + n\right]^2 + \left[((m+2k)-2)T_{n-1} + n\right]^2$$

$$= \left[(m-2)^2 + ((m+2k)-2)^2\right]T_{n-1}^2 + 2n^2 + 2\left[(m-2) + ((m+2k)-2)\right]nT_{n-1}$$

$$= \left[2m^2 + 4m(k-2) + 2k^2 + 2(k^2 - 4k + 4)\right]T_{n-1}^2 + 2n^2 + 4\left[m + (k-2)\right]nT_{n-1}$$

$$= \left[2\left[m^2 + 2m(k-2) + (k-2)^2\right] + 2k^2\right]T_{n-1}^2 + 2n^2 + 4\left[m + (k-2)\right]nT_{n-1}$$

$$= 2\left\{\left[((m+k)-2)T_{n-1} + n\right]^2 + k^2T_{n-1}^2\right\} = 2z^2 + \omega^2$$

$$\therefore x^2 + y^2 = 2z^2 + \omega^2$$

Taking $\omega = kT_{n-1}$ it is clear that the triple $P(m,n), P(m+k,n), P(m+2k,n)$ form the solution of the given equation in the order x, z, y.

2.5. Proposition

$$P^2(m,n) + P^2(m,n+2) = \frac{1}{2}\left[\left[n^2(m-2) + nm\right]^2 + n^2(16-gonal) + 2n(12-gonal)\right]$$

Proof

$$P^2(m,n) + P^2(m,n+2) = \left[\frac{(m-2)n(n-1)}{2} + n\right]^2 + \left[\frac{(m-2)(n+2)(n+1)}{2} + (n+2)\right]^2$$

$$= \frac{(m-2)^2}{4}\left[n^2(n-1)^2 + (n+2)^2(n+1)^2\right] + \left[n^2 + (n+2)^2\right] + (m-2)\left[n^2(n-1) + (n+1)(n+2)^2\right]$$

$$= \frac{(m-2)^2}{4}\left[2n^4 + 4n^3 + 14n^2 + 12n + 4\right] + \left[2n^2 + 4n + 4\right] + (m-2)\left[2n^3 + 4n^2 + 8n + 4\right]$$

$$= \frac{1}{4}\left[m^2(2n^4 + 4n^3 + 14n^2 + 12n + 4) - m(8n^4 + 8n^3 + 40n^2 + 16n) + (8n^4 + 32n^2)\right]$$

$$= \frac{1}{4}\left[2n^4(m^2 - 4m + 4) + 4n^3m(m-2) + 2n^2(7m^2 - 20m + 16) + 4n(3m^2 - 4m) + 4m^2\right]$$

$$= \frac{1}{2}\left[n^4(m-2)^2 + 2n^3m(m-2) + n^2\left[3m^2 + 4(m-1)(m-4)\right] + 2n\left[m^2 + 2m(m-2)\right] + 2m^2\right]$$

$$= \frac{1}{2}\left[\left[n^2(m-2) + nm\right]^2 + n^2\left[2m^2 + 4(m-1)(m-4)\right] + 2n\left[m^2 + 2m(m-2)\right] + 2m^2\right]$$

$$P^2(m,n) + P^2(m,n+2) = \frac{1}{2}\left[\left[n^2(m-2) + nm\right]^2 + n^2(16-gonal) + 2n(12-gonal)\right]$$

2.6. Proposition

$$\sum_{l=m-r}^{m+r+4} P^2(l,n) = (2(r+2)+1)\left[mT_n + n\right]^2 + 2\sum_{i=1}^{r+2} i^2T_n^2$$

Proof

$$\sum_{l=m-r}^{m+r+4} P^2(l,n) = P^2(m-r,n) + P^2(m-r+1,n) + P^2(m-r+2,n) + \cdots + P^2(m,n) + P^2(m+1,n)$$
$$+ P^2(m+2,n) + \cdots + P^2(m+r,n) + P^2(m+r+1,n) + \cdots + P^2(m+r+4,n)$$
$$= (m-r-2)^2 T_n^2 + n^2 + 2nT_n(m-r-2) + (m-r-1)^2 T_n^2 + n^2 + 2nT_n(m-r-1) + \cdots$$
$$+ (m-2)^2 T_n^2 + n^2 + 2nT_n(m-2) + (m-1)^2 T_n^2 + n^2 + 2nT_n(m-1) + \cdots$$
$$+ (m+r-1)^2 T_n^2 + n^2 + 2nT_n(m+r-1) + \cdots + (m+r+2)^2 T_n^2 + n^2 + 2nT_n(m+r+2)$$

$$\therefore \sum_{l=m-r}^{m+r+4} P^2(l,n) = (2(r+2)+1)[mT_n + n]^2 + 2\sum_{i=1}^{r+2} i^2 T_n^2.$$

3. Conclusion

It is observed that the polygonal numbers of consecutive ranks constitute the solution of the Diophantine equation $x^2 + y^2 = 2z^2 + \omega^2$ in the order (x, z, y). Also we try to find that sums of squares of polygonal numbers are general.

References

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[2] Kwak, J.H. and Hong, S. (2004) Linear Algebra. 2nd Edition, Birkhauser, xvi+390 pp.

[3] Gopalan, M.A. and Gnanam, A. (2009) Magna Numbers. *Indian Journal of Mathematical Sciences*, **5**, 33-34.

Notations

$P(m,n)$: Polygonal number of order *m* rank *n*.
T_{n-1} : Triangular Number.
$16MAG_n$: Magna Number order *n*.