On the Structure of Infinitesimal Automorphisms of the Poisson-Lie Group $SU(2,\mathbb{R})$

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Received 6 January 2014; revised 6 February 2014; accepted 15 February 2014

Abstract

We study the Poisson-Lie structures on the group $SU(2,\mathbb{R})$. We calculate all Poisson-Lie structures on $SU(2,\mathbb{R})$ through the correspondence with Lie bialgebra structures on its Lie algebra $su(2,\mathbb{R})$. We show that all these structures are linearizable in the neighborhood of the unity of the group $SU(2,\mathbb{R})$. Finally, we show that the Lie algebra consisting of all infinitesimal automorphisms is strictly contained in the Lie algebra consisting of Hamiltonian vector fields.

Keywords

Poisson-Lie Structure, Lie Bialgebra, Hamiltonian, Poisson Automorphism, Linearization

1. Introduction

Let $G$ be a Lie group. A Poisson-Lie structure on $G$ is a Poisson structure on $G$ for which the group multiplication is a Poisson map. Then as is usual in [1]-[3], this is equal to giving an antisymmetric contravariant 2-tensor $\pi$ on $G$ which satisfies Jacobi identity and the relation

$$\pi(xy) = l_x \pi(y) + r_y \pi(x), \quad \forall x, y \in G,$$

where $l_x$ and $r_y$ respectively denote the left and right translations in $G$ by $x$ and $y$. We note that a Poisson-Lie structure $\pi$ has rank zero at a neutral element $e$ of $G$, i.e., $\pi(e) = 0$.

If we choose local coordinates $(x_1, x_2, \cdots, x_n)$ in a neighborhood $U$ of neutral element $e$ of $G$, the Poisson-Lie structure $\pi$ reads

\[ \pi(x) = \sum \pi_{ij}(x) \partial_i \wedge \partial_j, \quad x \in U, \] (2)

where \( \pi_{ij} \) are smooth functions vanishing at \( e \) and

\[ \{x_i, x_j\}(x) = \pi_{ij}(x), \quad x \in U, \] (3)

where \( \{,\} \) is the Poisson bracket associated to \( \pi \). By this Poisson bracket, \( C^\infty(G) \) becomes a Lie algebra.

Let \( \mathcal{G} \) be a Lie algebra of \( G \). The derivative of \( \pi \) at \( e \) defines a skew-symmetric co-commutator map \( \delta : \mathcal{G} \to \mathcal{G} \wedge \mathcal{G} \) such that:

1) The map \( \delta \) is a 1-cocycle, i.e.,

\[ \delta([X,Y]) = ad_x \delta(Y) - ad_y \delta(X), \quad \forall X,Y \in \mathcal{G}. \] (4)

2) The dual map \( \delta^* : \mathcal{G}^* \wedge \mathcal{G}^* \to \mathcal{G}^* \) is a Lie bracket on \( \mathcal{G}^* \).

The bialgebra structure \( \delta \) is called a coboundary one when there exists a skew-symmetric element \( r \) of \( \mathcal{G} \wedge \mathcal{G} \) (the classical r-matrix) such that

\[ \delta(S) = ad_s r, \quad \forall S \in \mathcal{G}. \] (5)

Both properties 1) and 2) imply that the element \( r \) has to be a constant solution of the modified classical Yang-Baxter equation (mCYBE) [4]-[6]:

\[ ad_s [r,r] = 0, \quad S \in \mathcal{G}. \] (6)

Therefore, a constant solution of mCYBE \( r \) on a given Lie algebra \( \mathcal{G} \) provide a coboundary Poisson-Lie structure \( \pi \) on (connected and simply connected) group \( G \) given by

\[ \pi(s) = r_s - l_s r, \quad \forall s \in G, \] (7)

where \( l_s \) and \( r_s \) denote respectively the left and right translations in \( G \) by \( s \).

Finally, recall that for semisimple Lie algebras, all Lie bialgebra structures are coboundaries, and the corresponding Poisson-Lie structures can be fully solved through the classical r-matrices.

In this work, We shall treat the case of the Poisson-Lie group \( SU(2,\mathbb{R}) \). We will calculate, firstly, all Poisson-Lie structures through the correspondence with Lie bialgebra; secondly, we will show that these Poisson-Lie structures are linearizable in a neighborhood of the unity \( e \) of the group \( SU(2,\mathbb{R}) \) and, finally, we shall study infinitesimal automorphism of \( SU(2,\mathbb{R}) \) with a linear Poisson-Lie structure, and show that the Lie algebra \( A \), consisting of all infinitesimal automorphisms is strictly contained in the Lie algebra \( \mathfrak{h} \) consisting of Hamiltonian vector fields.

2. The Group \( SU(2,\mathbb{R}) \) and Lie Algebra \( su(2,\mathbb{R}) \)

The special unitary group \( SU(2,\mathbb{R}) \) is defined by

\[ SU(2,\mathbb{R}) = \left\{ \begin{pmatrix} \alpha & \beta \\ -\bar{\beta} & \bar{\alpha} \end{pmatrix} : \alpha, \beta \in \mathbb{C}, \alpha \bar{\alpha} + \beta \bar{\beta} = 1 \right\}. \]

Let \( \alpha = x + iy \) and \( \beta = z + it \). \( SU(2,\mathbb{R}) \) can be identified with the unit sphere \( \mathbb{S}^3 \) in \( \mathbb{R}^4 \) with the unity \( e = (1,0,0,0) \).

The Lie algebra \( su(2,\mathbb{R}) \) of group \( SU(2,\mathbb{R}) \) is defined by

\[ su(2,\mathbb{R}) = \left\{ S \in \mathbb{C}^{2 \times 2} : \bar{S} + S = 0 \text{ and } Tr(S) = 0 \right\}. \]

Let

\[ e_1 = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} ; \quad e_2 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} ; \quad e_3 = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}. \]
a basis of \( su(2, \mathbb{R}) \). The Lie bracket on \( su(2, \mathbb{R}) \) is defined by
\[
\begin{align*}
[e_1, e_1] &= 2e_1; \\
[e_1, e_2] &= -2e_2; \\
[e_2, e_1] &= 2e_2.
\end{align*}
\]

Through a straightforward computation, the left invariant fields associated to this basis had this local expression
\[
\begin{align*}
X &= -y \partial_y + x \partial_x + t \partial_z - z \partial_t, \\
Y &= -z \partial_z - t \partial_y + x \partial_z + y \partial_t, \\
Z &= -t \partial_t + z \partial_z - y \partial_x + x \partial_t.
\end{align*}
\]

3. The Lie Bialgebra Structure on \( su(2, \mathbb{R}) \) and the Poisson Lie Structure on \( SU(2, \mathbb{R}) \)

3.1. Lie Bialgebra Structures on \( su(2, \mathbb{R}) \)
Recall that the Lie algebra \( su(2, \mathbb{R}) \) is semisimple. Then, all Lie bialgebra structures on \( su(2, \mathbb{R}) \) are co-boundaries, there exists an skew symmetric element \( r \) of \( su(2, \mathbb{R}) \) such that the cocommutator \( \delta \) is given by
\[
\delta(S) = ad_r, \quad \forall S \in su(2, \mathbb{R}).
\]

We stress that the element \( r \) satisfies the classical Yang-Baxter Equation (CYBE) (6). Through a long but straightforward computation, we show that these solutions are of the form
\[
r = k \cdot e_1 \wedge e_2, \quad k \in \mathbb{R}^*.
\]

So any Lie bialgebra structure of \( su(2, \mathbb{R}) \) can be written as
\[
\begin{align*}
\delta(e_1) &= -2k e_1 \wedge e_2, \\
\delta(e_2) &= 2k e_2 \wedge e_1, \\
\delta(e_3) &= 0.
\end{align*}
\]

3.2. Poisson-Lie Structures on \( SU(2, \mathbb{R}) \)
Since the Lie bialgebra structures \( \delta \) on \( su(2, \mathbb{R}) \) are coboundaries, the Poisson-Lie structures on \( SU(2, \mathbb{R}) \) corresponding to \( \delta \) are given by
\[
\pi(s) = r_s - l_s r, \quad \forall s \in SU(2, \mathbb{R}),
\]
where \( r \) is the solution of Yang-Baxter equation given by (8) and \( r_s \) and \( l_s \) respectively denote the right and left translations in \( SU(2, \mathbb{R}) \) by \( s \). Then, using \( \alpha = x + iy, \quad \beta = z + it \) and \( x^2 + y^2 + z^2 + t^2 = 1 \), one gets
\[
\pi(x, y, z, t) = 2k(zx - yt)Y \wedge Z - 2k(xy + zt)Z \wedge X + 2k(y^2 + z^2)X \wedge Y.
\]

Let
\[
\begin{align*}
\pi_1 &= 2k(zx - yt); \\
\pi_2 &= -2k(xy + zt); \\
\pi_3 &= 2k(y^2 + z^2),
\end{align*}
\]
be the components of \( \pi \) in the basis \( (Y \wedge X, Z \wedge X, X \wedge Y) \) of the bivector field.

4. Linearization of Poisson-Lie Structures on \( SU(2, \mathbb{R}) \)
By taking back the formula (2), The Taylor series of the functions \( \pi_{ij} \) reads
\[
\pi_{ij}^k(x) = c^k_{ij} x^k + \theta^k_i (x) x^i,
\]
where \( c^k_{ij} = \frac{\partial \pi_{ij}}{\partial x^k}(e) \) are the structure constants of a Lie algebra \( \mathcal{G} \), dual of a Lie algebra \( \mathcal{G}^* \), and the \( \theta^k_i \) are smooth functions vanishing at \( e \).

The term \( c^k_{ij} x^k \) of (12) defines a linear Poisson structure, called the linear part of \( \pi \). The linearization
problem for a structure $\pi$ around $e$ is the following [7] [8]:

**Linearization problem.** Are there new coordinates where the functions $\theta^i_j$ vanish identically, so that the Poisson structure is linear in these coordinates?

Let us notice that the Lie bialgebra structure $\delta$ associated to $\pi$ defines a linear Poisson-Lie structure on the additive group $G \ (G = \mathbb{R}^n)$ that can be expressed as follows

$$\delta(a) = \sum_i c^i_j a_i \partial_i \wedge \partial_j, \quad a = (a_1, \cdots, a_n) \in \mathbb{R}^n,$$

where $(\partial_1, \cdots, \partial_n)$ is the canonical basis of $\mathbb{R}^n$.

Let us notice that (13) coincides with the linear part of $\pi$, so, the linearization problem would be the following:

**There is a local Poisson diffeomorphism $G \rightarrow G$ of a neighborhood in $e$ of $G$ to a neighborhood of $0$ in $G$?**

If $(\phi_1, \cdots, \phi_n)$ are the components of $\varphi$, then $\varphi$ is solution of the system of equations

$$\{ \varphi_i, \varphi_j \} = \sum c^i_j \varphi_k, \quad 1 \leq i < j \leq n.$$  

(14)

For the Poisson-Lie structure on $SU(2, \mathbb{R})$ given by (10), the system of equations (14) would be

$$\{ \varphi_1, \varphi_2 \} = 0, \quad \{ \varphi_2, \varphi_3 \} = 2k \varphi_2, \quad \{ \varphi_3, \varphi_1 \} = -2k \varphi_1.$$  

(15)

With the identification of the subgroups of the singular points and the symplectic leaves of $SU(2, \mathbb{R})$ and $\text{SU}(2, \mathbb{R})$, we have:

**Proposition 1.** The map $\varphi = (\varphi_1, \varphi_2, \varphi_3) : (x, y, z, t) = \left( y, z, \text{Arctan} \frac{t}{x} \right)$ is a diffeomorphism in the neighborhood of $e = (1, 0, 0, 0)$ such that $\varphi(e) = 0$ and

$$\{ \varphi_1, \varphi_2 \} = 0, \quad \{ \varphi_2, \varphi_3 \} = 2k \varphi_2, \quad \{ \varphi_3, \varphi_1 \} = -2k \varphi_1.$$  

(16)

So, the Poisson-Lie structure $\pi$ on $SU(2, \mathbb{R})$ is linear in the new variables

$$u = y; \quad v = z; \quad w = \text{Arctan} \frac{t}{x}.$$  

(17)

and will be written

$$\pi(u, v, w) = 2kv \cdot \partial_v \wedge \partial_w - 2ku \cdot \partial_u \wedge \partial_w.$$  

(18)

5. Casimir Functions and Infinitesimal Automorphisms on $SU(2, \mathbb{R})$

Recall that for $f \in C^\infty(SU(2, \mathbb{R}))$, $\{ f, \cdot \}$ defines a derivation of $C^\infty(SU(2, \mathbb{R}))$. Hence there corresponds a vector field $\chi_f$, which we call the Hamiltonian vector field. We denote by $\mathcal{H}$ the Lie algebra of Hamiltonian vector fields.

A Casimir function on $SU(2, \mathbb{R})$ is a function $C$ such that $\{ C, f \} = 0$ for all function $f$. On the other words, $C$ is an element of the center of the Lie algebra $C^\infty(SU(2, \mathbb{R}))$. By simple consideration, we know that for each Casimir function $C$ there exists a function $\phi$ of one variable such that $C(u, v, w) = \phi \left( \frac{u}{v} \right)$.

Each symplectic leaf is the common level manifold of casimir functions. So, these have for equation:

$$\lambda u + \mu v = 0 \quad (\lambda, \mu \in \mathbb{R}; (\lambda, \mu \neq (0, 0)),$$

and hence are spheres.

By an automorphism of $SU(2, \mathbb{R})$, we mean a smooth vector field $\xi$ on $SU(2, \mathbb{R})$ such that

$$\mathcal{L}_\xi \pi = 0,$$  

(19)

where $\mathcal{L}_\xi$ denotes the Lie derivative along $\xi$. 
If we denote by \( \mathcal{A} \) the Lie algebra consisting of all infinitesimal automorphism, it is easy to see that \( \mathcal{H} \) is an ideal of \( \mathcal{A} \). Let \( \xi = f \partial_u + g \partial_v + h \partial_w \) be a vector field of \( \mathcal{A} \). Then three function \( f, g \) and \( h \) must satisfy:

\[
\begin{align*}
    f &= u \partial_u f + v \partial_v f + u \partial_u h; \\
    g &= u \partial_u g + v \partial_v g + v \partial_v h; \\
    v \partial_v f &= u \partial_u g.
\end{align*}
\]  

(20)

Now we shall clarify the gap between \( \mathcal{H} \) and \( \mathcal{A} \).

We consider the vector field

\[
\mathcal{U} = \pi_1 [Y, Z] + \pi_2 [Z, X] + \pi_3 [X, Y],
\]

(21)

where \( (\pi_1/\pi_2/\pi_3) \) are the components of the structure \( \pi \) in the basis \( (Y \wedge Z, Z \wedge X, X \wedge Y) \) given by (11). In the local coordinates \( (u, v, w) \) given by (14), this vector field reads

\[
\mathcal{U} = -4kv \partial_u + 4ku \partial_v.
\]

(22)

A simple check shows that the components of \( \mathcal{U} \) satisfy the relations (20). So, the vector field \( \mathcal{U} \) belongs to \( \mathcal{A} \). In other hand, \( \mathcal{U} \) is locally Hamiltonian if and only if there exist a smooth function \( F \) in a neighborhood of the unity \( e \) of the group \( SU(2, \mathbb{R}) \) such that \( \mathcal{U} = \chi_F \), this is translated by the fact that \( F \) is a solution of the following system of equations

\[
\begin{align*}
    u \partial_u &= -v, \\
    v \partial_v &= u \\
    u \partial_u + v \partial_v &= 0.
\end{align*}
\]

(23)

It is easy to see that (23) does not admit solutions. Hence \( \mathcal{U} \) does not belong \( \mathcal{H} \). Thus we have proved:

**Proposition 2.** The ideal \( \mathcal{H} \) is strictly contained in the Lie algebra \( \mathcal{A} \).

In terms of Poisson cohomology [9], recall that the first Poisson cohomology group \( H^1_\pi(SU(2, \mathbb{R})) \) is the quotient of the Lie algebra \( \mathcal{A} \) by its ideal \( \mathcal{H} \). Then, by Proposition 2, we show that the vector field \( \mathcal{U} \) defines a non trivial class \( [\mathcal{U}] \in H^1_\pi(SU(2, \mathbb{R})) \). On the other hand, this result shows that the classical result due to Conn [10] [11] stating that for a Poisson structure formally linearizable around a singular point any local Poisson automorphism is Hamiltonian, and not just in the \( C^\infty \) category.

**References**


