A Modified Wallman Method of Compactification

Hueytzen J. Wu¹, Wan-Hong Wu²

¹Department of Mathematics, Texas A&M University, Kingsville, USA
²University of Texas at San Antonio, San Antonio, USA
Email: hueytzen.wu@tamuk.edu, ddi273@yahoo.com

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ABSTRACT

Closed øϕ, and basic closed C*ϕ-filters are used in a process similar to Wallman method for compactifications of the topological spaces Y, of which, there is a subset D of C*(Y) containing a non-constant function, where C*(Y) is the set of bounded real continuous functions on Y. An arbitrary Hausdorff compactification (Z,h) of a Tychonoff space X can be obtained by using basic closed C*ϕ-filters from D={ϕ ∩ h | ϕ ∈ C(D)} in a similar way, where C(Z) is the set of real continuous functions on Z.

Keywords: Closed øϕ-Filter; Open and Closed C*ϕ-Filter Bases; Basic Open and Closed C*ϕ-Filters; Compactification; Stone-Cech and Wallman Compactifications

1. Introduction

Throughout this paper, [T]ca will denote the collection of all finite subsets of the set T. For the other notations and the terminologies in general topology which are not explicitly defined in this paper, the readers will be referred to the reference [1].

Let C*(Y) be the set of bounded real continuous functions on a topological space Y. For any subset D of C*(Y), we will show in Section 2 that there exists a unique rϕ in Cl(f(Y)) for each f in D so that for any

H ∈ [D]ca , ε > 0, ϕ ≠ ∩f∈H f⁻¹((rϕ−ε,rϕ+ε))

∩ f∈H f⁻¹([rϕ−ε,rϕ+ε]) ≠ ϕ

Let K be the set

{∩f∈H f⁻¹([rϕ−ε,rϕ+ε]) | ∩f∈H f⁻¹([rϕ−ε,rϕ+ε]) ≠ ϕ}

for any H ∈ [D]ca , ε > 0

and let V be the set

{∩f∈H f⁻¹((rϕ−ε,rϕ+ε)) | ∩f∈H f⁻¹((rϕ−ε,rϕ+ε)) ≠ ϕ}

for any H ∈ [D]ca , ε > 0

K and V are called a closed C*ϕ-filter base and an open C*ϕ-filter base on Y, respectively. A closed filter (or an open filter) on Y generated by a K (or a V) is called a basic closed C*ϕ-filter (or a basic open C*ϕ-filter), denoted by E (or A). If rϕ = f(x) for all f in D at some x in Y, then K, V, E and A are denoted by Kx, Vx, Ex, and Ax, respectively. Let Y be a topological space, of which, there is a subset D of C*(Y) containing a non-constant function. A compactification {Y", S} of Y is obtained by using closed øϕ and basic closed C*ϕ-filters in a process similar to the Wallman method, where Y" = Yζ ∪ Yₚ, Yₚ is the set {N | N is a closed øϕ-filter, x is in Y}, Yζ is the set of all basic closed C*ϕ-filter that does not converge in Y, S is the topology induced by the base τ = {F∩E | F is a nonempty closed set in Y} for the closed sets of Y", and Fζ is the set of all C in Y" such that F ∩ T ≠ Ø for all T in C. Similarly, an arbitrary Hausdorff compactification (Z,h) of a Tychonoff space X can be obtained by using the basic closed C*ϕ-filters on X from D={ϕ ∩ h | ϕ ∈ C(D)}, where C(D) is the set C*(Z).

2. Open and Closed C*ϕ-Filter Bases, Basic Open and Closed C*ϕ-Filters

For an arbitrary topological space Y, let D be a subset
of $\mathcal{C}^\ast(Y)$.

**Theorem 2.1** Let $\mathcal{F}$ be a filter on $Y$. For each $f \in D$ there exists a $r_j$ in $\text{Cl}(f(Y))$ such that

$$f^{-1}\left((r_j - \varepsilon, r_j + \varepsilon)\right) \cap F \neq \phi$$

for any $F \in \mathcal{F}$ and any $\varepsilon > 0$ (See Thm. 2.1 in [2, p.1164]).

**Proof.** If the conclusion is not true, then there is an $f$ in $D$ such that for each $r_j$ in $\text{Cl}(f(Y))$ there exist an $F_j$ in $\mathcal{F}$ and an $\varepsilon_j > 0$ such that

$$F_j \cap f^{-1}\left((r_j - \varepsilon_j, r_j + \varepsilon_j)\right) = \phi.$$

Since $\text{Cl}(f(Y))$ is compact and $\text{Cl}(f(Y))$ is contained in

$$\bigcup\left\{(r_j - \varepsilon, r_j + \varepsilon) | r_j \in \text{Cl}(f(Y))\right\},$$

there exist $r_1, \ldots, r_n$ in $\text{Cl}(f(Y))$ such that $Y$ is contained in

$$\bigcup\left\{f^{-1}\left((r_i - \varepsilon, r_i + \varepsilon)\right) | i = 1, \ldots, n\right\}.$$

Let $F = \cap F_j$ if $i = 1, \ldots, n$, then $F_j$ is in $\mathcal{F}$ and

$$F_j \subseteq \bigcup\left\{f^{-1}\left((r_i - \varepsilon, r_i + \varepsilon)\right) | i = 1, \ldots, n\right\} = \phi,$$

contradicting that $\phi$ is not in $\mathcal{F}$.

**Corollary 2.2** Let $\mathcal{F}$ (or $\mathcal{Q}$) be a closed (or an open) ultralimit of $Y$. For each $f$ in $D$, there exists a unique $r_j$ in $\text{Cl}(f(Y))$ such that (1) for any $H \in [D]^{\omega}$, any $\varepsilon > 0$,

$$\bigcap_{f \in H} f^{-1}\left((r_j - \varepsilon, r_j + \varepsilon)\right) \in \mathcal{F}$$

(or $\bigcap_{f \in H} f^{-1}\left((r_j - \varepsilon, r_j + \varepsilon)\right) \in \mathcal{Q}$)

and (2) for any $H \in [D]^{\omega}$, any $\varepsilon > 0$,

$$\bigcap_{f \in H} f^{-1}\left((r_j - \varepsilon, r_j + \varepsilon)\right) \neq \phi$$

(or $\bigcap_{f \in H} f^{-1}\left((r_j - \varepsilon, r_j + \varepsilon)\right) \neq \phi$).

(See Cor. 2.2 & 2.3 in [2, p.1164].)

Therefore, for a given closed ultralimit $\mathcal{F}$ (or open ultralimit $\mathcal{Q}$), there exists a unique $r_j$ in $\text{Cl}(f(Y))$ for each $f$ in $D$ such that for any $H \in [D]^{\omega}$, any $\varepsilon > 0$,

$$\bigcap_{f \in H} f^{-1}\left((r_j - \varepsilon, r_j + \varepsilon)\right) \neq \phi$$

(or $\bigcap_{f \in H} f^{-1}\left((r_j - \varepsilon, r_j + \varepsilon)\right) \neq \phi$).

Let $K$ be the set

$$\left\{\bigcap_{f \in H} f^{-1}\left((r_j - \varepsilon, r_j + \varepsilon)\right) | H \in [D]^{\omega}, \varepsilon > 0\right\}$$

and let $V$ be the set

$$\left\{\bigcap_{f \in H} f^{-1}\left((r_j - \varepsilon, r_j + \varepsilon)\right) | H \in [D]^{\omega}, \varepsilon > 0\right\}$$

$K$ and $V$ are called a closed and an open $\mathcal{C}^\ast$-filter bases, respectively. If for all $f$ in $D$, $r_j = f(x)$ for some $x$ in $Y$, then $K$ and $V$ are called the closed and open $\mathcal{C}^\ast$-filter bases at $x$, denoted by $K_x$ and $V_x$, respectively. Let $\mathcal{E}$ and $\mathcal{E}_x$ (or $\mathcal{A}$ and $\mathcal{A}_x$) be the closed (or open) filters generated by $K$ and $K_x$ (or $V$ and $V_x$), respectively, then $\mathcal{E}$ and $\mathcal{E}_x$ (or $\mathcal{A}$ and $\mathcal{A}_x$) are called a basic closed $\mathcal{C}^\ast$-filter and the basic closed $\mathcal{C}^\ast$-filter at $x$ (or a basic open $\mathcal{C}^\ast$-filter and the basic open $\mathcal{C}^\ast$-filter at $x$), respectively.

**Corollary 2.3** Let $\mathcal{F}$ and $\mathcal{Q}$ be a closed and an open ultralimits on a topological space $Y$, respectively. Then there exist a unique basic closed $\mathcal{C}^\ast$-filter $\mathcal{E}$ and a unique basic open $\mathcal{C}^\ast$-filter $\mathcal{A}$ on $Y$ such that $\mathcal{E}$ is contained in $\mathcal{F}$ and $\mathcal{A}$ is contained in $\mathcal{Q}$.

### 3. A Closed $\mathcal{C}^\ast$-Filter and a Modified Wallman Method of Compactification

Let $Y$ be a topological space, of which, there is a subset $D$ of $\mathcal{C}^\ast(Y)$ containing a non-constant function. For each $x$ in $Y$, let $N$ be the union of $\{x\}$ and $\mathcal{E}_x$, if $V_x$ is an open nhoo filter base at $x$; let $N_x$ be the union of $\{x\}$ and $\{F | F$ is closed, $x$ is in $F\}$, if $V_x$ is not an open nhoo filter base at $x$. For each $x$ in $Y$, $N_x$ is a $\phi$-filter with $\phi$ being $N_x$. (See 12E in [1, p.82] for definition and convergence). $N_x$ is called a closed $\phi$-filter. It is clear that $K_x$ is contained in $\mathcal{E}_x$ and $\mathcal{E}_x$ is contained in $N_x$. $N_x$ converges to $x$ for each $x$ in $Y$. Let $Y_x$ be the set of all $N_x$ in $Y$. Let $Y_x$ be the set of all basic closed $\mathcal{C}^\ast$-filter $\mathcal{E}$ that does not converge to $Y_x$ in $Y$ and let $Y'' = Y_x \cup Y_x$.

**Definition 3.4** For each nonempty closed set $F$ in $Y$, let $F^\ast$ be the set of $C$ in $Y''$ such that the intersection of $F$ and $T$ is not an empty set for all $T$ in $C$.

From the Def. 3.4, the following Cor. 3.5 can be readily proved. We omit its proofs.

**Corollary 3.5** For a closed set $F$ in $Y$, (i) $x$ is in $F$ if $N_x$ is in $F$; (ii) $F$ is equal to $Y$ if $F^\ast$ is equal to $Y''$; (iii) if $F$ is in $\mathcal{C}$, then $C$ is in $F^\ast$; (iv) $C$ is in $Y'' - F^\ast$) if there is a $T$ in $\mathcal{C}$ such that $T$ is contained in $Y - F$.

**Lemma 3.6** For any two nonempty closed sets $E$ and $F$ in $Y$.

(i) $E \subseteq F \iff E^\ast \subseteq F^\ast$;

(ii) $E \cap F^\ast \subseteq (E^\ast \cap F^\ast)$;

(iii) $(E \cup F)^\ast = (E^\ast \cup F^\ast)$.

**Proof.** (i) For $E \in F$, pick an $x$ in $E - F$, by

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Cor. 3.5 (i), \( N_\varepsilon \) is in \( E^* \) and \( N_\varepsilon \) is not in \( F^* \); i.e., \( E^* \subsetneq F^* \). For \((\Rightarrow)\) is obvious. (ii) is clear from (i). (iii) For \([\subseteq]\): If \( C \) belongs to \( (E \cup F)^* \) and does not belong \( E^* \cup F^* \), then pick \( T_1, T_2 \) in \( C \) such that
\[
E \cap T_1 = F \cap T_2 = \emptyset.
\]
Since \( T_1 \cap T_2 \) is in \( C \) and
\[
(E \cup F) \cap (T_1 \cap T_2) \subseteq (E \cap T_1) \cup (F \cap T_2) = \emptyset.
\]
Thus, \( C \) does not belong to \( (E \cup F)^* \), contradicting the assumption. For \([\supseteq]\) is obvious from (i).

Proposition 3.7 \( \tau = \{F^* | F \text{ is a nonempty closed set in } Y^*\} \) is a base for the closed sets of \( Y^* \).

Proof. Let \( B \) be the set \( \{Y^w - F^* | F^* \in \tau\} \). We show that \( B \) is a base for \( Y^w \). For (a) of Thm. 5.3 in [1, p.38], if \( C \in Y^w \), then there exist an \( f \) in \( D \), a \( \delta > 0 \) such that
\[
S = f^{-1}\left(\left[ r_j - \delta, r_j + \delta \right]\right) \subseteq E \subseteq C
\]
and
\[
E = Y - f^{-1}\left(\left[ r_j - 2\delta, r_j + 2\delta \right]\right) \neq \emptyset,
\]
otherwise, if for all \( f \) in \( D \), all \( \delta > 0 \), \( E = \emptyset \), then for all \( f \) in \( D \), \( f(Y) = \{r_j\} \), contradicting that \( D \) contains a non-constant function. Thus \( E \neq \emptyset \), \( E \) is closed, \( S \) is in \( C \) and \( S \cap E = \emptyset \) imply that \( C \) is in \( Y^w - E^* \). So,
\[
X^w = \bigcup \left\{ (Y^w - E^*) | E^* \in \tau \right\}.
\]
For (b) of Thm. 5.3, if \( C \) belongs to \( (Y^w - E^*) \cap (Y^w - F^*) \), then \( E \cup F \) is closed, \( (E \cup F)^* \in \tau \) and \( (Y^w - E^*) \cap (Y^w - F^*) = Y^w - (E \cup F)^* \) is in \( B \). Thus, \( C \) is in \( Y^w - (E \cup F)^* \subseteq (Y^w - E^*) \cap (Y^w - F^*) \).

Equip \( Y^w \) with the topology \( \Delta \) induced by \( \tau \). For each \( f \in D \), define \( f^* : Y^w \to R \) by \( f^*(C) = r_j \), if
\[
f^{-1}\left(\left[ r_j - \varepsilon, r_j + \varepsilon \right]\right) \subseteq E \subseteq C
\]
for all \( \varepsilon > 0 \). Since (i) if \( C \) is equal to \( N_\varepsilon \) for some \( N_\varepsilon \) in \( Y_k \), then
\[
f^{-1}\left(\left[ f(x) - \varepsilon, f(x) + \varepsilon \right]\right) \subseteq E \subseteq C
\]
is in \( N_\varepsilon \) for all \( \varepsilon > 0 \), (ii) if \( C \) is \( E \) which is in \( Y_k \), then
\[
f^{-1}\left(\left[ r_j - \varepsilon, r_j + \varepsilon \right]\right) \subseteq E \subseteq C
\]
is in \( E \) for all \( \varepsilon > 0 \), (iii) by Cor. 2.2, the \( r_j \) is unique for each \( f \) in \( D \) and (iv) the \( K \) that is contained in \( C \) is unique. Thus, \( f^* \) is well-defined for each \( f \) in \( D \). For all \( f \) in \( D \), all \( x \) in \( Y \),
\[
f^{-1}\left(\left[ f(x) - \varepsilon, f(x) + \varepsilon \right]\right) \subseteq \emptyset \]
is in \( N_\varepsilon \) for all \( \varepsilon > 0 \), thus \( f^*(N_\varepsilon) \) is equal to \( f(x) \) for all \( f \) in \( D \) and all \( x \) in \( Y \).

Lemma 3.8 For each \( f \) in \( D \), let \( r \) be in \( Cl(f(Y)) \), then
\[(i) \quad f^{-1}\left(\left[ r - \delta, r + \delta \right]\right) \subseteq f^{-1}\left(\left[ r - \varepsilon, r + \varepsilon \right]\right)
\]
and
\[(ii) \quad f^{-1}\left(\left[ r - \varepsilon, r + \varepsilon \right]\right) \subseteq f^{-1}\left(\left[ r - \delta, r + \delta \right]\right)
\]
for any \( \varepsilon > \delta > 0 \).

Proof. (i): If \( C \) is in \( f^{-1}\left(\left[ r - \delta, r + \delta \right]\right) \) and \( f^*(C) \) is \( t_f \), then
\[
f^{-1}\left(\left[ r - \delta, r + \delta \right]\right) \cap f^{-1}\left(\left[ t_f - \gamma, t_f + \gamma \right]\right) \neq \emptyset
\]
for all \( \gamma > 0 \), where \( f^{-1}\left(\left[ t_f - \gamma, t_f + \gamma \right]\right) \subseteq E \subseteq C \) for all \( \gamma > 0 \). Thus,
\[
\left[ r - \delta, r + \delta \right] \cap \left[ t_f - \gamma, t_f + \gamma \right] \neq \emptyset
\]
for all \( \gamma > 0 \) i.e., \( f^*(C) \) is \( t_f \) in \( \left[ r - \delta, r + \delta \right] \subseteq (r - \varepsilon, r + \varepsilon) \), so \( C \) is in \( f^{-1}\left(\left[ r - \varepsilon, r + \varepsilon \right]\right) \). For (ii): If \( C \) is in \( f^{-1}\left(\left[ r - \varepsilon, r + \varepsilon \right]\right) \) and \( f^*(C) \) is \( t_f \), then
\[t_f \in (r - \varepsilon, r + \varepsilon)\]
\[f^{-1}\left(\left[ t_f - \delta, t_f + \delta \right]\right) \subseteq f^{-1}\left(\left[ r - \varepsilon, r + \varepsilon \right]\right).
\]
Since
\[
f^{-1}\left(\left[ t_f - \delta, t_f + \delta \right]\right) \subseteq E \subseteq C \]
thus \( f^{-1}\left(\left[ r - \varepsilon, r + \varepsilon \right]\right) \subseteq f^*(C) \). By Cor. 3.5 (iii), \( C \) is in \( \left( f^{-1}\left(\left[ r - \varepsilon, r + \varepsilon \right]\right) \right) \).
\[ E = f^{-1}\left((\mathbb{R}, t_j - \varepsilon/2, t_j + \varepsilon/2)\right) \cap f^{-1}\left((t_j + \varepsilon/2, \mathbb{R})\right) \]
and \( U = Y^w - E^* \). Since
\[ P = f^{-1}\left((t_j - \varepsilon/3, t_j + \varepsilon/3)\right) \subseteq \mathcal{C} \subseteq \mathcal{C} \]
and \( P \subseteq Y - E \), by Cor. 3.5 (iv), \( \mathcal{C} \subseteq U \). Next, for any \( \mathcal{C} \), if \( x \) is an element of \( \mathcal{C} \), then \( S' \) is \( f^{-1}\left((t_j - \varepsilon/2, t_j + \varepsilon/2)\right) \) is dense in \( U \).

Lemma 3.10 Let \( k: Y \to Y^w \) be defined by \( k(x) = f(x) \). Then, \( i \) is an embedding from \( Y \) to \( Y^w \); \( ii \) for all \( f \) in \( D \), \( f \cdot k = f \); and \( iii \) \( k(Y) \) is dense in \( Y^w \).

Proof. \( i \) By the setting, \( N_1 = N_2 \) if \( x = y \). Thus \( k \) is a well-defined and one-one. Let \( k^{-1} \) be a function from \( k(Y) \) into \( Y \) defined by \( k^{-1}(k(x)) = x \). To show the continuity of \( k \) and \( k^{-1} \), for any \( F^* \) in \( \alpha \), \( x \) is in \( k^{-1}\left((Y^w - F^*)\right) \cap k(Y) \).

iff \( b \): \( k(x) = N_1 \) is in \( (Y^w - F^*) \). By Cor. 3.5 (i), (b)

iff \( c \): \( x \) is not in \( F \). So,
\[ Y - F = k^{-1}\left((Y^w - F^*)\right) \cap k(Y) \]
i.e.,
\[ k(Y - F) = k(Y) \cap (Y^w - F^*) \]

So, \( k \) and \( k^{-1} \) are continuous. \( ii \) is obvious. \( iii \)

For any \( F^* \) in \( \alpha \) such that \( Y^w - F^* \neq \Phi \), pick a \( C \) in \( Y^w - F^* \). By Cor. 3.5 (iv), there is a \( T \) in \( C \) such that \( T \subseteq Y - F \). Pick an \( x \) in \( T \), by Cor. 3.5 (i), \( k(x) = N_1 \), which is not in \( F^* \), so \( N_1 = k(x) \) is both \( k(Y) \) and \( (Y^w - F^*) \); i.e., \( k(Y) \) \( (Y^w - F^*) \neq \Phi \). Thus, \( k(Y) \) is dense in \( Y^w \).

Let \( D^* = \{f'| f \in D\} \). Then \( D^* \subseteq \mathbb{C}(Y^w) \). Let
\[ K^* = \left(\cap_{f \in D^*} f^{-1}\left((f(x) - \varepsilon, f(x) + \varepsilon)\right)\right) \cap \left(\cap_{f \in D^*} f^{-1}\left((f(x) - \varepsilon, f(x) + \varepsilon)\right)\right) \neq \Phi \]be a closed \( C^*_{\alpha} \)-filter base on \( Y^w \) and let \( E^* \) be the basic closed \( C^*_{\alpha} \)-filter on \( Y^w \) generated by \( K^* \). Since \( k \) and \( k^{-1} \) are one-one, \( f \cdot k = f \) for all \( f \) in \( D \) and \( k(Y) \) is dense in \( Y^w \), so
\[ \bigcap_{f \in D^*} f^{-1}\left((f(x) - \varepsilon, f(x) + \varepsilon)\right) \subseteq Y - F \]for any \( H^* \in [D^*]^{\infty} \), \( H = \{f | f \in H^*\} \) (or any \( H \in [D]^{\infty} \), \( H^* = \{f | f \in H\} \) and all \( \varepsilon > 0 \). Thus,
\[ \bigcap_{f \in D^*} f^{-1}\left((f(x) - \varepsilon, f(x) + \varepsilon)\right) \neq \Phi \]
iff \[ \bigcap_{f \in D^*} f^{-1}\left((f(x) - \varepsilon, f(x) + \varepsilon)\right) \neq \Phi \]
and
\[ \bigcap_{f \in D^*} f^{-1}\left((f(x) - \varepsilon, f(x) + \varepsilon)\right) \subseteq Y - F \]for any \( H^* \in [D^*]^{\infty} \), \( H = \{f | f \in H^*\} \) (or any \( H \in [D]^{\infty} \), \( H^* = \{f | f \in H\} \) and all \( \varepsilon > 0 \). Therefore, the \( K^* \) or \( E^* \) defined as above is well-defined, so is \( K \) or \( E \) defined as in Section 2 well-defined and vice versa.

Lemma 3.11 Let \( \mathcal{E} \) be a basic closed \( C^*_{\alpha} \)-filter on \( Y \) defined as in Section 2. If \( \mathcal{E} \) converges to a point in \( Y \), then \( i \) \( x = f(x) \) for all \( f \in D \); i.e., \( \mathcal{E} = \mathcal{E}_x \), (ii) \( V \) is an open nhood base at \( x \in Y \)
\[ V^* = \left(\bigcap_{f \in D^*} f^{-1}\left((f(x) - \varepsilon, f(x) + \varepsilon)\right)\right) \cap \left(\bigcap_{f \in D^*} f^{-1}\left((f(x) - \varepsilon, f(x) + \varepsilon)\right)\right) \subseteq Y - F \]is an open nhood base at \( k(x) \) in \( Y^w \).

Proof. If \( \mathcal{E} \) converges to \( x \), \( i \) for each \( f \in D \), \[ x \subseteq \bigcap_{f \in D^*} f^{-1}\left((f(x) - \varepsilon, f(x) + \varepsilon)\right) \subseteq \mathcal{C} \]for all \( \varepsilon > 0 \) and \( x \neq f(x) \); i.e., \( \mathcal{E} = \mathcal{E}_x \). (ii) Since \( \mathcal{E} \) converges to \( x \) in \( Y \), for any open nhood \( U \) of \( x \), there is
\[ E = \bigcap_{f \in D^*} f^{-1}\left((f(x) - \varepsilon, f(x) + \varepsilon)\right) \subseteq \mathcal{C} \]
which is contained in \( E_x = \mathcal{E}_x \) for some \( H \in [D]^{\infty} \), \( \varepsilon > 0 \) such that \( \mathcal{E} \subseteq U \). Since \( x \) is in \( U \), \( S = \bigcap_{f \in D^*} f^{-1}\left((f(x) - \varepsilon, f(x) + \varepsilon)\right) \subseteq \mathcal{C} \)
and \( S \) is in \( V \), thus \( V \) is an open nhood base at \( x \); (iii): For any \( f^* \) in \( V \), \( f_0 \) is not in \( F^* \), by Cor. 3.5 (i), \( x \) is not in \( F \), and by (ii) of Lemma 3.11 above, \( x \) is in
\[ O = \bigcap_{f \in D^*} f^{-1}\left((f(x) - \varepsilon, f(x) + \varepsilon)\right) \subseteq Y - F \]
for some $H \in [D]^*\omega, \delta > 0$. Since

$$x \in P = \cap_{f \in H} f^{-1}\left(\left[ f(x) - \delta, f(x) + \delta \right]\right) \in \mathbb{N}_e,$$

Cor. 3.5 (i), Lemmas 3.6 (ii) and 3.8 (i) imply that

$$N_e \in P \in \cap_{f \in H} f^{-1}\left(\left( f(x) - \delta, f(x) + \delta \right) \right) \Rightarrow \mathbb{N}_e \in \alpha,$$

where $H = \{ f^* | f \in H \}$. We claim that $T \subseteq Y^* - F*$.

For any $C \in \mathbb{N}_e$, let $s = f^{-1}\left( f(x) - \delta, f(x) + \delta \right)$ for all $f \in H$. Pick a $\rho > 0$ such that $[s]_\rho \in C$, for all $f \in H$, then

$$L = \cap_{f \in H} f^{-1}\left( f(x) - \delta, f(x) + \delta \right)$$

is a filter base in $Y^* - F^*$. So $k(x) \in T \subseteq Y^* - F^*$.

Thus $V^*_k(k)$ is an open nhood base at $k(x)$.

**Lemma 3.12** Let $E$ be a basic $C^*_b$-filter on $Y$ defined as in Section 2. If $E$ does not converge in $Y$,

$$V^*_k = \{ f^{-1}\left( f(x) - \delta, f(x) + \delta \right) | f \in D^* \},$$

then $E^*$ converges to $Y^* - F^*$.

For any $F^* \in \tau$ such that $E \subseteq Y^* - F^*$, by Cor. 3.5 (iv) there exists a

$$E = \cap_{f \in H} f^{-1}\left( [f(x) - \delta, f(x) + \delta] \right) \in \mathbb{N}_e$$

for some $H \in [D]^*\omega, \delta > 0$ such that $E \subseteq Y - F$. For $H = \{ f^* | f \in H \}$, let

$$U = \cap_{f \in H} f^{-1}\left( [f(x) - \delta, f(x) + \delta] \right),$$

then $E \subseteq U \in V^*$. We claim that $U \subseteq Y^* - F^*$. For any $E^* \in U$, let $f^*(E) = t_f$ for each $f^* \in H^*$. Then for each $f$ in $H$, $t_f$ is in

$$(t_f - \delta, t_f + \delta)$$

for all $f \in H$. Then

$$L = \cap_{f \in H} f^{-1}\left( [t_f - \delta, t_f + \delta] \right) \subseteq E \subseteq Y - F.$$
For a nonempty closed set \( F \) in \( X \), (i) \( x \) is in \( F \) if \( E_x \) is in \( F^* \); (ii) \( F \) is \( X \) if \( F^* = X^* \); (iii) for each \( E \) in \( X^* \), \( F \) is in \( E \) implying \( E \) is in \( F^* \); (iv) \( E \in X^* - F^* \) \( \Leftrightarrow \) there is a \( S \) in \( E \) such that \( S \subset X - F \).

**Proof.** (i) (\( \Leftarrow \)) If \( E_x \) is in \( F^* \), then

\[
F \cap f^{-1}\left( (f(x) - \varepsilon, f(x) + \varepsilon) \right) = \emptyset
\]

and for all \( f \) in \( D \), \( \varepsilon > 0 \). Since \( V_x \) is a nhhood base at \( x \), thus \( X \) is a cluster point of \( F \), so \( X \) is in \( F \). (i) implying (ii), (iii) and (iv) are obvious.

**Proof.** (ii) For any \( F \) and \( E \) in \( X \),

\[
(i) \quad E \subseteq F \Leftrightarrow E^* \subseteq F^* ;
(ii) \quad (E \cap F)^* \subseteq E^* \cap F^* ;
(iii) \quad (E \cup F)^* = (E^* \cup F^*) .
\]

**Prop.** 3.7 **Theorem 3.8** \( g^*(E) \) is \( f(x) \) for all \( f \) in \( D \) and all \( x \) in \( X \).

**Proof.** (i) \( f^{-1}\left( [r - \delta, r + \delta] \right) \subseteq f^{-1}\left( [r - \varepsilon, r + \varepsilon] \right) \)

and (ii) \( f^{-1}\left( [r - \varepsilon, r + \varepsilon] \right) \subseteq f^{-1}\left( [r - \varepsilon, r + \varepsilon] \right) \)

for any \( \varepsilon > 0 \).

**Prop.** 3.9 **Theorem 3.10** Let \( k : X \rightarrow X^* \) be defined by \( k(x) = E_x \). Then, (i) \( k \) is an embedding from \( X \) into \( X^* \); (ii) \( f^* k = f \) for all \( f \) in \( D \); and (iii) \( k \) is dense in \( X^* \).

**Theorem 4.17** \( (X^*, k) \) is a Hausdorff compactification of \( X \).

**Proof.** By 4.15.10 (i) and (iii), 4.15.14 and Lemma 4.16, \( (X^*, k) \) is a Hausdorff compactification of \( X \).

**5. The Homeomorphism between \((X^*, \kappa)\) and \((Z, h)\)**

Let \( (Z, h) \) be an arbitrary Hausdorff compactification of \( X \), then \( X \) is a Tychonoff space. Let \( \partial D \) denote \( C(Z) \) which is the family of real continuous functions on \( Z \), and let \( D = \{ f | f = 0, f \circ h, f \in \partial D \} \). Then \( D \) is a subset of \( C(X) \) such that \( D \) separates points of \( X \), the topology on \( X \) is the weak topology induced by \( D \) and \( D \) contains a non-constant function.

Let \( (X^*, k) \) be the Hausdorff compactification of \( X \) obtained by the process in Section 4 and \( D \) is defined as above. For each basic closed \( C_{\partial D} \)-filter \( E \) in \( X^* \), let \( E \) be generated by

\[
K = \left\{ \bigcap_{f \in D} f^{-1}\left( [r_j - \varepsilon, r_j + \varepsilon] \right) \mid \bigcap_{f \in D} f^{-1}\left( [r_j - \varepsilon, r_j + \varepsilon] \right) \neq \emptyset \right\}
\]

for any \( H \in [\partial D]^{\omega} \), \( \varepsilon > 0 \).
\[ \mathcal{K} = \left\{ f \circ h \circ f^{-1} \left( [r_j - \varepsilon, r_j + \varepsilon] \right) \mid f \circ h \circ f^{-1} \left( [r_j - \varepsilon, r_j + \varepsilon] \right) \neq \emptyset \right\} \]

and let \( h^{-1} \) be the function from \( h(\mathcal{X}) \) to \( \mathcal{X} \) defined by \( h^{-1}(h(x)) = x \). Since \( h \) and \( h^{-1} \) are one-one, \( f = f \circ h \) and \( h(\mathcal{X}) \) is dense in \( \mathcal{X} \), similar to the arguments in the paragraphs prior to Lemma 3.11, we have that

\[ \bigcap_{f \in h^{-1}} f^{-1} \left( [r_j - \varepsilon, r_j + \varepsilon] \right) \neq \emptyset \text{ iff } \bigcap_{f \in h^{-1}} f^{-1} \left( [r_j - \varepsilon, r_j + \varepsilon] \right) \neq \emptyset \]

for any \( H \in [D]^{\omega} \) (or \( \mathcal{H} \in [D]^{\omega} \)).

is the \textbf{closed} \( C^{*} \)-filter base at \( z \). The filter \( \mathcal{Z} \), generated by \( \mathcal{K} \), is the \textbf{basic closed} \( C^{*} \)-filter at \( z \).

Since \( \mathcal{K} \) is compact Hausdorff, each \( \mathcal{E} \) on \( Z \) converges to a unique point \( z \) in \( Z \). We define \( \mathcal{T}: X^{w} \to Z \) by \( \mathcal{T}(\mathcal{E}) = z \), where \( \mathcal{E} \) is in \( X^{w} \) and \( z \) is the unique point in \( Z \) such that the basic closed \( C^{*} \)-filter \( \mathcal{E} \) on \( Z \) induced by \( \mathcal{E} \) converges to \( z \). For \( \mathcal{E}, \mathcal{E}' \in X^{w} \), let

\[ \mathcal{K}_{z} = \left\{ \bigcap_{f \in h^{-1}} f^{-1} \left( [s_j - \varepsilon, s_j + \varepsilon] \right) \bigg| \bigcap_{f \in h^{-1}} f^{-1} \left( [s_j - \varepsilon, s_j + \varepsilon] \right) \neq \emptyset \right\} \]

and similarly for \( \mathcal{K}_{z} \), such that \( \mathcal{E} \) and \( \mathcal{E}' \) are generated by \( \mathcal{K} \) and \( \mathcal{K}_{z} \), respectively. Assume that \( \mathcal{E} \) and \( \mathcal{E}' \) converge to \( z_{1} \) and \( z_{2} \) in \( Z \), respectively. Then \( \mathcal{E}_{1} \) is not equal to \( \mathcal{E}_{1} \), \( \mathcal{E}_{1} \) is not equal to \( \mathcal{E}_{2} \), and \( z_{1} \) is not equal to \( z_{2} \) is equivalent. Hence \( \mathcal{T} \) is well-defined and one-one. For each \( z \in Z \), let \( \mathcal{E}_{z} \) be the basic closed \( C^{*} \)-filter at \( z \), since \( Z \) is compact Hausdorff and

\[ \mathcal{V}_{z} = \left\{ \bigcap_{f \in h^{-1}} f^{-1} \left( [s_j - \varepsilon, s_j + \varepsilon] \right) \bigg| \bigcap_{f \in h^{-1}} f^{-1} \left( [s_j - \varepsilon, s_j + \varepsilon] \right) \neq \emptyset \right\} \]

is an open nhood base at \( z \), thus \( \mathcal{E}_{z} \) converges to \( z \). Let \( \mathcal{E}_{z} \) be the element in \( X^{w} \) induced by \( \mathcal{E}_{z} \), then \( \mathcal{T}(\mathcal{E}_{z}) = z \). Hence, \( \mathcal{T} \) is one-one and onto.

**Theorem 5.18** \( (X^{w}, k) \) is homeomorphic to \( (Z, h) \) under the mapping \( T \) such that \( T(k(x)) = h(x) \).

**Proof.** We show that \( T^{-1} \) is continuous. For each \( \mathcal{E} \) in \( F^{*} \) which is in \( r \), let \( \mathcal{E} \) be the basic closed \( C^{*} \)-filter on \( Z \) induced by \( \mathcal{E} \). If \( \mathcal{E} \) converges to \( z \) in \( Z \), \( f \circ h \circ f^{-1} \) for each \( f \) in \( D \) and

\[ \mathcal{K} = \left\{ \bigcap_{f \in h^{-1}} f^{-1} \left( [r_j - \varepsilon, r_j + \varepsilon] \right) \bigg| \bigcap_{f \in h^{-1}} f^{-1} \left( [r_j - \varepsilon, r_j + \varepsilon] \right) \neq \emptyset \right\} \]

for any \( H \in [D]^{\omega} \) (or \( \mathcal{H} \in [D]^{\omega} \)).

Then (a): \( \mathcal{E} \) is in \( F^{*} \) iff (b):

\[ F \cap \left\{ \bigcap_{f \in h^{-1}} f^{-1} \left( [r_j - \varepsilon, r_j + \varepsilon] \right) \bigg| \bigcap_{f \in h^{-1}} f^{-1} \left( [r_j - \varepsilon, r_j + \varepsilon] \right) \neq \emptyset \right\} \]

for any \( H \in [D]^{\omega} \) (or \( \mathcal{H} \in [D]^{\omega} \)).

Since \( h \) is one-one, \( f = f \circ h \) for all \( f \) in \( D \), so (b) iff (c):

\[ h(F) \cap \left\{ \bigcap_{f \in h^{-1}} f^{-1} \left( [r_j - \varepsilon, r_j + \varepsilon] \right) \bigg| \bigcap_{f \in h^{-1}} f^{-1} \left( [r_j - \varepsilon, r_j + \varepsilon] \right) \neq \emptyset \right\} \]

for any \( F \in [D]^{\omega} \) (or \( \mathcal{H} \in [D]^{\omega} \)).

and all \( \varepsilon > 0 \). Since

\[ f^{-1} \left( [r_j - \varepsilon, r_j + \varepsilon] \right) = h \left( F \cap \left\{ \bigcap_{f \in h^{-1}} f^{-1} \left( [r_j - \varepsilon, r_j + \varepsilon] \right) \bigg| \bigcap_{f \in h^{-1}} f^{-1} \left( [r_j - \varepsilon, r_j + \varepsilon] \right) \neq \emptyset \right\} \right) \]

for any \( F \in [D]^{\omega} \) (or \( \mathcal{H} \in [D]^{\omega} \)).

and all \( \varepsilon > 0 \). Since

\[ f^{-1} \left( [r_j - \varepsilon, r_j + \varepsilon] \right) = h \left( F \cap \left\{ \bigcap_{f \in h^{-1}} f^{-1} \left( [r_j - \varepsilon, r_j + \varepsilon] \right) \bigg| \bigcap_{f \in h^{-1}} f^{-1} \left( [r_j - \varepsilon, r_j + \varepsilon] \right) \neq \emptyset \right\} \right) \]

for any \( F \in [D]^{\omega} \) (or \( \mathcal{H} \in [D]^{\omega} \)).

is an arbitrary basic open nhood of \( z \) in \( Z \). So, (d) iff \( z \) is in \( \text{Cl}_Z(h(F)) \); i.e., \( \mathcal{E} \) in \( F^{*} \) if \( \mathcal{T}(z) \) is equal to \( z \) which belongs to \( \text{Cl}_Z(h(F)) \). Hence, \( T(F^{*}) = \text{Cl}_Z(h(F)) \) is closed in \( Z \) for all \( F^{*} \) in \( r \). Thus, \( T^{-1} \) is continuous. Since \( T \) is one-one, onto and both \( Z \) and \( X^{w} \) are compact Hausdorff, by Theorem 17.14 in [1, p.123], \( T \) is a homeomorphism. Finally, from the definitions of \( k \) and \( h \), it is clear that \( T(k(x)) = h(x) \) for all \( x \) in \( X \).

**Corollary 5.19** \( (\mathcal{X}, h_{\beta}) \) be the Stone-Cech compactification of a Tychonoff space \( \mathcal{X} \),

\[ D = \{ f \mid f = f \circ h \circ f \in C(\mathcal{X}) \} \]

and \( \mathcal{T}_{\beta}: X^{w} \to \beta X \) is defined similarly to \( T \) as above. Then \( (\mathcal{X}, h_{\beta}) \) is homeomorphic to \( (X^{w}, k) \) such that \( T_{\beta}(k(x)) = h(x) \).
Corollary 5.20 Let $(\gamma X, h)$ be the Wallman compactification of a normal $T_1$-space $X$.

\[ D = \{ f \mid f = \circ h \circ f, f \in \mathcal{C}(\gamma X) \} \]

and $T_x : X^w \to \gamma X$ is defined similarly to $T$ as above. Then $(\gamma X, h)$ is homeomorphic to $(X^w, k)$ such that $T_x(k(x)) = h(x)$.

REFERENCES
