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Property S[a,b]: A Direct Approach

George Nickolaos Miliaras
American University of Athens, Athens, Greece
E-mail: gmiliara@yahoo.gr, gmiliaras@aua.edu
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Abstract

In this paper we prove directly that the property $S[a,b]$, implies $[a,b)$-compact, and under certain conditions it implies $[a,b]$-compact.

Keywords: Compactness number, $[a,b]$-compact, $[a,b)$-compact, property $S[a,b]$

1. Introduction

Compactness is one of the oldest and the most famous notions in mathematical analysis and especially in topology. A partial generalization is $[a,b]$-compactness [1-8]. This has been shaped to property $S[a,b]$ by Vaughan in 1975 [8], (page 253 and 256-257) who proved that $[a,b]$-compactness is equivalent to $S[a,b]$ if $cf(b) \geq a$ (Theorem 2C) using the Corollary of Lemma 2 (pages 254-255).

In this paper we are going to prove directly something stronger, which we will need the following definitions:

Definition 1. The compactness number $Cn(X)$ of a space X is the least cardinal k such that every open cover of X has a subcover of cardinality less than k.

Definition 2. A space X is called $[a,b]$-compact ($[a,b)$-compact) if every open cover U of X with $|U| \leq b$ ($|U| < b$) has a subcover of cardinality strictly less than a.

Definition 3. A space X is said to have property $S[a,b]$ if every open cover of X of regular cardinality less than b, has a subcover of cardinality strictly less than a.

2. Main Result

Theorem. Let X have the property $S[a,b]$, then X is $[a,b)$-compact, and if b is regular or if $cf(b) \geq a$, then X is $[a,b]$-compact, furthermore either $Cn(X) > b$, or, $a = b = Cn(X)$.

Proof. We study the following three cases:

Case 1. $Cn(X) > b$

Let U_* be an open cover of X with $|U_*| = k$ be the first singular cardinal greater than a, with the property that U_* has no subcover with cardinality less than a.

If $k > b$, then clearly X is $[a,b]$-compact.

If $k = b$, then clearly X is $[a,b)$-compact.

Assume that $k < b$. Since $k < b < Cn(X)$, there exists an open cover V_* of X which $|V_*| = \lambda > k$. Then at least one $U \in U_*$ is covered by a collection $V'_* \subset V_*$ with $|V'_*| = \mu > k$, such that no subcollection of V'_* with less than μ elements can cover U. If such a U didn't exist it would contradict the hypothesis $Cn(X) > b > k$. Consider the collection of open sets $W_* = \{V_\gamma \cap U \mid V_\gamma \in V'_*\}$. Let a collection $W'_* = \{W_\gamma \mid \gamma \in k^+\}$ of k^+ elements of W_* and let W be the union of the rest of the remaining elements of W_* (if there are any left). We have $U = W \cup (\cup W'_*)$, put $U'_* = U_* - \{U\}$, then $S_* = U'_* \cup W'_* \cup \{W\}$ is an open cover of X with $|S_*| = k^+$ since $|U'_*| = k$, $|W'_*| = k^+$ and $|\{W\}| = 1$, then S_* has a subcover with cardinality less than a, since $k^+ \leq b$ is a regular cardinal and X has property $S[a,b]$. Now, since S_* refines U_*, U_* must have a subcover with cardinality less than a, thus X is $[a,b)$-compact.

Now, if b is a regular cardinal then clearly X is $[a,b]$-compact, since it is $[a,b)$-compact, and has property $S[a,b]$.

Assume that b is singular and $cf(b) \geq a$. Let $U_* = \{U_\gamma \mid \gamma < b\}$, be an open cover of X, with $|U_*| = b$, let $cf(b) = k \geq a$, choose cardinals b_β, $\beta < k$ with $\sup\{b_\beta\} = b$. For every $\beta < k$, let $V_\beta = \cup\{U_\gamma \mid \gamma < b_\beta\}$, and let $V_* = \{V_\beta \mid \beta < k\}$. Then V_* has a subcover V_* with $|V_*| = \mu < a$, since k is regular, $k \geq a$ and X has property $S[a,b]$. Let

$V_*' = \{V_\beta | \beta < \mu\}$, then $V_{\mu+1} = X$, but $V_{\mu+1} = \cup\{U_\gamma | \gamma < b_{\mu+1}\}$. Put $U_*' = \{U_\gamma | \gamma < b_{\mu+1}\}$, then $|U_*'| = |b_{\mu+1}| < b$, then since X is $[a,b)$ -compact, if $|U_*'| \geq a$, it has a subcover U_*' with $|U_*'| < a$, thus X is $[a,b]$-compact.

Case 2. $Cn(X) = b$

Assume that b is a limit cardinal. Since X has property $S[a,b]$, for every $\lambda < b$, X has property $S[a,\lambda]$, and therefore X is $[a,\lambda)$ -compact for every $\lambda < b$, and if λ is regular or $cf(\lambda) \geq a$, X is $[a,\lambda]$-compact, by case 1. Let U_* be an open cover of X, since $Cn(X) = b$, U_* has a subcover U_*' such that $|U_*'| = k < b$. Now, since b is a limit cardinal, $k^+ < b$, it follows from the above that X is $[a,k^+]$-compact, so U_*' has a subcover U_*'' such that $|U_*''| < a$, thus X is $[a,b]$-compact and since $Cn(X) = b$, we must have $a = b = Cn(X)$. Assume that b is a successor cardinal k^+, then X has property $S[a,k^+]$, therefore X has property $S[a,k]$ and therefore X is $[a,k)$-compact. If k is regular or $cf(k) \geq a$, X is $[a,k]$-compact, by case 1, and since $Cn(X) = k^+$, X is $[a,k^+]$-compact, thus X is $[a,b]$-compact, and since $Cn(X) = b$ we have $a = b = Cn(X)$.

Assume that $cf(k) < a$. Let U_* be an open cover of X with $|U_*| = k$, with no subcover with cardinality less than k. Let V_* be an open cover of X with $|V_*| = k^+$. Consider the open cover $W_* = \{W = U \cap V | U \in U_*, V \in V_*\}$. Then $|W_*| = k^+$ and also W_* refines U_*, since X has property $S[a,k^+]$, W_* has a subcover W_*' with $|W_*'| < a$ and since W_*' refines U_*, U_* has a subcover U_*' with $|U_*'| < a$, thus X is $[a,k]$ -compact and since $Cn(X) = k^+ = b$, X is $[a,b]$-compact and using the previous argument $a = b = Cn(X)$.

Case 3. $Cn(X) < b$

Let $Cn(X) = \lambda < b$, then X has property $S[a,\lambda]$, so $Cn(X) = \lambda = b = a$ by case 2. Now since $Cn(X) = a$, X is $[a,c]$-compact for every $c \geq a$, therefore X is $[a,b]$-compact.

The proof is complete.

3. References

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