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Some Equivalences and Dualities via Static Modules

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Abstract

For a ring A, an extension ring B, a fixed right A-module M, the endomorphism ring D formed by M, the endomorphism ring E formed by $M \otimes_A B$, and the endomorphism ring F formed by $Hom_A(B,M)$, we present equivalences and dualities between subcategories of B-modules which are finitely cogenerated injective as A-modules and E-modules and F-modules which are finitely generated projective as D-modules.

Keywords: Static Modules, Finitely Cogenerated Injective Modules, Finitely Generated Projective Modules

1. Introduction

Let A be any ring, M a fixed right A-module and $D = End_A(M)$. An object V in *Mod-A* (respectively W in *Mod-D*) is said to be M-static (respectively M-adstatic), in case V remains invariant under the composite covariant functor $Hom_A(M,-)\otimes_D M$ (respectively $Hom_A(M,-\otimes_D M)$. We denote by *Stat*(M) and *Adst*(M) the classes of all static and adstatic objects of *Mod-A* and *Mod-D*, respectively. We will use the notation *FCI-A* and *FGP-D* (*D-FGP*) for the classes of all finitely cogenerated injective and finitely generated projective objects in *Mod-A*, *Mod-D* (*D-Mod*) respectively.

It is clear that

$$Hom_A(M,-): Stat(M) \rightleftarrows Adstat(M): -\otimes_D M$$

is an equivalence and a special case of this equivalence is an equivalence between *Mod*(A:*weak* M) and *Mod*(D: *weak* D) where *Mod*(A:*weak* M) is the full additive subcategory of all those objects which weakly divide (*i.e.* divide some finite direct sum of copies of) M in *Mod-A* and *Mod*(D:*weak* D) = *FGP-D*.

In [1], which is an extension of the work of Xue in [2], it is proved that M is a finitely cogenerated injective cogenerator of *Mod-A* iff *Mod*(A:*weak* M) = *FCI-A* and that this fact is equivalent to the existence of an equivalence or a duality between *FCI-A* and *FGP-D* or *FCI-A* and *D-FGP*, respectively.

Let B be another ring and $\alpha:A\to B$ a ring homomorphism. Suppose that $E = End_B(M \otimes_A B)$. Then the ring homomorphism $\sigma: D \to E$ defined via $\sigma(d)(m\otimes b) = d(m)\otimes b$, for all $(d,m,b)\in D \times M \times B$ is clearly identity preserving. Similarly if $F = End_B(Hom_A(B,M))$, then the ring homomorphism $\alpha: D \to F$ defined via $\alpha(\mathrm{d})(f) = d \circ f$, for all $d\in D$ and $f \in Hom_A(B,M)$, is clearly identity preserving.

Let us set

Mod(B:*weak* M) = {$V\in$*Mod-B*:V_A *weakly divides* M *in Mod-A*},

Mod(B:*FCI-A*) = {$V\in$*Mod-B*: $V_A\in$*FCI-A* },

Mod(E:*FGP-D*) = {$W\in$*Mod-E*: $W_D\in$*FGP-D*},

Mod(E:*D-FGP*) = {$W\in$*E-Mod*: $W_D\in$*D-FGP*},

Mod(F:*FGP-D*) = {$W\in$*Mod-F*: $W_D\in$*FGP-D*},

Mod(F:*D-FGP*) = {$W\in$*F-Mod*: $W_D\in$*D-FGP*}.

With the assumption that $M \otimes_A B$ is M-static in *Mod-A*, an equivalence between subcategories *Mod*(B: *weak* M) and *Mod*(E:*FGP-D*) of *Mod-B* and *Mod-E*, respectively, is established in [3]. This in fact is a generalization of the work of Cline [4] and Dade [5] on stable Clifford theory. In this work using the same assumption it is proved that *Mod*(B:*FCI-A*) and *Mod*(E:*FGP-D*) are equivalent and with some additional assumption dualities between *Mod*(B:*FCI-A*) and both *Mod*(F:*D-FGP*) and *Mod*(E:*D-FGP*) are deduced.

We assume that the rings are associative with identity, the ring homomorphisms are identity preserving, all (left, right) modules are unital, and all subcategories are full and additive.

2. Equivalences and Dualities

We fix here all the notations and terminology from the previous section.

The following Theorem is proved in [1], Theorem 3.

Theorem 2.1 *The following statements are equivalent for a right A-module M.*

1) *M is a finitely cogenerated injective cogenerator in Mod-A,*

2) *Mod-(A:weak M) = FCI-A,*

3) $Hom_A(M,-): FCI-A \rightleftarrows FGP-D: -\otimes_D M$ *define an equivalence,*

4) $Hom_A(-,M): FCI-A \rightleftarrows D-FGP: Hom_D(-,M)$ *define a duality.*

Corollary 2.2 *Let M be a right A-module. If M is a finitely cogenerated injective cogenerator in Mod-A, then Mod(B:weak M) = Mod(B:FCI-A).*

Proof. By Theorem 2.1-(2) and the definitions of the subcategories *Mod*(*B*:*FCI-A*) and *Mod*(*B*:*weak M*), the result follows. ☐

Remark 2.3 *From now on we assume that* $M\otimes_A B$ *is M-static as an A-module. With this assumption, one can deduce that* $M\otimes_A B \cong E\otimes_D M$ *in D-Mod-A. The details of internal maps of this isomorphism and their proofs can be seen in* [3].

The following Theorem is proved in [3], Theorem 5.5.

Theorem 2.4 *For a right A-module M, the restrictions of the additive functors*

$$Hom_B(M\otimes_A B,-) \text{ and } -\otimes_E(M\otimes_A B)$$

form an equivalence of the full additive subcategories Mod(B:weak M) and Mod(E:FGP-D) of Mod-B and Mod-E, respectively.

Proposition 2.5 *For a right A-module M, the following statements are equivalent.*

1) *Mod(B:weak M) = Mod(B:FCI-A),*

2) *The restrictions of the additive functors* $Hom_B(M\otimes_A B,-)$ and $-\otimes_E(M\otimes_A B)$ *form an equivalence of the full additive subcategories Mod(B:FCI-A) and Mod(E:FGP-D) of Mod-B and Mod-E, respectively.*

Proof. (1)⇒(2) By Theorem 2.4.

(2)⇒(1) By Theorem 2.4, it is clear that

$$Mod(B: weak\ M) = \mathrm{Im}\left(-\otimes_E (M\otimes_A B)\big|_{Mod(E:FGP-D)}\right)$$
$$= Mod(B: FCI-A)$$

Remark 2.6 *Recall that* $E = End_B(M\otimes_A B)$ *and* $F = End_B(Hom_A(B,M))$

1) *For the following one can see* [6], *Lemma* 3.2. *We have*

$$Hom_A(Hom_A(B,M),M) \cong F,$$

and in this case

$$Hom_D(Hom_A(Hom_A(B,M),M),M) \cong Hom_D(F,M)$$

So if $Hom_A(B,M)$ *is M-reflexive, then*

$$Hom_A(B,M) \cong Hom_D(F,M),$$

therefore

$$Hom_F(W, Hom_A(B,M)) \cong Hom_F(W, Hom_D(F,M)).$$

But by the adjoint associativity theorem we have

$$Hom_F(W, Hom_D(F,M)) \cong Hom_D(W,M)$$

thus

$$Hom_F(W, Hom_A(B,M)) \cong Hom_D(W,M)$$

2) *Suppose that* $(M\otimes_A B) \cong Hom_A(B,M)$ *as B-modules and hence as A-modules. Then we have the following sequence of D-isomorphisms:*

$$\begin{aligned} E &\cong Hom_B((M\otimes_A B),(M\otimes_A B)) \\ &\cong Hom_B((M\otimes_A B), Hom_A(B,M)) \\ &\cong Hom_A((M\otimes_A B)\otimes_B B, M) \\ &\cong Hom_A((M\otimes_A B), M). \end{aligned}$$

This means that

$$Hom_D(E,M) \cong Hom_D(Hom_A((M\otimes_A B),M),M)$$

in E-Mod-A. Further, if $(M\otimes_A B)$ *is M-reflexive, then*

$$Hom_D(E,M) \cong M\otimes_A B \cong E\otimes_D M$$

in E-Mod-A.

3) *On the other hand, according to the assumption that* $(M\otimes_A B) \cong Hom_A(B,M)$, *it is easy to see that* $E \cong F$ *in D-Mod.*

The following Proposition is proved in [6, Proposition 3.3].

Proposition 2.7 *Let M be a right A-module. If* $Hom_A(B,M)$*-reflexive is M-reflexive, then*

1) V_B *is* $Hom_A(B,M)$*-reflexive if and only if* V_A *is M-reflexive.*

2) ${}_F W$ *is* $Hom_A(B,M)$*-reflexive if and only if* ${}_D W$ *is M-reflexive.*

Theorem 2.8 *Let M be a right A-module. Let* $Hom_A(B,M)$ *be M-reflexive. Then the restrictions of the additive functors* $Hom_B(-, Hom_A(B,M))$ *and* $Hom_F(-, Hom_A(B,M))$ *form a duality of the full additive subcategories Mod(B:weak M) and Mod(F:D-FGP) of Mod-B and F-Mod, respectively.*

Proof. Let $V\in$ *Mod*(*B*:*weak M*), hence $V\in$ *Mod*(*A*:*weak M*). This means that $M^n \cong V\oplus U$, for some *U* and some positive integer *n*. Now we have

$$D^n \cong (M^n)^* \cong V^* \oplus U^*,$$

where $(-)^* = Hom_A(-,M)$. Thus $V^* = Hom_A(V,M) \in$ *D-FGP*. We have the following isomorphisms,

$$\begin{aligned} Hom_B(V, Hom_A(B,M)) &\cong Hom_A((V\otimes_B B),M) \\ &\cong Hom_A(V,M) \end{aligned}$$

Hence $Hom_B\left(V, Hom_A\left(B,M\right)\right) \in$ *D-FGP* and so belongs to *Mod*(*F*:*D-FGP*). Let $W \in$ *Mod*(*F*:*D-FGP*). Then $W \in$ *D-FGP* and so $D^n \cong W \oplus Q$, for some *Q* in *D-Mod* and some positive integer *n*. Now we have

$$W^* \oplus Q^* \cong \left(D^n\right)^* \cong M^n,$$

where $(-)^* = Hom_D\left(-,M\right)$. Therefore $W^* = Hom_D\left(W,M\right)$ weakly divides *M* in *Mod-A*. *i.e.* $W^* \in$ *Mod*(*A*:*weak M*) and hence it belongs to *Mod*(*B*: *weak M*). Now as we can see from Remark 2.6-(1),

$$Hom_F\left(W, Hom_A\left(B,M\right)\right) \cong Hom_D\left(W,M\right)$$

We deduce that $Hom_F\left(W, Hom_A\left(B,M\right)\right) \in$ *Mod*(*B*: *weak M*).

According to the fact that M^n and D^n are *M*-reflexive, it is clear that for every *V* in *Mod*(*A*:*weak M*), *V* is *M*-reflexive and for every *W* in D-*FGP*, *W* is *M*-reflexive. Applying Proposition 2.7, since $Hom_A\left(B,M\right)$ is *M*-reflexive, every *V* in *Mod*(*B*:*weak M*) = *Mod*(*B*:*FCI-A*) is $Hom_A\left(B,M\right)$-reflexive, and every *W* in *Mod*(*F*:*D-FGP*) is $Hom_A\left(B,M\right)$-reflexive. □

Theorem 2.9 *Let M be a right A-module. Let* $Hom_A\left(B,M\right)$ *be M-reflexive. Then the following statements are equivalent for M.*

1) *Mod*(*B*:*weak M*) = *Mod*(*B*:*FCI-A*),

2) *The restrictions of the additive functors* $Hom_B\left(-, Hom_A\left(B,M\right)\right)$ *and* $Hom_F\left(-, Hom_A\left(B,M\right)\right)$ *form a duality of the full additive subcategories Mod* (*B*:*FCI-A*) *and Mod*(*F*:*D-FGP*) *of Mod-B and F-Mod*, *respectively.*

Proof. (1)⇒(2) By Theorem 2.8.

(2)⇒(1) Consider the following isomorphisms obtained by the adjoint associativity theorem and Remark 2.6, we have

$$Hom_B\left(V, Hom_A\left(B,M\right)\right) \cong Hom_A\left(V,M\right)$$

and

$$Hom_F\left(W, Hom_A\left(B,M\right)\right) \cong Hom_D\left(W,M\right) \quad (2)$$

Let $V \in$ *Mod*(*B*:*FCI-A*), then $V^* = Hom_B\left(V, Hom_A\left(B,M\right)\right) \in$ *Mod(F*:*D-FGP)*.

So $D^n \cong V^* \oplus Q$, for some *Q* and some positive integer *n*. Now we have the following sequence of isomorphisms,

$$\begin{aligned} M^n &\cong Hom_D\left(D^n,M\right) \\ &\cong Hom_D\left(V^* \oplus Q, M\right) \\ &\cong Hom_D\left(V^*,M\right) \oplus Hom_D\left(Q,M\right) \\ &\cong Hom_F\left(V^*, Hom_A\left(B,M\right)\right) \oplus Hom_D\left(Q,M\right) \\ &\cong V \oplus Hom_D\left(Q,M\right), \end{aligned}$$

where the fourth isomorphism is due to (2) and the fifth isomorphism is due to the fact that *V* is $Hom_A\left(B,M\right)$-reflexive. So $V \in$ *Mod*(*A*:*weak M*) and therefor it belongs to *Mod*(*B*:*weak M*).

Conversely suppose that $V \in$ *Mod*(*B*:*weak M*). This means that $M^n \cong V \oplus U$, for some *U* and some positive integer *n*. Now we have the following isomorphisms,

$$\begin{aligned} D^n &\cong Hom_A\left(M^n,M\right) \\ &\cong Hom_A\left(V \oplus U, M\right) \\ &\cong Hom_A\left(V,M\right) \oplus Hom_A\left(U,M\right) \end{aligned}$$

Thus $Hom_A\left(V,M\right) \in$ *D-FGP*. By (1) it is clear that

$$V^* = Hom_B\left(V, Hom_A\left(B,M\right)\right) \in Mod(F{:}D\text{-}FGP).$$

Therefore by the given duality

$$Hom_F\left(V^*, Hom_A\left(B,M\right)\right) \in Mod(B{:}FCI\text{-}A). \quad (3)$$

Now since $V \in$ *Mod*(*A*:*weak M*), it is *M*-reflexive, so by Proposition 2.7 it is $Hom_A\left(B,M\right)$-reflexive. Hence the result follows by (3). □

Theorem 2.10 *Let M be a right A-module. Let* $\left(M \otimes_A B\right) \cong Hom_A\left(B,M\right)$ *as B-modules and let* $M \otimes_A B$ *be M-reflexive. Then the following statements are equivalent for M.*

1) *Mod*(*B weak M*) = *Mod*(*B*:*FCI-A*),

2) *The restrictions of the additive functors* $Hom_B\left(M \otimes_A B, -\right)$ *and* $- \otimes_E \left(M \otimes_A B\right)$ *form an equivalence of the full additive subcategories Mod*(*B*:*FCI-A*) *and Mod*(*E*:*FGP-D*) *of Mod-B and Mod-E*, *respectively.*

3) *The restrictions of the additive functors* $Hom_B\left(-, M \otimes_A B\right)$ *and* $Hom_E\left(-, M \otimes_A B\right)$ *form a duality of the full additive subcategories Mod*(*B*:*FCI-A*) *and Mod*(*E*:*D-FGP*) *of Mod-B and E-Mod*, *respectively.*

Proof.

(1)⇔(2) By Proposition 2.5.

(1)⇔(3) With the assumption

$$\left(M \otimes_A B\right) \cong Hom_A\left(B,M\right),$$

This is clear from Theorem 2.9 (see Remark 2.6-(2)-(3)). □

Corollary 2.11 *Let M be a right A-module such that M is a finitely cogenerated injective cogenerator in Mod-A. Let* $\left(M \otimes_A B\right) \cong Hom_A\left(B,M\right)$ *as B-modules and let* $M \otimes_A B$ *be M-reflexive. Then the restrictions of the additive functors* $Hom_B\left(-, M \otimes_A B\right)$ *and* $Hom_E\left(-, M \otimes_A B\right)$ *form a duality of the full additive subcategories Mod*(*B*:*FCI-A*) *and Mod*(*E*:*D-FGP*) *of Mod-B and E-Mod*, *respectively.*

Corollary 2.12 *Let M be a right A-module. Let* $\left(M \otimes_A B\right) | M$ *and* $Hom_A\left(B,M\right) \cong \left(M \otimes_A B\right)$ *as D-B-bimodules. Then the following statements are equivalent for M.*

1) *Mod*(*B*:*weak M*) = *Mod*(*B*:*FCI*-*A*),

2) *The restrictions of the additive functors* $Hom_B(M \otimes_A B, -)$ *and* $- \otimes_E (M \otimes_A B)$ *form an equivalence of the full additive subcategories Mod*(*B*:*FCI*-*A*) *and Mod*(*E*:*FGP*- *D*) *of Mod-B and Mod-E, respectively.*

3) *The restrictions of the additive functors* $Hom_B(-, M \otimes_A B)$ *and* $Hom_E(-, M \otimes_A B)$ *form a duality of the full additive subcategories Mod*(*B*:*FCI*-*A*) *and Mod*(*E*:*D*-*FGP*) *of Mod-B and E-Mod, respectively.*

Proof. Since $Hom_A(B, M) \cong (M \otimes_A B)$ as *D*-*B*-bimodules, V is $Hom_A(B, M)$-reflexive if and only if V is $M \otimes_A B$-reflexive. The assumption $(M \otimes_A B) | M$, implies the fact that $(M \otimes_A B)$ is *M*-reflexive. Hence the proof follows by Theorem 2.10. □

Corollary 2.13 *Let M be a right A-module such that M is a finitely cogenerated injective cogenerator in Mod-A. Let* $(M \otimes_A B) | M$ *and* $Hom_A(B, M) \cong (M \otimes_A B)$ *as D-B-bimodules. Then the restrictions of the additive functors* $Hom_B(-, M \otimes_A B)$ *and* $Hom_E(-, M \otimes_A B)$ *form a duality of the full additive subcategories Mod*(*B*:*FCI*-*A*) *and Mod*(*E*:*D*-*FGP*) *of Mod-B and E-Mod, respectively.*

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4. References

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