The p.q.-Baer Property of Skew Group Rings under Finite Group Action*

Bo Li, Hailan Jin

Department of Mathematics, College of Sciences, Yanbian University, Yanji, China
Email: #hljin98@ybu.edu.cn, hljin98@hanmail.net

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ABSTRACT

In this paper, Let $R$ is a ring, $G$ be a finite group of ring automorphisms of $R$. $R^*G$ denote the skew group ring of $R$ under $G$. We investigate the right p.q.-Baer property of skew group rings under finite group action. Assume that $R$ is a semiprime ring with a finite group $G$ of X-outer ring automorphisms of $R$, then 1) $R^*G$ is p.q.-Baer if and only if $R$ is $G$-p.q.-Baer; 2) if $R$ is p.q.-Baer, then $R^*G$ is p.q.-Baer.

Keywords: p.q.-Baer Property; Skew Group Ring; Group Action

1. Introduction

Throughout this paper all rings are associative with identity unless otherwise stated. Let $R$ is a ring, for a non-empty subset $X$ of a ring $R$, $r_X(X)$ (resp., $l_X(X)$) denote a right (resp., left) annihilator of $X$ in $R$. A ring $R$ is called right principally quasi-Baer (simply, right p.q.-Baer) if the right annihilator of every principal right ideal of $R$ is generated, as a right ideal by an idempotent of $R$ in [1]. A left principally quasi-Baer (simply, left p.q.-Baer) ring is defined similarly. Right p.q.-Baer rings have been initially studied in [1]. For more details on (right) p.q.-Baer rings, see [1-6]. A ring $R$ is called quasi-Baer if the right annihilator of every right ideal is generated, as a right ideal by an idempotent of $R$ in [7] (see also [8]. A ring $R$ is called biregular, if for each $x \in R$, $RxR = eR$ for some central idempotent $e \in R$.

We note that the class of right p.q.-Baer rings is a generalization of classes of quasi-Baer rings and biregular rings. $Q(R)$ denote a fixed maximal right ring of quotients of $R$. Recall from [9] an idempotent $e$ of a ring $R$ is called left (resp., right) semicentral if $ae = eae$ (resp., $ea = eae$) for all $a \in R$. Equivalently, an idempotent $e$ is left (resp., right) semicentral if and only if $eR$ (resp., $Re$) is a two-sided ideal of $R$. $S_X(R)$ (resp., $S^*_X(R)$) denote the set of all left (resp., right) semicentral idempotents. An idempotent $e$ of a ring $R$ is called semicentral reduced if $Su_e(R) = \{0, e\}$. According to [2] a ring $R$ is called semicentral reduced if $S(R) = \{0, 1\}$, i.e., 1 is a semicentral reduced idempotent of $R$.

If $R$ is a semiprime ring and $I$ is a two-sided ideal of $R$, then $r^*_R(I) = l^*_R(I)$. For a right $R$-module $M$ and a submodule $N$ of $M$, we use $M_R^{\text{ess}}$ and $M_R^{\text{ess}}$ to denote that $N_R$ is essential in $M_R$ and $N_R$ is dense in $M_R$, respectively.

Let $R$ is a ring, $Aut(R)$ denote a group of ring automorphisms of $R, G$ be a subgroup of $Aut(R)$.

The skew group ring $R^*G$ is defined to be

$R^*G = \bigoplus_{g \in G} Rg$

with addition given componentwise and multiplication given as follows: if $a, b \in R$ and $g, h \in G$, then

$(ab)(gh) = ab^{e^{-1}}gh \in Rgh$.

We begin with the following example.

2. Preliminary

Example 2.1 There exist a ring $R$ and a finite group $G$ of ring automorphisms of $R$ such that $R$ is right p.q.-Baer but $R^*G$ is not right p.q.-Baer.

Let $R = \begin{bmatrix} F & F \\ 0 & F \end{bmatrix}$ with a field $F$ of characteristic 2, then $R$ is right p.q.-Baer. Define $g \in Aut(R)$ by

$g \begin{bmatrix} a & b \\ 0 & c \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}^{-1} \begin{bmatrix} a & b \\ 0 & c \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$. 

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*Corresponding author.
Since characteristic of $F$ is $2$, Then $g^2 = 1$.

Now we show that $R \ast G$ is not right p. q.-Baer. Consider the right ideal $(1 + g)(R \ast G)$ of $R \ast G$ generated by $1 + g$. By computation, we have

$$r_{nG}(1 + g)(R \ast G) = \begin{bmatrix} x & y \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} x & x + y \\ 0 & 0 \end{bmatrix}g, \quad x, y \in F.$$

Suppose that

$$r_{nG}(1 + g)(R \ast G) = e(R \ast G)$$

for some $e = e^2 \in R \ast G$. Note that the idempotents of $R \ast G$ are $0, 1$.

$$\begin{bmatrix} 1 & a \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & b \\ 0 & 0 \end{bmatrix}g, \quad \begin{bmatrix} 0 & a \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} 0 & b \\ 0 & 0 \end{bmatrix}g$$

with $a, b \in F$. Since $e = r_{nG}(1 + g)(R \ast G)$, the only possible choice for $e$ is $0$. Thus if $R \ast G$ is right p. q.-Baer, then it follows that $r_{nG}(1 + g)(R \ast G) = 0$. This is a contradiction. Therefore $R \ast G$ is not right p. q.-Baer. Also we see that $R \ast G$ is not left p. q.-Baer.

**Definition 2.2** Let $R$ be a semiprime ring. For $g \in \text{Aut}(R)$, let

$$\phi_g = \{x \in Q_n(R) | x^g = rx \text{ for each } r \in R \},$$

where $Q_n(R)$ is the Martindale right ring of quotients of $R$ (see [10] for more on $Q_n(R)$). We say that $g$ is X-outer if $\phi_g = 0$. A subgroup of $\text{Aut}(R)$ is called X-outer on $R$ if every $1 \neq g \in G$ is X-outer. Assume that $R$ is a semiprime ring, then for $g \in \text{Aut}(R)$, let

$$\Phi_g = \{x \in Q(R) | x^g = rx \text{ for each } r \in R \}.$$

For $g \in \text{Aut}(R)$, we claim that $\Phi_g = \phi_g$. Obviously $\phi_g \subseteq \Phi_g$. Conversely, if $x \in \Phi_g$ then $x^g = rx$. There exists $I_g \subseteq \text{Rad}_R$ such that $x \in R$. Therefore $RI < R$, $(RI)_g \subseteq \text{Rad}_R$, and $xRI = Rx \subseteq R$. Thus $x \in Q_n(R)$, hence $x \in \phi_g$. Therefore $\Phi_g = \phi_g$. So if $G$ is X-outer on $R$, then $G$ can be considered as a group of ring automorphisms of $Q(R)$ and $G$ is X-outer on $Q(R)$. For more details for X-outer ring automorphisms of a ring, etc., see [10, p. 396] and [11].

We say that a ring $R$ has no nonzero $n$-torsion ($n$ is a positive integer) if $na = 0$ with $a \in R$ implies $a = 0$.

**Lemma 2.3** Let $R$ be a semiprime ring and $G$ a group of ring automorphisms of $R$.

1) [11,12] If $G$ is X-outer, then every nonzero two-sided ideal of $R \ast G$ intersects $R$ nontrivially. Hence $R \ast G$ is semiprime.

2) [11] If $G$ is finite and $R$ has no nonzero $|G|$-torsion, then $R \ast G$ is semiprime.

For a ring $R$, we use $\text{Cen}(R)$ to denote the center of $R$.

**Lemma 2.4** For a semiprime ring $R$, let $G$ be a group of X-outer ring automorphisms of $R$. Then $\text{Cen}(R \ast G) = \text{Cen}(R_G)$.

**Proof.**

Let $\alpha = a_1 + \alpha_2 a_2 + \cdots + a_n g_n \in \text{Cen}(R)$ with $a_i \in R$, $g_i$ the identity of $G$, and $g_n \in G$. The

$$(a_1 + a_2 a_3 + \cdots + a_g g_n) = b(a_1 + a_2 a_3 + \cdots + a_g g_n)$$

for all $b \in R$. So $ab = ba_1, ab^2 = ba_2, \cdots, ab^{e_n} = ba_n$ for all $b \in R$. Since $G$ is X-outer, it follows that $a_2 = \cdots = a_n = 0$. Hence $\alpha = a_1 \in R$. Also since $ab = ba_1$ for all $b \in R$, we have that $a_1 \in \text{Cen}(R)$. Note that for all $g \in G$, $a_i = g a_i = g a_i \in \text{Cen}(R_G)$. Thus

$$\text{Cen}(R \ast G) \subseteq \text{Cen}(R_G).$$

Conversely, $\text{Cen}(R) \subseteq \text{Cen}(R \ast G)$ is clear. Therefore $\text{Cen}(R \ast G) = \text{Cen}(R_G)$.

**Lemma 2.5** [13,14] Let $R$ be a ring and $G$ a finite group of ring automorphisms of $R$. Then $Q(R) \ast G$ is the maximal right ring of quaternions of $R$.

Assume that a group $G$ of ring automorphisms of a ring $R$ is finite. Then for $a \in R$, let $tr(a) = \sum_{g \in G} a^g$, which is called the trace of $a$. Also for a right ideal $I$ of $R$, the right ideal $tr(I) = \sum_{g \in G} a^g$, of $R^G$ is called the trace of $I$. Say $G = \{g_1, \cdots, g_n\}$, we put $t = g_1 \cdots g_n \in R^G$. For $r \in R$ and $\alpha = a_1 g_1 + a_2 g_2 + \cdots + a_n g_n \in R^G$ with $a_i \in R$, define $r \cdot \alpha = r^g a_1 g_1 + r^g a_2 g_2 + \cdots + r^g a_n g_n$. Then $R$ is a right $R^G$-module. Moreover, we see that $\forall^g : R^G \rightarrow R^G$ is an $(R^G, R \ast G)$-bimodule.

**Lemma 2.6** Assume that $R$ is a semiprime ring and $e \in B(Q(R))$. Let $I$ be a two-sided ideal of $R$ such that $I_R \leq \text{Ess} eR$ and $r(e) = fR$ with $r \in B(Q(R))$. Then $e = 1 - f$.

**Proof.** Since $R$ is semiprime,

$I_R \leq \text{Ess} (1 - f)Q(R)_R$.

Thus

$I_R \leq \text{Ess} (1 - f)Q(R)_R$.

As $I_R \leq \text{Ess} eR$ and $I_R \leq \text{Ess} Q(R)_R$. We note that $e$ and $1 - f$ are in $B(Q(R))$. So we have that $e = 1 - f$.

**Proposition 2.7** [1] Let $R$ be a semiprime ring. Then the following are equivalent.

1) $R$ is right p. q.-Baer.

2) Every principal two-sided ideal of $R$ is right essential in a ring direct summand of $R$. 

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3) Every finitely generated two-sided ideal of $R$ is right essential in a ring direct summand of $R$.
4) Every principal two-sided ideal of $R$ that is closed as a right ideal is a direct summand of $R$.
5) For every principal two-sided ideal $I$ of $R$, $r_e(I)$ is right essential in a direct summand of $R$.
6) $R$ is left p.q.-Baer.

For a ring $R$ with a group $G$ of ring automorphisms of $R$, we say that a right ideal $I$ of $R$ is $G$-invariant if $I^g \subseteq I$ for every $g \in G$, where $I^g = \{ag | a \in I\}$. Assume that $R$ is a semiprime ring with a group $G$ of ring automorphisms of $R$. We say that $R$ is $G$-p.q.-Baer if the right annihilator of every finitely generated $G$-invariant two-sided ideal is generated by an idempotent, as a right ideal.

As was shown in [15], a ring $R$ is p.q.-Baer if and only if $R$ is $G$-p.q.-Baer.

Lemma 2.8 [15] Assume that $R$ is a semiprime ring. Then:
1) The ring $\hat{Q}_{p.q.}(R)$ is the smallest right ring of quotients of $R$ which is p.q.-Baer.
2) $R$ is p.q.-Baer if and only if $B_p(Q(R)) \subseteq R$.

With these preparations, in spite of Example 2.1, we have the following result for p.q.-Baer property of $R\ast G$ on a semiprime ring $R$ for the case when $G$ is finite and X-outer.

3. Main Results

Theorem 3.1 Let $R$ be a semiprime ring with a finite group $G$ of X-outer ring automorphisms of $R$. Then $R\ast G$ is p.q.-Baer if and only if $R$ is $G$-p.q.-Baer.

**Proof.** Assume that $R\ast G$ is p.q.-Baer. Say
$I = Ra_1R + \cdots + Ra_nR$
is a finitely generated $G$-invariant two-sided ideal of $R$ with $a_i \in R$. Then $I \ast G$ is a two-sided ideal of $R \ast G$.
Moreover,
$I \ast G = (R \ast G)a_1(R \ast G) + \cdots + (R \ast G)a_n(R \ast G)$.

Note that $R \ast G$ is semiprime by Lemma 2.3, So Proposition 2.7 yields that there exists $e \in S_i(R \ast G)$ such that
$(I \ast G)_{R \ast G} \subseteq e (R \ast G)_{R \ast G}$.

Since $R \ast G$ is semiprime, $e \in B(R \ast G)$ by [9]. Hence by Lemma 2.4, $e \in Cen(R \ast G)$. First, we see that $I_g \subseteq eR_g$. For this, let $0 \neq er \in eR$ with $r \in R$. As $(I \ast G)_{R \ast G} \subseteq e(R \ast G)_{R \ast G}$, there exists $\beta \in R \ast G$ such that $0 \neq er \beta \in I \ast G$.

Say $\beta = b_1g_1 + \cdots + b_ng_n$ with $b_i \in R$ and $g_i \in G$ for $i = 1, \cdots, n$. Then
$er\beta = (erb_1)g_1 + \cdots + (erb_n)g_n \in I \ast G$.

Hence $0 \neq erb_j \in I$ for some $j$, so $I_g \subseteq eR_g$. As $e = e^2 \in Cen(R \ast G)$, $I \subseteq eR$, and so $I_{eR} \subseteq eR_{eR}$.

Now we show that $r_{\alpha}(I) = (1 - e)R$. If $e = 0$, then $r_{\alpha}(I) = \emptyset$. So we may assume that $e \neq 0$. Note that $eR$ is semiprime and $I_{eR} \subseteq eR_{eR}$, and so $r_{eR}(I) = 0$.

Hence
$eR \cap r_{\alpha}(I) = eR \cap (1 - e)R = 0$.

As $I \subseteq eR$, $(1 - e)R \subseteq r_{\alpha}(I)$. From the modular law,
$r_{\alpha}(I) = (1 - e)R \oplus (eR \cap r_{\alpha}(I))$.

But since $eR \cap r_{\alpha}(I) = 0$, $r_{\alpha}(I) = (1 - e)R$. Therefore $R$ is $G$-p.q.-Baer.

Conversely, let $R$ be $G$-p.q.-Baer. Take
$e \in B_p(Q(R) \ast G)$.

Then
$e \in \left[Cen(Q(R))\right]^G$

by Lemma 2.4 since $G$ is also X-outer on $Q(R)$ as was noted. Also there exists $\alpha \in R \ast G$ such that
$(R \ast G)\alpha (R \ast G)_{R \ast G} \subseteq e(R \ast G)_{R \ast G}$
because $Q(R) \ast G$ is the maximal right ring of quotients of $R \ast G$ (Lemma 2.5) and $e \in B_p(Q(R) \ast G)$. Say
$\alpha = a_1g_1 + a_2g_2 + \cdots + a_ng_n$ with $a_i \in R$ and $g_i \in G$ for $i = 1, 2, \cdots, n$. Then $\alpha \in e(R \ast G)(eR) \ast G$ and so $a_i \in eR$
for each $i = 1, 2, \cdots, n$. Consider $K = \sum_{i=1}^n Ra_iR$. Then $K$ is a finitely generated $G$-invariant two-sided ideal of $R$.

Further, $K \subseteq eR$ because $e \in \left[Cen(Q(R))\right]^G$. By the preceding argument, we see that $K_{eR} \subseteq eR_{eR}$. From the assumption, there exists $f \in S_i(R) = B(R)$ such that
$r_{\alpha}(K) = fR$. Thus $e = 1 - f \in eR$ by Lemma 2.6. Therefore $eR \subseteq R \ast G$, so $B_p(Q(R) \ast G) \subseteq R \ast G$.

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Lemma 2.8, $R^*G$ is p.q.-Baer.

**Corollary 3.2** Let $R$ be a semiprime ring with a finite group $G$ of $X$-outer ring automorphisms of $R$. If $R$ is p.q.-Baer, then $R^*G$ is p.q.-Baer.

**Proof.** The proof follows immediately by Theorem 3.1.

4. Conclusion

In [16] researched quasi-Baer property of skew group rings under finite group actions on a semiprime ring and their applications to $C^*$-algebras (see also [17,18]). In this paper, we investigate the right p.q.-Baer property of skew group rings under finite group action. Assume that $R$ is a semiprime ring with a finite group $G$ of $X$-outer ring automorphisms of $R$, then 1) $R^*G$ is p.q.-Baer if and only if $R$ is $G$-p.q.-Baer; 2) if $R$ is p.q.-Baer, then $R^*G$ is p.q.-Baer.

**REFERENCES**


