Generalized Löb’s Theorem. Strong Reflection Principles and Large Cardinal Axioms

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ABSTRACT

In this article, a possible generalization of the Löb’s theorem is considered. Main result is: let \( \kappa \) be an inaccessible cardinal, then \( \neg \text{Con}(ZFC + (V = H_\kappa)) \).

Keywords: Löb’s Theorem; Second Godel Theorem; Consistency; Formal System; Uniform Reflection Principles; \( \omega \)-Model of ZFC; Standard Model of ZFC; Inaccessible Cardinal

1. Introduction

Let \( Th \) be some fixed, but unspecified, consistent formal theory.

**Theorem 1** [1]. (Löb’s Theorem).

If \( Th \vdash \exists x \text{Prov}_{Th}(x, n) \rightarrow \phi_n \) where \( x \) is the Gödel number of the proof of the formula with Gödel number \( n \), and \( \bar{n} \) is the numeral of the Gödel number of the formula \( \phi_n \), then \( Th \vdash \phi_n \). Taking into account the second Gödel theorem it is easy to be able to prove \( \forall x \exists n \text{Prov}_{Th}(x, n) \rightarrow \phi_n \), for disprovable (refutable) and undecidable formulas \( \phi_n \). Thus summarized, Löb’s theorem says that for refutable or undecidable formula \( \phi_n \), the intuition “if exists proof of \( \phi_n \) then \( \phi_n \)” is fails.

**Definition 1.** Let \( M^h_\omega \) be an \( \omega \)-model of the \( Th \). We said that, \( Th^h \) is a nice theory over \( Th \) or a nice extension of the \( Th \) iff:

1) \( Th^h \) contains \( Th \);
2) Let \( \Phi \) be any closed formula, then

\[
[Th \vdash Pr_{T_h}(\Phi)] \& [M^h_\omega \vdash \Phi]\]

implies \( Th^h \vdash \Phi \).

**Definition 2.** We said that, \( Th^h \) is a maximally nice theory over \( Th \) or a maximally nice extension of the \( Th \) iff \( Th^h \) is consistent and for any consistent nice extension \( Th' \) of the \( Th \): \( \text{Ded}(Th^h) \subseteq \text{Ded}(Th') \) implies \( \text{Ded}(Th^h) = \text{Ded}(Th') \).

**Theorem 2.** (Generalized Löb’s Theorem). Assume that 1) \( \text{Con}(Th) \) and 2) \( Th \) has an \( \omega \)-model \( M^h_\omega \). Then theory \( Th \) can be extended to a maximally consistent nice theory \( Th^h \).

2. Preliminaries

Let \( Th \) be some fixed, but unspecified, consistent formal theory. For later convenience, we assume that the encoding is done in some fixed formal theory \( S \) and that \( Th \) contains \( S \). We do not specify \( S \)—it is usually taken to be a formal system of arithmetic, although a weak set theory is often more convenient. The sense in which \( S \) is contained in \( Th \) is better exemplified than explained: If \( S \) is a formal system of arithmetic and \( Th \) is, say, \( ZFC \), then \( Th \) contains \( S \) in the sense that there is a well-known embedding, or interpretation, of \( S \) in \( Th \). Since encoding is to take place in \( S \), it will have to have a large supply of constants and closed terms to be used as codes. (e.g. in formal arithmetic, one has \( 0, 1, \ldots \) \( S \) will also have certain function symbols to be described shortly. To each formula, \( \Phi \), of the language of \( Th \) is assigned a closed term, \( [\Phi]^S \), called the code of \( \Phi \). [N.B. If \( \Phi(x) \) is a formula with a free variable \( x \), then \( [\Phi(x)]^S \) is a closed term encoding the formula \( \Phi(x) \) with \( x \) viewed as a syntactic object and not as a parameter.] Corresponding to the logical connectives and quantifiers are function symbols, \( \neg(\cdot) \), \( \text{imp}(\cdot) \), etc., such that, for all formulae \( \Phi, \Psi : S \neg \neg \left([\Phi]^S \right) = \left([\neg \Phi]^S \right) \).

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Of particular importance is the substitution operator, represented by the function symbol \( \text{sub}(\cdot, \cdot) \). For formulæ \( \Phi(x) \), terms \( t \) with codes \( [t] \):

\[
S \vdash \text{sub}\left(\left[\Phi(x)\right], [t]\right) = \left[\Phi(t)\right].
\]  

(2.1)

Iteration of the substitution operator \( \text{sub} \) allows one to define function symbols \( \text{sub}_{s}, \text{sub}_{a}, \ldots, \text{sub}_{n} \) such that

\[
S \vdash \text{sub}_{s}\left(\left[\Phi(x, x_{2}, \ldots, x_{n})\right], [t], [t_{2}], \ldots, [t_{n}]\right) = \left[\Phi(t, t_{2}, \ldots, t_{n})\right].
\]  

(2.2)

It is well known \([2,3]\) that one can also encode derivations and have a binary relation \( \text{Prov}_{\alpha}(x, y) \) (read “\( x \) proves \( y \)” or “\( x \) is a proof of \( y \)” such that for closed \( t_{1}, t_{2} : S \vdash \text{Prov}_{\alpha}(t_{1}, t_{2}) \) iff \( t_{1} \) is the code of a derivation in \( \alpha \) of the formula with code \( t_{2} \). It follows that

\[
\alpha \vdash \Phi \iff S \vdash \text{Prov}_{\alpha}(t_{1}, \left[\Phi\right]).
\]  

(2.3)

for some closed term \( t \). Thus one can define predicate \( \text{Pr}_{\alpha}(y) \):

\[
\alpha \vdash \text{Pr}_{\alpha}(y) \iff \exists x \text{Prov}_{\alpha}(x, y).
\]  

(2.4)

and therefore one obtain a predicate asserting provability.

Remark 2.1. We note that not always the case that \([2,3]\):

\[
\alpha \vdash \Phi \iff \exists x \text{Prov}_{\alpha}(x, y).
\]  

(2.5)

It is well known \([3]\) that the above encoding can be carried out in such a way that the following important conditions \( D1, D2 \) and \( D3 \) are met for all sentences \([2,3]\):

\[
\begin{align*}
D1. & \quad \alpha \vdash \Phi \text{ implies } S \vdash \text{Pr}_{\alpha}\left(\left[\Phi\right]\right), \\
D2. & \quad S \vdash \text{Pr}_{\alpha}\left(\left[\Phi\right]\right) \rightarrow \text{Pr}_{\alpha}\left(\left[\text{Pr}_{\alpha}\left(\left[\Phi\right]\right)\right]\right), \\
D3. & \quad S \vdash \text{Pr}_{\alpha}\left(\left[\Phi\right]\right) \land \text{Pr}_{\alpha}\left(\left[\Phi \rightarrow \Psi\right]\right) \\
& \quad \rightarrow \text{Pr}_{\alpha}\left(\left[\Psi\right]\right).
\end{align*}
\]  

(2.6)

Conditions \( D1, D2 \) and \( D3 \) are called the Derivability Conditions.

Assumption 2.1. We assume now that:

1) the language of \( \alpha \) consists of:

- numerals \( 0, 1, 2, \ldots \)
- countable set of the numerical variables: \( \{v_{0}, v_{1}, \ldots\} \)
- countable set of the function variables: \( F = \{x, y, z, X, Y, Z, \lambda, \ldots\} \)
- countable set of the \( n \)-ary function symbols: \( f_{n}^{n}, f_{n}^{a}, \ldots \)
- countable set of the \( n \)-ary relation symbols: \( R_{n}^{a}, R_{n}^{i}, \ldots \)
- connectives: \( \neg, \rightarrow \)
- quantifier: \( \forall \).

2) \( \alpha \) contains

\[ \text{Th} \triangleq ZFC + \exists(\omega - \text{model of ZFC}) \]

3) \( \alpha \) has an \( \omega \)-model \( M_{\alpha}^{\omega} \).

Theorem 2.1. (Łöb’s Theorem). Let be 1) \( \text{Con}(\alpha) \) and 2) \( \phi \) be closed. Then

\[
\alpha \vdash \text{Pr}_{\alpha}\left(\left[\phi\right]\right) \rightarrow \phi \text{ iff } \alpha \vdash \phi.
\]  

(2.7)

It well known that replacing the induction scheme in Peano arithmetic \( PA \) by the \( \omega \)-rule with the meaning “if the formula \( A(n) \) is provable for all \( n \), then the formula \( A(x) \) is provable”:

\[
A(0), A(1), \ldots, A(n), \ldots, \quad \forall x A(x),
\]  

(2.8)

leads to complete and sound system \( PA_{\omega} \) where each true arithmetical statement is provable. S. Feferman showed that an equivalent formal system \( Th^{\omega} \) can be obtained by erecting on \( Th = PA \) a transfinite progression of formal systems \( PA_{\omega} \) according to the following scheme

\[
\begin{align*}
PA_{0} & = PA, \\
PA_{\omega+1} & = PA + \left\{ \forall x \text{Pr}_{\alpha}\left(\left[A(x)\right]\right) \rightarrow \forall x A(x) \right\}, \\
PA_{\omega} & = \bigcup_{\lambda < \omega} PA_{\lambda},
\end{align*}
\]  

(2.9)

where \( A(x) \) is a formula with one free variable and \( \lambda \) is a limit ordinal. Then \( Th = \bigcup_{\alpha \in \omega} PA_{\alpha} \) being Kleene’s system of ordinal notations, is equivalent to \( Th^{\omega} = PA_{\omega} \). It is easy to see that \( Th^{\omega} \) is a maximally nice extension of the \( PA \).

3. Generalized Łöb’s Theorem

Definition 3.1. An \( \alpha \)-wff \( \Phi \) (well-formed formula \( \Phi \) ) is closed i.e., \( \Phi \) is a \( \alpha \)-sentence iff it has no free variables; a wff \( \Psi \) is open if it has free variables. We’ll use the slang “\( k \)-place open wff” to mean a wff with \( k \) distinct free variables. Given a model \( M_{\alpha}^{\omega} \) of the \( \alpha \) and a \( \alpha \)-sentence \( \Phi \), we assume known the meaning of \( M^{\omega} \models \Phi \) —i.e. \( \Phi \) is true in \( M_{\alpha}^{\omega} \), (see for example [4-6]).

Definition 3.2. Let \( M_{\alpha}^{Th} \) be an \( \omega \)-model of the \( \alpha \).

We said that, \( Th^{\omega} \) is a nice theory over \( Th \) or a nice extension of the \( Th \) iff:

1) \( Th^{\omega} \) contains \( Th \);

2) Let \( \Phi \) be any closed formula, then

\[
\left[ Th \vdash \text{Pr}_{\alpha}\left(\left[\Phi\right]\right) \right] \land \left[ M_{\alpha}^{Th} \models \Phi \right]
\]

implies \( Th^{\omega} \models \Phi \).

Definition 3.3. We said that \( Th^{\omega} \) is a maximally nice theory over \( Th \) or a maximally nice extension of the \( Th \) iff \( Th^{\omega} \) is consistent and for any consistent nice exten-
sion \( Th' \) of the \( Th: \) \( \text{Ded}(Th') \subseteq \text{Ded}(Th) \) implies \( \text{Ded}(Th') = \text{Ded}(Th) \).

**Lemma 3.1.** Assume that: 1) \( \text{Con}(Th) \); and 2) \( Th \vdash \text{Pr}_{Th}(\phi^+ \phi) \), where \( \phi \) is a closed formula. Then
\( Th \nvdash \text{Pr}_{Th}(\neg \phi) \).

Proof. Let \( \text{Con}_{Th}(\phi) \) be the formula
\[
\text{Con}_{Th}(\phi) \triangleq \forall t_1 \forall t_2 \left[ \text{Prov}_{Th}(t_1, \phi^+) \land \text{Prov}_{Th}(t_2, \neg \phi) \right] \implies \neg \exists t_1 \exists t_2 \left[ \text{Prov}_{Th}(t_1, \phi^+) \land \text{Prov}_{Th}(t_2, \neg \phi) \right],
\]
where \( t_1, t_2 \) is a closed term. We note that under canonical observation, one obtains
\( Th + \text{Con}(Th) \vdash \text{Con}_{Th}(\phi) \) for any closed wff \( \phi \).

Suppose that \( Th \vdash \text{Pr}_{Th}(\neg \phi) \), then assumption (ii) gives
\[
Th \vdash \text{Pr}_{Th}(\neg \phi) \land \text{Pr}_{Th}(\neg \phi).
\]
From (3.1) and (3.2) one obtains
\[
\exists t_1 \exists t_2 \left[ \text{Prov}_{Th}(t_1, \phi^+) \land \text{Prov}_{Th}(t_2, \neg \phi) \right].
\]
But the Formula (3.3) contradicts the Formula (3.1). Therefore: \( Th \nvdash \text{Pr}_{Th}(\neg \phi) \).

**Lemma 3.2.** Assume that: 1) \( \text{Con}(Th) \); and 2) \( Th \vdash \text{Pr}_{Th}(\neg \phi) \), where \( \phi \) is a closed formula. Then
\( Th \nvdash \text{Pr}_{Th}(\phi) \).

**Theorem 3.1.** [7,8]. (Generalized L"ob’s Theorem). Assume that: \( \text{Con}(Th) \). Then theory \( Th \) can be extended to a maximally consistent nice theory \( Th^* \) over \( Th \).

Proof. Let \( \phi_1, \ldots, \phi_n, \ldots \) be an enumeration of all wff’s of the theory \( Th \) (this can be achieved if the set of propositional variables can be enumerated). Define a chain \( \nu \triangleq \{ Th \in \mathbb{N}, Th = Th \text{ of consistent theories inductively as follows: assume that theory } Th \text{ is defined.} \}

1) Suppose that a statement (3.4) is satisfied
\[
Th \vdash \text{Pr}_{Th}(\phi_i^+) \quad \text{and} \quad [Th \nvdash \phi_i \land \{M^Th \vdash \phi_i\}].
\]
Then we define theory \( Th_{i+1} \) as follows:
\[
Th_{i+1} \triangleq Th_i \cup \{\phi_i\}.
\]
2) Suppose that a statement (3.5) is satisfied
\[
Th \vdash \text{Pr}_{Th}(\neg \phi_i^+) \quad \text{and} \quad [Th \nvdash \phi_i \land \{M^Th \vdash \phi_i\}].
\]
Then we define theory \( Th_{i+1} \) as follows:
\[
Th_{i+1} \triangleq Th_i \cup \{\neg \phi_i\}.
\]
3) Suppose that a statement (3.6) is satisfied
\[
Th \vdash \text{Pr}_{Th}(\neg \phi_i) \quad \text{and} \quad [Th \nvdash \phi_i \land \{M^Th \vdash \phi_i\}].
\]
Then we define theory \( Th_{i+1} \) as follows:
\[
Th_{i+1} \triangleq Th_i \cup \{\neg \phi_i\}.
\]
4) Suppose that a statement (3.7) is satisfied
\[
Th \vdash \text{Pr}_{Th}(\neg \phi_i^+) \quad \text{and} \quad [Th \nvdash \phi_i \land \{M^Th \vdash \phi_i\}].
\]
Then we define theory \( Th_{i+1} \) as follows:
\[
Th_{i+1} \triangleq Th_i \cup \{\neg \phi_i\}.
\]
First, notice that each \( Th \) is consistent. This is done by induction on \( i \) and by Lemmas 3.1-3.2. By assumption, the case is true when \( i = 1 \). Now, suppose \( Th \) is consistent. Then its deductive closure \( \text{Ded}(Th) \) is also consistent. If a statement (3.6) is satisfied i.e.,
\[
Th \vdash \text{Pr}_{Th}(\neg \phi_i^+) \quad \text{and} \quad [Th \nvdash \phi_i \land \{M^Th \vdash \phi_i\}],
\]
Then we define theory \( Th_{i+1} \) as follows:
\[
Th_{i+1} \triangleq Th_i \cup \{\neg \phi_i\}.
\]
We define now theory \( Th^* \) as follows:
\[
Th^* \triangleq \bigcup_{i \in \mathbb{N}} Th_i.
\]
is the union of a chain of consistent sets. To see that
\( \text{Ded}(T^h) \) is maximal, pick any wff \( \Phi \). Then \( \Phi \) is
some \( \Phi_i \) in the enumerated list of all wff’s. Therefore
for any \( \Phi \) such that \( T^h \vdash \text{Pr}_{T^h}(\Phi) \) or
\( T^h \vdash \text{Pr}_{T^h}(\neg \Phi) \), either \( \Phi \in T^h \) or \( \neg \Phi \in T^h \).

Since \( \text{Ded}(T_{\text{Con}}) \subset \text{Ded}(T^h) \), we have
\( \Phi \in \text{Ded}(T^h) \) or \( \neg \Phi \in \text{Ded}(T^h) \), which implies that
\( \text{Ded}(T^h) \) is maximally consistent nice extension of the
\( \text{Ded}(T) \).

**Lemma 3.3.** The union of a chain \( \varphi = \{ \Gamma_i \mid i \in \mathbb{N} \} \) of
the consistent sets \( \Gamma_i \), ordered by \( \subseteq \), is consistent.

**Definition 3.4.** (a) Assume that a theory \( T^h \) has an \( \omega \)-model \( M^h_\omega \) and \( \Phi \) is a \( \omega \)-sentence. Let \( \Phi^h_\omega \) be a \( \omega \)-sentence \( \Phi \) with all quantifiers relativized to the \( \omega \)-model \( M^h_\omega \) [9];
(b) Assume that a theory \( T^h \) has a standard model \( SM^h \)
And \( \Phi \) is a \( T \)-sentence. Let \( \Phi^h \) be a \( T \)-sentence \( \Phi \) with all quantifiers relativized to the model \( SM^h \) [9].

**Definition 3.5.** (a) Assume that \( Th \) has an \( \omega \)-model \( M^h_\omega \) . Let \( Th^h_\omega \) be a theory \( Th \) relativized to a model \( M^h_\omega \) —i.e., any \( Th^h_\omega \)-sentence has a form \( \Phi^h_\omega \) for some \( \omega \)-sentence \( \Phi \) [9];
(b) Assume that \( Th \) has a standard model \( SM^h \). Let \( Th^h \) be a theory \( Th \) relativized to a model \( SM^h \) —i.e., any \( Th^h \)-sentence has a form \( \Phi^h \) for some \( \omega \)-sentence \( \Phi [9] \).

**Definition 3.6.** (a) For a given \( \omega \)-model \( M^h_\omega \) of the
\( Th \) and for any \( Th^h \)-sentence \( \Phi^h \), we define
\( M^h_\omega \models \Phi^h \) such that the equivalence:
\[
M^h_\omega \models \Phi^h \Leftrightarrow \Phi^h \models \Phi^h \wedge
\left( Th^h \vdash \text{Pr}_{Th^h}(\Phi^h) \Rightarrow T^h \models \Phi^h \right).
\]
where \( T^h \models \Phi^h \) is satisfied;
(b) For a given standard model \( SM^h \) of the \( Th \) and
for any \( Th^h \)-sentence \( \Phi^h \), we define
\( SM^h \models \Phi^h \) such that the equivalence:
\[
SM^h \models \Phi^h \Leftrightarrow \Phi^h \models \Phi^h \wedge
\left( Th^h \vdash \text{Pr}_{Th^h}(\Phi^h) \Rightarrow T^h \models \Phi^h \right).
\]
where \( T^h \models \Phi^h \) is satisfied.

**Theorem 3.2. (Strong Reflection Principle).** Assume that:
1) \( \text{Con}(Th) \), 2) \( Th \) has an \( \omega \)-model \( M^h_\omega \) and 3) \( M^h_\omega \models \Phi^h \). Then
\[
Th^h \models \text{Pr}_{Th^h}(\Phi^h) \iff Th^h \models \Phi^h.
\]

Proof. The one direction is obvious. For the other, assume that
\[
Th^h \models \text{Pr}_{Th^h}(\Phi^h) \iff Th^h \models \Phi^h.
\]

\[
Th^h \not\models \Phi^h \text{ and } Th^h \not\models \neg \Phi^h \text{. Then}
\]
\[
Th^h \not\models \text{Pr}_{Th^h}(\neg \Phi^h) \iff Th^h \models \neg \Phi^h.
\]
(b) Assume that $\text{Th}$ has a strong standard model $SM_{\omega}^{Th}$ and $\Phi$ is a $\text{Th}$-sentence. Let $\Phi_{\omega}^{Th}$ be a $\text{Th}$-sentence $\Phi$ with all quantifiers relativized to $SM_{\omega}^{Th}$.

**Definition 3.10.** Assume that $\text{Th}$ has a strong $\omega$-model $M_{\omega}^{Th}$. Let $\text{Th}_{\omega}^{\Phi}$, be a theory $\text{Th}$ relativised to $M_{\omega}^{Th}$, i.e., any $\text{Th}_{\omega}^{\Phi}$-sentence has the form $\Phi_{\omega}^{Th}$ for some $\text{Th}$-sentence $\Phi$.

Let $\text{Th}$ be a theory such that Assumption 1.1 is satisfied. Let $\text{Con}(\text{Th}; M_{\omega}^{Th})$ be a sentence in $\text{Th}$ asserting that $\text{Th}$ has a strong $\omega$-model $M_{\omega}^{Th}$. Let $\text{Th}^*$ be a theory: $\text{Th}^* = \text{Th} + \text{Con}(\text{Th}; M_{\omega}^{Th})$.

Let $\text{Con}(\text{Th}^*; M_{\omega}^{Th})$ be a sentence in $\text{Th}^*$ asserting that $\text{Th}^*$ has a strong $\omega$-model $M_{\omega}^{Th^*}$. We assume throughout that $\text{Th}^*$ is a strongly consistent, i.e., a sentence $\text{Con}(\text{Th}^*; M_{\omega}^{Th^*})$ is true in any $\omega$-model $M_{\omega}^{Th}$ of the $\text{Th}^*$. Note that:

$$\text{Con}(\text{Th}; M_{\omega}^{Th^*}) \iff \text{Con}(\text{Th}_{\omega}^{\Phi})$$  \hspace{1cm} (3.16)

where a sentence $\Phi_{\omega}^{Th}$ is refutable in $\text{Th}_{\omega}^{\Phi}$ and

$$\text{Con}(\text{Th}^*; M_{\omega}^{Th^*}) \iff \text{Con}(\text{Th}_{\omega}^{\Phi^*})$$  \hspace{1cm} (3.17)

where a sentence $\Phi_{\omega}^{Th^*}$ is refutable in $\text{Th}_{\omega}^{\Phi^*}$.

**Lemma 3.4.** $\text{Th}^*$ is a strongly consistent.

Proof. Assume that $\text{Th}^*$ is no strongly consistent, that is, has no any strong $\omega$-model $M_{\omega}^{Th^*}$. This means that there is no any $\omega$-model $M_{\omega}^{Th^*}$ of the $\text{Th}^*$ in which $\text{Con}(\text{Th}; M_{\omega}^{Th^*})$ is true and therefore from Formula (3.16) one obtain, that a formula $\neg \text{Con}(\text{Th}_{\omega}^{\Phi})$ is true in any $\omega$-model $M_{\omega}^{Th}$ of the $\text{Th}$. So from Formula (3.16) by using a Strong Reflection Principle (Theorem 3.2) one obtain that a sentence $\neg \text{Con}(\text{Th}; M_{\omega}^{Th^*})$ is provable in $\text{Th}_\omega$, i.e. $\text{Th}_\omega \vdash \neg \text{Con}(\text{Th}; M_{\omega}^{Th^*})$. But a sentence $\neg \text{Con}(\text{Th}; M_{\omega}^{Th^*})$ contrary to the assumption that $\text{Th}^*$ is a strongly consistent. This contradiction completed the proof.

**Theorem 3.3.** $\text{Th}$ has no any strong $\omega$-model $M_{\omega}^{Th^*}$. Proof. By Lemma 3.4 and Formula (3.17) one obtain that $\text{Th}_\omega \vdash \neg \text{Con}(\text{Th}; M_{\omega}^{Th^*})$. But Godel’s Second Incompleteness Theorem applied to $\text{Th}_\omega^{\Phi^*}$ asserts that $\text{Con}(\text{Th}_{\omega}^{\Phi^*})$ is unprovable in $\text{Th}_\omega$. This contradiction completed the proof.

**Theorem 3.4.** $\text{ZFC}$ has no any strong $\omega$-model $M_{\omega}^{\text{ZFC}}$. Proof. Immediately follows from Theorem 3.3 and definitions.

**Theorem 3.5.** $\text{ZFC}$ has no any strong standard model. $SM_{\omega}^{\text{ZFC}}$. Proof. Immediately follows from Theorem 3.4 and definitions.

**Theorem 3.6.** $\text{ZFC} + \text{Con}(\text{ZFC})$ is incompatible with all the usual large cardinal axioms [10,11] which imply the existence of a standard model of $\text{ZFC}$. Therefore Theorem 3.7 immediately follows from Theorem 3.6.

**Theorem 3.7.** Let $\kappa$ be an inaccessible cardinal. Then $\neg \text{Con}(\text{ZFC} + \exists \kappa)$. Proof. Let $H_\kappa$ be a set of all sets having hereditary size less then $\kappa$. It easy to see that $H_\kappa$ forms a strong standard model of $\text{ZFC}$. Therefore Theorem 3.7 immediately follows from Theorem 3.6.

**4. Conclusion**

In this paper we proved so-called strong reflection principles corresponding to formal theories $\text{Th}$ which has $\omega$-models $M_{\omega}^{Th}$ and in particular to formal theories $\text{Th}$, which has a standard models $SM_{\omega}^{Th}$. The assumption that there exists a standard model of $\text{Th}$ is stronger than the assumption that there exists a model of $\text{Th}$. This paper examined some specified classes of the standard models of $\text{ZFC}$ so-called strong standard models of $\text{ZFC}$. Such models correspond to large cardinals axioms. In particular we proved that theory $\text{ZFC} + \text{Con}(\text{ZFC})$ is incompatible with existence of any inaccessible cardinal $\kappa$. Note that the statement: $\text{Con}(\text{ZFC} + \exists \kappa)$ some inaccessible cardinal $\kappa$ is $\text{T}_\omega$. Thus Theorem 3.6 asserts there exist numerical counterexample which would imply that a specific polynomial equation has at least one integer root.

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