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On Maximal Regularity and Semivariation of α-Times Resolvent Families*

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ABSTRACT

Let $1<\alpha<2$ and A be the generator of an α-times resolvent family $\{S_\alpha(t)\}_{t\geq 0}$ on a Banach space X. It is shown that the fractional Cauchy problem $\mathbf{D}_t^\alpha u(t)=Au(t)+f(t)$, $t\in(0,r]$; $u(0),u'(0)\in D(A)$ has maximal regularity on $C([0,r];X)$ if and only if $S_\alpha(\cdot)$ is of bounded semivariation on $[0,r]$.

Keywords: α-Times Resolvent Family; Maximal Regularity; Semivariation

1. Introduction

Many initial and boundary value problems can be reduced to an abstract Cauchy problem of the form

$$\begin{aligned} &u'(t)=Au(t)+f(t),\ t\in[0,r] \\ &u(0)=x\in D(A) \end{aligned} \tag{1.1}$$

where A is the generator of a C_0-semigroup. One says that (1.1) has maximal regularity on $C([0,r];X)$ if for every $f\in C([0,r];X)$ there exists a unique $u\in C^1([0,r];X)$ satisfying (1.1). From the closed graph theorem it follows easily that if there is maximal regularity on $C([0,r];X)$, then there exists a constant $C>0$ such that

$$\|u'\|_{C([0,r];X)}+\|Au\|_{C([0,r];X)}\leq\|f\|_{C([0,r];X)}.$$

Travis [1] proved that the maximal regularity is equivalent to the C_0-semigroup generated by A being of bounded semivariation on $[0,r]$.

Chyan, Shaw and Piskarev [2] gave similar results for second order Cauchy problems. More precisely, they showed that the second order Cauchy problem

$$\begin{aligned} &u''(t)=Au(t)+f(t),\ t\in(0,r] \\ &u(0)=x,u'(0)=y,\quad x,y\in D(A) \end{aligned} \tag{1.2}$$

has maximal regularity on $[0,r]$ if and only if the cosine opeator function generated by A is of bounded semivariation on $[0,r]$.

In this paper, we will consider the maximal regularity for fractional Cauchy problem

$$\begin{aligned} &\mathbf{D}_t^\alpha u(t)=Au(t)+f(t),\ t\in(0,r] \\ &u(0)=x,u'(0)=y,\quad x,y\in D(A) \end{aligned} \tag{1.3}$$

where $\alpha\in(1,2)$, A is the generator of an α-times resolvent family (see **Definition 2.2**) and $\mathbf{D}_t^\alpha u$ is understood in the Caputo sense. We show that (1.3) has maximal regularity on $C([0,r];X)$ if and only if the corresponding α-times resolvent family is of bounded semivariation on $[0,r]$.

2. Preliminaries

Let $1<\alpha<2$, $g_0(t):=\delta(t)$ and

$$g_\beta(t):=\frac{t^{\beta-1}}{\Gamma(\beta)}\ (\beta>0)$$

for $t>0$. Recall the Caputo fractional derivative of order $\alpha>0$

$$\mathbf{D}_t^\alpha f(t):=\int_0^t g_{2-\alpha}(t-s)\frac{\mathrm{d}^2}{\mathrm{d}s^2}f(s)\mathrm{d}s,\quad t\in[0,r]$$

for $f\in C^2([0,r];X)$. The condition that

$$f\in C^2([0,r];X)$$

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can be relaxed to $f \in C^1([0,r];X)$ and

$$g_{2-\alpha} * \left(f - f(0) - f'(0) g_2\right) \in C^2([0,r];X),$$

for details and further properties see [3] and references therein. And in the above we denote by

$$(g_\beta * f)(t) = \int_0^t g_\beta(t-s) f(s) \mathrm{d}s$$

the convolution of g_β with *f*. Note that $g_\alpha * g_\beta = g_{\alpha+\beta}$.

Consider a closed linear operator *A* densely defined in a Banach space *X* and the fractional evolution Equation (1.3).

Definition 2.1 A function $u \in C([0,r];X)$ is called a strong solution of (1.3) if

$$u \in C([0,r];D(A)) \cap C^1([0,r];X),$$
$$g_{2-\alpha} * (u(t) - x - ty) \in C^2([0,r];X)$$

and (1.3) holds on $[0,r]$. $u \in C([0,r];X)$ is called a mild solution of (1.3) if $g_\alpha * u \in D(A)$ and

$$u(t) - x - ty = A(g_\alpha * u)(t) + (g_\alpha * f)(t)$$

for $t \in [0,r]$.

Definition 2.2 Assume that *A* is a closed, densely defined linear operator on *X*. A family $\{S_\alpha(t)\}_{t\geq 0} \subset B(X)$ is called an *α*-times resolvent family generated by A if the following conditions are satisfied:

(a) $S_\alpha(\cdot)$ is strongly continuous on $\mathbb{R}_+$ and $S_\alpha(0) = I$;

(b) $S_\alpha(t) D(A) \subset D(A)$ and $AS_\alpha(t)x = S_\alpha(t)Ax$ for all $x \in D(A), t \geq 0$;

(c) For all $x \in D(A)$ and $t \geq 0$, $S_\alpha(t)x = x + (g_\alpha * S_\alpha)(t)Ax$.

Remark 2.3 Since *A* is closed and densely defined, it is easy to show that for all $x \in X$, $(g_\alpha * S_\alpha)(t)x \in D(A)$ and $A(g_\alpha * S_\alpha)(t)x = S_\alpha x - x.$

The *α*-times resolvent families are closely related to the solutions of (1.3). It was shown in [3] that if *A* generates an *α*-times resolvent family $S_\alpha(\cdot)$, then (1.3) has a unique strong solution given by $S_\alpha(t)x + \int_0^t S_\alpha(s) y \mathrm{d}s$.

Next, we recall the definition of functions of bounded semivariation (see e.g. [4]). Given a closed interval $[a,b]$ of the real line, a subdivision of $[a,b]$ is a finite sequence $d: a = d_0 < d_1 < \cdots < d_n = b$. Let $D[a,b]$ denote the set of all subdivisions of $[a,b]$.

Definition 2.4 For

$$G:[a,b] \to B(X) \quad \textit{and} \quad d \in D[a,b],$$

define

$$SV_d[G] = \sup\left\{\left\|\sum_{n=1}^{n}\left[G(d_i) - G(d_{i-1})\right]x_i\right\| : x_i \in X, \|x_i\| \leq 1\right\}$$

and

$$SV[G] = \sup\{SV_d[G] : d \in D[a,b]\}.$$

We say *G* is of bounded semivariation if $SV[G] < \infty$.

3. Main Results

We begin with some properties on α-times resolvent families which will be needed in the sequel.

Proposition 3.1 Let $1 < \alpha < 2$ and $\{S_\alpha(t)\}_{t\geq 0}$ be the α-times resolvent family with generator A. Define

$$P_\alpha(t)x = (g_{\alpha-1} * S_\alpha)(t)x = \int_0^t g_{\alpha-1}(t-s) S_\alpha(s) x \mathrm{d}s, \quad x \in X,$$

then the following statements are true.

(a) For every $x \in X$, $\int_0^t P_\alpha(s) x \mathrm{d}s \in D(A)$ and

$$A\int_0^t P_\alpha(s) x \mathrm{d}s = S_\alpha(t)x - x;$$

(b) For every $x \in X$, $0 \leq a, b \leq t$,

$$\int_a^b s P_\alpha(t-s) x \mathrm{d}x \in D(A)$$

and

$$A\int_a^b s P_\alpha(t-s) x \mathrm{d}s = a S_\alpha(t-a)x - b S_\alpha(t-b)x + \int_a^b S_\alpha(t-s) x \mathrm{d}s;$$

(c) For every $x \in X$,

$$\int_0^t g_\alpha(t-s) s P_\alpha(s) x \mathrm{d}s \in D(A)$$

and

$$A\left(\int_0^t g_\alpha(t-s) s P_\alpha(s) x \mathrm{d}s\right) = -\alpha(g_\alpha * S_\alpha)(t)x + tP_\alpha(t)x;$$

(d) If $f \in C([0,r];X)$, then $g_\alpha * S_\alpha * f \in D(A)$ and

$$A(g_\alpha * S_\alpha * f) = (S_\alpha - 1) * f. \tag{3.1}$$

Proof. (a) follows from the fact that

$$\int_0^t P_\alpha(s) x \mathrm{d}s = (g_1 * g_{\alpha-1} * S_\alpha)(t)x = (g_\alpha * S_\alpha)(t)x \in D(A)$$

and $A(g_\alpha * S_\alpha)(t)x = S_\alpha(t)x - x$ by **Remark 2.3**.

(b) By integration by parts we have

$$\begin{aligned}
\int_a^b s P_\alpha(t-s) x \mathrm{d}s &= \int_a^b s \mathrm{d}_s\left[\int_0^s P_\alpha(t-\tau) x \mathrm{d}\tau\right] \\
&= \int_a^b s \mathrm{d}_s\left[(g_\alpha * S_\alpha)(t-s)x\right] \\
&= -s(g_\alpha * S_\alpha)(t-s)x\Big|_a^b + \int_a^b (g_\alpha * S_\alpha)(t-s) x \mathrm{d}s \\
&= a(g_\alpha * S_\alpha)(t-a)x - b(g_\alpha * S_\alpha)(t-b)x \\
&\quad + \int_a^b (g_\alpha * S_\alpha)(t-s) x \mathrm{d}s,
\end{aligned}$$

since $(g_\alpha * S_\alpha)(t)x\mathrm{d}s \in D(A)$ by **Remark 2.3**, operating A on both sides of the above identity gives (b).

(c) follows from the fact that

$$\begin{aligned}
&\int_0^t g_\alpha(t-s)sP_\alpha(s)x\mathrm{d}s \\
&=\int_0^t g_\alpha(t-s)(s-t)P_\alpha(s)x\mathrm{d}s + t\int_0^t g_\alpha(t-s)P_\alpha(s)x\mathrm{d}s \\
&=-\alpha\int_0^t g_{\alpha+1}(t-s)P_\alpha(s)x\mathrm{d}s + t(g_\alpha * P_\alpha)(t)x \\
&=-\alpha(g_{\alpha+1} * P_\alpha)(t)x + t(g_\alpha * P_\alpha)(t)x \\
&=-\alpha(g_{\alpha+1} * g_{\alpha-1} * S_\alpha)(t)x + t(g_\alpha * g_{\alpha-1} * S_\alpha)(t)x \\
&=-\alpha(g_\alpha * g_\alpha * S_\alpha)(t)x + t(g_{\alpha-1} * g_\alpha * S_\alpha)(t)x
\end{aligned}$$

belongs to $D(A)$ and

$$\begin{aligned}
&A\left(\int_0^t g_\alpha(t-s)sP_\alpha(s)x\mathrm{d}s\right) \\
&=-\alpha\left(g_\alpha * A(g_\alpha * S_\alpha)\right)(t)x + t\left(g_{\alpha-1} * A(g_\alpha * S_\alpha)\right)(t)x \\
&=-\alpha\left(g_\alpha * (S_\alpha - 1)\right)(t)x + t\left(g_{\alpha-1} * (S_\alpha - 1)\right)(t)x \\
&=-\alpha(g_\alpha * S_\alpha)(t)x + \alpha g_{\alpha+1}(t)x \\
&\quad + t(g_{\alpha-1} * S_\alpha)(t) - tg_\alpha(t)x \\
&=-\alpha(g_\alpha * S_\alpha)(t)x + tP_\alpha(t)x.
\end{aligned}$$

(d) (3.1) is true for step functions, and then for continuous functions by the closedness of A.

The following two lemmas can be proved similarly as that in [1,2].

Lemma 3.2 If $f \in C([0,r];X)$ and the α-times resolvent family $S_\alpha(t)$ is of bounded semivariation on $[0,r]$, then $(P_\alpha * f)(t) \in D(A)$ *and*

$$A(P_\alpha * f)(t) = -\int_0^t \mathrm{d}_s\left[S_\alpha(t-s)\right]f(s).$$

Lemma 3.3 If $f \in C([0,r];X)$ and the α-times resolvent family $S_\alpha(t)$ is of bounded semivariation on $[0,r]$, then $\int_0^t \mathrm{d}_s\left[S_\alpha(t-s)\right]f(s)$ is continuous in t on $[0,r]$.

We next turn to the solution of

$$\begin{aligned}
&\mathbf{D}_t^\alpha u(t) = Au(t) + f(t), \quad t \in (0,r], \\
&u(0) = 0, u'(0) = 0,
\end{aligned} \tag{3.2}$$

where A is the generator of an α-times resolvent family. If $v(t)$ is a mild solution of (3.2), then by **Definition 2.1** $(g_\alpha * v)(t) \in D(A)$ and

$$v(t) = A(g_\alpha * v)(t) + (g_\alpha * f)(t).$$

It then follows from the properties of α-times resolvent family that

$$\begin{aligned}
1 * v &= \left(S_\alpha - A(g_\alpha * S_\alpha)\right) * v \\
&= S_\alpha * v - S_\alpha * A(g_\alpha * v) \\
&= S_\alpha * \left(v - A(g_\alpha * v)\right) \\
&= S_\alpha * g_\alpha * f,
\end{aligned}$$

which implies that $g_\alpha * S_\alpha * f$ is differentiable and

$$\begin{aligned}
v(t) &= \frac{\mathrm{d}}{\mathrm{d}t}(g_\alpha * S_\alpha * f)(t) \\
&= (g_{\alpha-1} * S_\alpha * f)(t) \\
&= (P_\alpha * f)(t).
\end{aligned}$$

Therefore, the mild solution of (1.3) is given by

$$u(t) = S_\alpha(t)x + \int_0^t S_\alpha(s)y\mathrm{d}s + (P_\alpha * f)(t). \tag{3.3}$$

Proposition 3.4 Let A be the generator of an α-times resolvent family $S_\alpha(\cdot)$, and let $f \in C([0,r];X)$ and $x, y \in D(A)$. Then the following statements are equivalent:

(a) (1.3) has a strong solution;

(b) $(S_\alpha * f)(\cdot) \in C^1([0,r];X)$;

(c) $(P_\alpha * f)(t) \in D(A)$ for $0 \le t \le r$ and $A(P_\alpha * f)(t)$ is continuous in t on $[0,r]$.

Proof. (a) If $u(t)$ is a strong solution of (1.3), then u is given by (3.3) since every strong solution is a mild solution. Therefore, by the definition of strong solutions,

$$g_{2-\alpha} * P_\alpha * f = g_1 * S_\alpha * f \in C^2([0,r];X);$$

it then follows that $S_\alpha * f \in C^1([0,r];X)$, this is (b).

(b)⇒(c). Suppose that $S_\alpha * f \in C^1([0,r];X)$. Since $g_1 * P_\alpha * f = g_\alpha * S_\alpha * f$, by **Proposition 3.1(d)**,

$$g_1 * P_\alpha * f \in D(A)$$

and

$$A(g_1 * P_\alpha * f) = A(g_\alpha * S_\alpha * f) = (S_\alpha - 1) * f. \tag{3.4}$$

Since A is closed and $S_\alpha * f \in C^1([0,r];X)$, we have $P_\alpha * f \in D(A)$ and $A(P_\alpha * f) = (S_\alpha * f)' - f$ is continuous.

(c)⇒(a). By (3.4),

$$g_1 * A(P_\alpha * f) = A(g_1 * P_\alpha * f) = (S_\alpha - 1) * f,$$

therefore $S_\alpha * f$ is differentiable and thus

$$g_{2-\alpha} * P_\alpha * f = g_1 * S_\alpha * f$$

is in $C^2([0,r];X)$. It is easy to check that $u(t)$ defined by (3.3) is a strong solution of (1.3).

Now we are in the position to give the main result of this paper. The proof is similar to that of **Proposition 3.1** in [1] or Theorem 4.2 in [2], we write it out for completeness.

Theorem 3.5 Suppose that A generates an α-times resolvent family $\{S_\alpha(t)\}_{t\ge 0}$. Then the function (3.3) is a strong solution of the Cauchy problem (1.3) for every pair $x, y \in D(A)$ and continuous function f if and only if $S_\alpha(\cdot)$ is of bounded semivariation on $[0,r]$.

Proof. The sufficiency follows from **Lemmas 3.2** and **3.3**.

Conversely, suppose that for $x, y \in D(A)$ and continuous function f, $u(t)$ given by (3.3) is a strong solution for (1.3). Define the bounded linear operator $L: C([0,r];X) \to X$ by $L(f) = (P_\alpha * f)(r)$. By **Proposition 3.4(c)** $Lf \in D(A)$, it thus follows from the closedness of A that $AL: C([0,r];X) \to X$ is bounded.

Let $\{d_i\}_{i=0}^n$ be a subdivision of $[0,r]$ and $\epsilon > 0$ such that $\epsilon < \min_{1\le i\le n}\{|d_i - d_{i-1}|\}$. For $x_i \in X$ with $\|x_i\| \le 1\,(i = 1,2,\cdots,n+1)$, define $f_{d,\epsilon} \in C([0,r];X)$ by

$$f_{d,\epsilon}(\tau) = \begin{cases} x_i, & d_{i-1} \le \tau \le d_i - \epsilon \\ x_{i+1} + \dfrac{\tau - d_i}{\epsilon}(x_{i+1} - x_i), & d_i - \epsilon \le \tau \le d_i \end{cases},$$

then $\|f_{d,\epsilon}\|_{C([0,r];X)} \le 1$. By **Proposition 3.1**,

$$\begin{aligned}
AL(f_{d,\epsilon}) &= A\int_0^r P_\alpha(r-s) f_{d,\epsilon}(s)\,\mathrm{d}s \\
&= \sum_{i=1}^n \left[A\int_{d_{i-1}}^{d_i-\epsilon} P_\alpha(r-s)x_i\,\mathrm{d}s + A\int_{d_i-\epsilon}^{d_i} P_\alpha(r-s)x_{i+1}\,\mathrm{d}s + A\int_{d_i-\epsilon}^{d_i} \frac{s-d_i}{\epsilon} P_\alpha(r-s)(x_{i+1}-x_i)\,\mathrm{d}x \right] \\
&= \sum_{i=1}^n \Big\{ \left[S_\alpha(r-d_{i-1})x_i - S_\alpha(r-d_i+\epsilon)x_i \right] + \left[S_\alpha(r-d_i+\epsilon)x_{i+1} - S_\alpha(r-d_i)x_{i+1} \right] \\
&\quad - \frac{d}{\epsilon}\left[S_\alpha(r-d_i+\epsilon)(x_{i+1}-x_i) - S_\alpha(r-d_i)(x_{i+1}-x_i) \right] \\
&\quad + \frac{1}{\epsilon}\left[(d_i-\epsilon) S_\alpha(r-d_i+\varepsilon)(x_{i+1}-x_i) - d_i S_\alpha(r-d_i)(x_{i+1}-x_i) \right] + \frac{1}{\epsilon}\int_{d_i-\epsilon}^{d_i} S_\alpha(r-s)(x_{i+1}-x_i)\,\mathrm{d}s \Big\} \\
&= \sum_{i=1}^n \left\{ \left[S_\alpha(r-d_{i-1})x_i - S_\alpha(r-d_i)x_{i+1} \right] + \frac{1}{\epsilon}\int_{d_i-\epsilon}^{d_i} S_\alpha(r-s)(x_{i+1}-x_i)\,\mathrm{d}s \right\} \\
&= \sum_{i=1}^n \left\{ \left[S_\alpha(r-d_{i-1}) - S_\alpha(r-d_i) \right] x_i - S_\alpha(r-d_i)(x_{i+1}-x_i) + \frac{1}{\epsilon}\int_{d_i-\epsilon}^{d_i} S_\alpha(r-s)(x_{i+1}-x_i)\,\mathrm{d}s \right\},
\end{aligned}$$

it then follows that

$$\left\| \sum_{i=1}^n \left[S_\alpha(r-d_{i-1}) - S_\alpha(r-d_i) \right] x_i \right\| \le \left\| AL(f_{d,\epsilon}) \right\| + \sum_{i=1}^n \left\| S_\alpha(r-d_i)(x_{i+1}-x_i) - \frac{1}{\epsilon}\int_{d_i-\epsilon}^{d_i} S_\alpha(r-s)(x_{i+1}-x_i)\,\mathrm{d}s \right\|.$$

By letting $\epsilon \to 0$, we obtain that S_α is of bounded semivariation on $[0,r]$.

Corollary 3.6 Suppose that $\{S_\alpha(t)\}_{t\ge 0}$ is an α-times resolvent family with generator A and $S_\alpha(\cdot)$ is of bounded semivariation on $[0,r]$ for some $r > 0$. Then $R(P_\alpha(t)) \subset D(A)$ for $t \in [0,r]$ and $\|tAP_\alpha(t)\|$ is bounded on $[0,r]$.

Proof. For $x \in X$, consider $f(t) = \alpha S_\alpha(t)x$. By **Proposition 3.1(c)**, $tP_\alpha(t)x$ is a mild solution of (3.2). Moreover, it follows from **Proposition 3.4** that $P_\alpha * f$ is a strong solution of (3.2). Since a strong solution must be a mild solution, we have $(P_\alpha * f)(t) = tP_\alpha(t)x$. Thus our claim follows from **Proposition 3.4**.

Remark 3.7 Let $\alpha = 1$. If A generates a C_0-semigroup $T(\cdot)$, then the condition that $tAT(t)$ is bounded on $[0,r]$ implies that $T(\cdot)$ is analytic (see [5]). When $\alpha = 2$ and A generates a cosine function $C(\cdot)$, then the condition that $tAC(t)$ is bounded on $[0,r]$ implies that A is bounded ([3]). However, since there is no semigroup properties for α-times resolvent family, it is not clear that one can get the analyticity of $S_\alpha(\cdot)$ from the local boundedness of $tAP_\alpha(t)$.

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