Retraction Notice

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Editor guiding this retraction: Prof. Dexing Kong (EiC of APM)
Elementary Operations on L-R Fuzzy Number

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Abstract

The aim of this paper is to find the formula for the elementary operations on L-R fuzzy number. In this paper we suggest and describe addition, subtraction, multiplication and division of two L-R fuzzy numbers in a brief.

Keywords

Fuzzy Number, L-R Fuzzy Number, Membership Function

1. Introduction

A fuzzy set [1] on \( \mathbb{R} \), set of real numbers is called a fuzzy number [2] which satisfies at least the following three properties:

1) \( A \) must be a normal fuzzy set [3].
2) \( A^\alpha \) must be a closed interval for every \( \alpha \in (0,1] \).
3) The support [1] of \( A \), \( A^{\alpha} \) must be bounded.

The fundamental idea of the L-R representation of fuzzy numbers is to split the membership function \( \mu_{\tilde{A}}(x) \) of a fuzzy number \( \tilde{A} \) into two curves \( \mu_{L}(x) \) and \( \mu_{R}(x) \), left and right of the modal value \( \tilde{x} \).

The membership function \( \mu_{\tilde{A}}(x) \) can be expressed through parameterized reference functions or shape function \( L \) and \( R \) in the form

\[
\mu_{\tilde{A}}(x) = \begin{cases} 
\mu_L(x) = L \left( \frac{x - \tilde{x}}{\alpha} \right) & \text{for } x < \tilde{x} \\
\mu_R(x) = R \left( \frac{x - \tilde{x}}{\beta} \right) & \text{for } x \geq \tilde{x} 
\end{cases}
\]

(1)
where \( \bar{x}_i \) is the modal value of the membership function and \( \alpha_i \) and \( \beta_i \) are the spreads corresponding to the left-hand and right-hand curve of the membership function [4] respectively.

As an abbreviated notation, we can define an L-R fuzzy number \( \tilde{p}_i \) with the membership function \( \mu_{\tilde{p}_i}(x_i) \) in (1) by

\[
\tilde{p}_i = \langle \bar{x}_i, \alpha_i, \beta_i \rangle_{L,R}
\]

where the subscripts \( L \) and \( R \) specify the reference functions [5].

2. Operations on L-R Fuzzy Number

In this section, the formulas for the elementary operations (addition, subtraction, multiplication, division) [5] between L-R fuzzy numbers [5] will be presented.

2.1. Addition of L-R Fuzzy Number

Suppose two fuzzy numbers \( \tilde{p}_1 \) and \( \tilde{p}_2 \), represented as L-R fuzzy numbers of the form

\[
\tilde{p}_1 = \langle \bar{x}_1, \alpha_1, \beta_1 \rangle_{L,R} \quad \text{and} \quad \tilde{p}_2 = \langle \bar{x}_2, \alpha_2, \beta_2 \rangle_{L,R}
\]

The sum \( \tilde{q} = \tilde{p}_1 + \tilde{p}_2 \) is again an L-R fuzzy number of the form

\[
\tilde{q} = \langle \bar{z}, \alpha, \beta \rangle_{L,R}
\]

with the modal value

\[
\bar{z} = \bar{x}_1 + \bar{x}_2
\]

and the spreads

\[
\alpha = \alpha_1 + \alpha_2 \quad \text{and} \quad \beta = \beta_1 + \beta_2
\]

In short we can write

\[
\langle \bar{x}_1, \alpha_1, \beta_1 \rangle_{L,R} + \langle \bar{x}_2, \alpha_2, \beta_2 \rangle_{L,R} = \langle \bar{x}_1 + \bar{x}_2, \alpha_1 + \alpha_2, \beta_1 + \beta_2 \rangle_{L,R}
\]

The left-hand reference functions of both fuzzy numbers \( \tilde{p}_1 \) and \( \tilde{p}_2 \) have to be given by \( L \), and the right-hand reference functions by \( R \).

The formula of the L-R addition in (7) is motivated by the following ways:

We first consider the right-hand curves

\[
\mu_1(x_i) = R\left[ \frac{x_i - \bar{x}_1}{\beta_1} \right] \quad \text{and} \quad \mu_2(x_i) = R\left[ \frac{x_i - \bar{x}_2}{\beta_2} \right]
\]

The degree of membership \( \mu^* \in [0,1] \) is taken on for the argument values

\[
x_1^* = \bar{x}_1 + \beta_1 R^{-1}(\mu^*) \quad \text{and} \quad x_2^* = \bar{x}_2 + \beta_2 R^{-1}(\mu^*)
\]

This implies

\[
z^* = x_1^* + x_2^* = \bar{x}_1 + \bar{x}_2 + (\beta_1 + \beta_2) R^{-1}(\mu^*)
\]

and we obtain for the right-hand curve \( \mu(x) \) of the fuzzy number \( \tilde{q} \)

\[
\mu(z) = \mu^* = R\left[ \frac{z^* - \bar{z}}{\beta} \right] \quad \text{with} \quad \bar{z} = \bar{x}_1 + \bar{x}_2 \quad \text{and} \quad \beta = \beta_1 + \beta_2
\]

The same reasoning holds for the left-hand curves of \( \tilde{p}_1 \), \( \tilde{p}_2 \) and \( \tilde{q} \), and we get

\[
\mu_1(z) = L\left[ \frac{z - \bar{z}}{\alpha} \right] \quad \text{with} \quad \bar{z} = \bar{x}_1 + \bar{x}_2 \quad \text{and} \quad \alpha = \alpha_1 + \alpha_2
\]
2.2. Subtraction of L-R Fuzzy Number

Suppose two fuzzy numbers $\tilde{p}_1$ and $\tilde{p}_2$, represented as L-R fuzzy numbers of the form

$$\tilde{p}_1 = \{x_1, \alpha_1, \beta_1\}_{L,R}$$

and

$$\tilde{p}_2 = \{x_2, \alpha_2, \beta_2\}_{L,R}$$

(13)

The opposite $-\tilde{p}$ of the L-R fuzzy number is defined as

$$-\tilde{p} = -\{x, \alpha, \beta\}_{L,R} = -\{x, \beta, \alpha\}_{R,L}$$

(14)

Now by using (7) we can deduce the following formula for the subtraction $\tilde{q} = E_q(\tilde{p}_1, \tilde{p}_2) = \tilde{p}_1 + \tilde{p}_2$ of the L-R fuzzy numbers:

$$\{x_1, \alpha_1, \beta_1\}_{L,R} - \{x_2, \alpha_2, \beta_2\}_{L,R} = \{x_1 - x_2, \alpha_1 + \beta_2, \beta_1 + \alpha_2\}_{L,R}$$

(15)

2.3. Multiplication of L-R Fuzzy Number

Let us consider two positive fuzzy numbers $\tilde{p}_1$ and $\tilde{p}_2$ of the same L-R type given by the L-R representations

$$\tilde{p}_1 = \{x_1, \alpha_1, \beta_1\}_{L,R}$$

and

$$\tilde{p}_2 = \{x_2, \alpha_2, \beta_2\}_{L,R}$$

(16)

We can construct the right-hand curve $\mu_r(z)$ of the product $\tilde{q} = E_{pq}(\tilde{p}_1, \tilde{p}_2) = \tilde{p}_1 \cdot \tilde{p}_2$ on the basis of the right-hand curves

$$\mu_n(x_1) = R \left[ \frac{x_1 - x_2}{\beta_1} \right]$$

and

$$\mu_2(x_2) = R \left[ \frac{x_1 - x_2}{\beta_2} \right]$$

(17)

of L-R fuzzy numbers $\tilde{p}_1$ and $\tilde{p}_2$. In accordance with the deduction of the formula for the L-R addition, the degree of membership $\mu^* \in [0,1]$ is taken on for the argument values

$$x_1^* = x_1 + \beta_1 R^{-1}(\mu^*)$$

and

$$x_2^* = x_2 + \beta_2 R^{-1}(\mu^*)$$

(18)

This implies

$$z^* = x_1^* x_2^* = x_1 x_2 \left[ (x_1 \beta_2 + x_2 \beta_1) R^{-1}(\mu^*) + \beta_1 \beta_2 \left( R^{-1}(\mu^*) \right)^2 \right]$$

(19)

Two approximations have been proposed, which is referred to as tangent approximation and secant approximation in the following:

2.3.1. Tangent Approximation

Let $\alpha_1$ and $\alpha_2$ are small compared to $x_1$ and $x_2$ and $\mu^*$ is in the neighborhood of 1. Then we can neglect the quadratic term $\left( R^{-1}(\mu^*) \right)^2$ in (19) and we obtain for the right-hand curve $\mu_r(z)$ of the approximated product $\tilde{q}_t$, an expression of the form

$$\mu_r(z^*) = \mu^* = R \left[ \frac{z^* - x_1}{\beta} \right]$$

(20)

with $\bar{z} = x_1 x_2$ and $\beta = x_1 \beta_2 + x_2 \beta_1$

Using the same reasoning for the left-hand curves of $\tilde{p}_1$, $\tilde{p}_2$ and $\tilde{q}_t$, we deduce the following formula for the multiplication of L-R fuzzy numbers

$$\{x_1, \alpha_1, \beta_1\}_{L,R} \cdot \{x_2, \alpha_2, \beta_2\}_{L,R} \approx \{x_1 x_2, \alpha_1 \alpha_2 + \alpha_1 \beta_2 + \beta_1 \alpha_2 + \beta_1 \beta_2\}_{L,R}$$

(21)

2.3.2. Secant Approximation

If the spreads are not negligible compared to the modal values $x_1$ and $x_2$, the rough shape of the product $\tilde{q} = \tilde{p}_1 \cdot \tilde{p}_2$ can be estimated by approximating quadratic term $\left[ R^{-1}(\mu^*) \right]^2$ in (19) by the linear term $\left[ R^{-1}(\mu^*) \right]$. This gives the right-hand curve $\mu_r(z)$ of the approximated product $\tilde{q}_t$ in the form

$$\mu_r(z^*) = \mu^* = R \left[ \frac{z^* - x_1}{\beta} \right]$$

(22)

with $\bar{z} = x_1 x_2$ and $\beta = x_1 \beta_2 + x_2 \beta_1 + \beta_1 \beta_2$
With the same reasoning for the left-hand curves of \( \tilde{p}_1 \), \( \tilde{p}_2 \) and \( \tilde{q}_1 \), the overall formula for the multiplication of L-R fuzzy numbers results in

\[
\langle x_1, \alpha_1, \beta_1 \rangle_{L,R} \cdot \langle x_2, \alpha_2, \beta_2 \rangle_{L,R} \approx \langle x_1 x_2, \alpha_1 \alpha_2 - \alpha_1 \beta_2 + \beta_1 \alpha_2 - \beta_1 \beta_2 \rangle_{L,R}
\tag{23}
\]

### 2.4. Division of L-R Fuzzy Number

An appropriate formulation for the quotient \( \tilde{q} = E_2(\tilde{p}_1, \tilde{p}_2) = \tilde{p}_1 / \tilde{p}_2 \) of two L-R fuzzy numbers \( \tilde{p}_1 \) and \( \tilde{p}_2 \) can be obtained by reducing the division of the fuzzy numbers \( \tilde{p}_1 \) and \( \tilde{p}_2 \) to the multiplication of the dividend \( \tilde{p}_1 \) with the inverse \( \tilde{p}_2^{-1} = 1 / \tilde{p}_2 \) of the divisor \( \tilde{p}_2 \).

When we consider a fuzzy number \( \tilde{p} \) which is either positive or negative, i.e., \( 0 \not\in \text{supp}(\tilde{p}) \), given by the L-R representation

\[
\tilde{p} = \langle x, \alpha, \beta \rangle_{L,R}
\]

the tangent approximation \( (\tilde{p}^{-1})_t \) for the inverse \( \tilde{p}^{-1} \) is defined by

\[
(\tilde{p}^{-1})_t = \left\{ \frac{1}{x}, \frac{\beta}{x}, \frac{\alpha}{x} \right\}_{L,R} \approx \tilde{p}^{-1}
\]

and the secant approximation \( (\tilde{p}^{-1})_s \) by

\[
(\tilde{p}^{-1})_s = \left\{ \frac{1}{x}, \frac{\beta}{x(x + \beta)}, \frac{\alpha}{x(x + \alpha)} \right\}_{L,R} \approx \tilde{p}^{-1}
\]

Using the above mentioned identity \( \tilde{p}_1 / \tilde{p}_2 = \tilde{p}_1 \tilde{p}_2^{-1} \) as well as the approximation formulas for the multiplication of L-R fuzzy numbers on one side and those for the inverse of an L-R fuzzy number on the other, a number of different approximated L-R representations for the quotient \( \tilde{p}_1 / \tilde{p}_2 \) can be formulated.

### 3. Example

We consider two L-R fuzzy number

\[
\tilde{p}_1 = \langle 2,1,1 \rangle_{ij} \quad \text{and} \quad \tilde{p}_2 = \langle 4,2,4 \rangle_{ij}
\]

Then using Equation (7) we get

\[
\tilde{p}_1 + \tilde{p}_2 = \tilde{q} = \langle 2 + 4,1 + 2,1 + 4 \rangle_{ij} = \langle 6,3,5 \rangle_{ij}
\]

Also can be written in the form

\[
\tilde{q} = \begin{cases} 
0; & x \leq 3 \\
\frac{x - 3}{3}; & 3 < x < 6 \\
\frac{11 - x}{5}; & 6 \leq x < 11 \\
0; & x \geq 11
\end{cases} = \text{tfn}(6,3,5).
\]

Using (15) we get

\[
\tilde{p}_1 - \tilde{p}_2 = \tilde{q} = \langle 2 - 4,1 + 4,1 + 2 \rangle_{ij} = \langle -2,5,3 \rangle_{ij}
\]

Also can be written in the form

\[
\tilde{q} = \begin{cases} 
0; & x \leq -7 \\
\frac{x + 7}{5}; & -7 < x < -2 \\
\frac{1 - x}{3}; & -2 \leq x < 1 \\
0; & x \geq 1
\end{cases} = \text{tfn}(-2,5,3).
\]
If we use the tangent approximation the product $\tilde{q} = \tilde{p}_1 \tilde{p}_2$ is approximated by the triangular L-R fuzzy number

$$\tilde{q}_i = \langle 8,8,12 \rangle_{i,j} = \text{tnf} (8,8,12) = \begin{cases} 0; & x \leq 0 \\ \frac{x}{8}; & 0 < x < 8 \\ \frac{20-x}{12}; & 8 \leq x < 20 \\ 0; & x \geq 20 \end{cases}$$

Again in the case of secant approximation the result $\tilde{q} = \tilde{p}_1 \tilde{p}_2$ is approximated by

$$\tilde{q}_s = \langle 8,6,16 \rangle_{i,j} = \text{tnf} (8,6,16) = \begin{cases} 0; & x \leq 2 \\ \frac{x-2}{6}; & 2 < x < 8 \\ \frac{24-x}{16}; & 8 \leq x < 24 \\ 0; & x \geq 24 \end{cases}$$

If we use the tangent approximation the inverse $p_2^{-1}$ is approximated by the triangular L-R fuzzy number

$$\langle p_2^{-1} \rangle = \langle \frac{1}{4}, \frac{1}{8}, \frac{1}{4} \rangle_{i,j} = \langle \frac{1}{4}, \frac{1}{8}, \frac{1}{4} \rangle_{i,j}$$

Thus

$$\frac{\tilde{p}_1}{p_2} = \tilde{q}_{ss} = \langle 2,1,1 \rangle_{i,j} \cdot \langle \frac{1}{4}, \frac{1}{8}, \frac{1}{4} \rangle_{i,j} = \langle \frac{2}{3}, \frac{1}{2}, \frac{1}{3} \rangle = \text{tnf} \langle \frac{2}{3}, \frac{1}{2}, \frac{1}{3} \rangle$$

But if we use the secant approximation the inverse $p_2^{-1}$ is approximated by the triangular L-R fuzzy number

$$\langle p_2^{-1} \rangle = \langle \frac{1}{4}, \frac{1}{8}, \frac{1}{4} \rangle_{i,j} = \langle \frac{1}{4}, \frac{1}{8}, \frac{1}{4} \rangle_{i,j}$$

Thus

$$\frac{\tilde{p}_1}{p_2} = \tilde{q}_{ss} = \langle 2,1,1 \rangle_{i,j} \cdot \langle \frac{1}{4}, \frac{1}{8}, \frac{1}{4} \rangle_{i,j} = \langle \frac{2}{3}, \frac{1}{2}, \frac{1}{3} \rangle = \text{tnf} \langle \frac{2}{3}, \frac{1}{2}, \frac{1}{3} \rangle$$

4. Conclusion

In this paper we have presented exact calculation formulas for addition, subtraction, multiplication and division.
of two L-R fuzzy numbers. Finally we have taken two L-R fuzzy numbers as an example and obtained results of addition, subtraction, multiplication and division. We have reviewed some research papers with proper references.

References


