

Optimal Coordinated Search for a Discrete Random Walker

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Abstract

This paper presents the search technique for a lost target. A lost target is random walker on one of two intersected real lines, and the purpose is to detect the target as fast as possible. We have four searchers start from the point of intersection, they follow the so called Quasi-Coordinated search plan. The expected value of the first meeting time between one of the searchers and the target is investigated, also we show the existence of the optimal search strategy which minimizes this first meeting time.

Keywords

Random Walk, Coordinate Search Technique, Lost Targets, Expected Value, Optimal Search

1. Introduction

The search problem for a randomly moving target is very interesting because it may arise in many real world situations such as searching for lost persons on roads, the cancer cells in the human body and missing black box of a plane crash in the depth of the sea or ocean, also searching for a gold mine underground, Landmines and navy mines, a faulty unit in a large linear system such as electrical power lines, telephone lines, and mining system, and so on (see [1], [2], [3], [4] and [5]).

The aim of search, in many cases (see [6], and [7]) is to calculate the expected cost of detecting the target and is to obtain the search plan, which minimizes this expected cost. In the case of linear search for stationary or randomly moving targets many studies are made (see [8]-[26]).

The coordinated search method is one of the famous search methods which consider the searchers starting together from the origin and moving, seeking for

a random walk target. Therefore, coordinated search technique is one of many techniques which studied previously on the line where the located targets have symmetric and unsymmetric distributions (see [27], [28], [29] and [30]), this technique has been illustrated on the circle with a known radius and the target equally likely to be anywhere on its circumference (see [31]), also this technique has been discussed in the plane when the located target has symmetric and asymmetric distribution (see [32] and [33]). There is obviously some similarity between this problem and the well known linear search problem.

In the present paper, we introduce the search problem for a random walk target motion on one of two intersected lines. This will happen by coordinating search between four searchers, all the searchers will start together at the same point of intersected their lines with zero as the starting and meeting point of the searchers. So that we may assume that two searchers always search to the right part and the other searchers search to the left part of intersected point. They return to zero after searching successively common distances until the target is found, we call this search as Quasi-Coordinated Linear Search Problem. We aim to minimize the expected value of the first meeting time between one of the searchers and the target. This paper is organized as follows. In Section 2 we formulate the problem and we give the conditions that make the expected value of the first meeting time between one of the searchers and the target which is finite. In Section 3 the existence of optimal search plan that minimizes the expected value of the first meeting time is presented. Finally, the paper concludes with a discussion of the results and directions for future research.

2. Problem Formulation

A target is assumed to move randomly on one of two intersected line according to a stochastic process $\{S(t), t \in I^+\}$, where I^+ is the set of non negative integers. Assume that $\{X_i\}_{i \geq 0}$ is a sequence of independent and identically distributed random variables such that for any $i \geq 1$: $P(X_i = 1) = p$ and $P(X_i = -1) = 1 - p = q, p, q > 0$. Thus, we have

$$S(t) = \sum_{i=1}^t X_i, t > 0 \text{ and } S(0) = 0 \quad (1)$$

as a Random Walk. Our aim is to calculate the expected value of the first meeting time between one of searcher and lost target, and investigate it, also we show the existence of a search plan which minimize this expected value.

In the present paper we take the search region to be the real lines. Assume that, we have four searchers S_1, S_2, S_3 and S_4 start together looking for the lost target from the intersected point $H_0 = 0$ on $L_1 \cap L_2$. The searchers coordinate their search to find the lost target, where each of the searchers S_1 and S_3 start search at H_0 and go to the right part of starting point as far as H_{11} and H_{21} respectively, and each of searchers S_2 and S_4 start search at H_0 and go the left part of the same lines as far as $-H_{11}$ and $-H_{21}$ respectively. Then, turn back to H_0 to tell the other searchers if the target is found or not.

Retrace the steps again to explore the right (left) part of H_{j1} ($-H_{j1}$) as far as H_{j2} ($-H_{j2}$), $j=1,2$ and so on, see **Figure 1**.

All the searchers S_1, S_2, S_3 and S_4 reach to $H_{11}, -H_{11}, H_{21}$ and $-H_{21}$ respectively in the same time G_1 , then they come back to H_0 again in the same time G_2 . If no one of the searchers do not find the lost target, then they begin their search from H_0 and reach to $H_{12}, -H_{12}, H_{22}$ and $-H_{22}$ in the time G_3 , then they come back to H_0 again in the same time G_4 and so on. A search plan ϕ_r with speed V_r is a function $\phi_r : R^+ \rightarrow R$, $r=1,2,3,4$, such that:

$$|\phi_r(t_1) - \phi_r(t_2)| = V_r |t_1 - t_2|, \forall t_1, t_2 \in R^+ \tag{2}$$

where $\phi_r(0) = 0, r=1,2,3,4$. Let the search plan be represented by $\phi = (\phi_1, \phi_2, \phi_3, \phi_4), \phi_0 \in \Phi_0$ where Φ_0 is the set of all search plan.

We assume that Z_0 is a random variable represented the initial position of the target and valued in $2I$ (or $2I + 1$) and independent with $S(t), t > 0$. If $Z_0 = Z_1$ then the target moves on L_1 and if $Z_0 = Z_2$ the target moves on L_2 such that $P(Z_0 = Z_1) + P(Z_0 = Z_2) = 1$. There is a known probability measures v_j , such that $v_1 + v_2 = 1$ on $L_1 \cup L_2$, where v_1 is probability measure induced by the position of the target on L_1 , while v_2 on L_2 . The first meeting time is a random variable valued in I^+ defined as:

$$\tau_{\phi_0} = \inf \left\{ t, \bigcup_{r=1}^2 |\phi_r(t)| = Z_1 + S(t) \text{ or } \bigcup_{r=3}^4 |\phi_r(t)| = Z_2 + S(t) \right\}.$$

At the beginning of the search suppose that the lost target is existing on any integer point on L_1 but more than H_{11} or less than $-H_{11}$ or the lost target is existing on any integer point on L_2 but more than H_{21} or less than $-H_{21}$. Let τ_{ϕ_0} be the first meeting time between one of the searchers and the target. The main objective is to find the search plan such that $E(\tau_{\phi_0}) < \infty$ and if $E(\tau_{\phi_0^*}) < E(\tau_{\phi_0})$ where E terms to expectation value, then we call ϕ_0^* is an optimal search plan. Given $n > 0$, if x is:

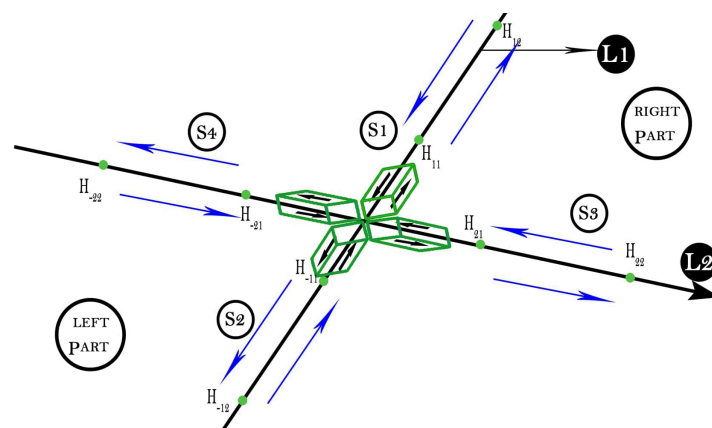


Figure 1. The searchers S_1 and S_2 start from the origin of L_1 after searching successively distances H_{11} and $-H_{11}$, respectively, they return to the origin (note the black arrow) and then they search the distances H_{12} and $-H_{12}$, respectively, they return to the origin (note the blue arrow) and so on also the same procedure for the searchers S_3 and S_4 on L_2 .

$$0 \leq k_1 \leq \frac{n+x}{2} \leq n$$

where k_1 is integer, then

$$P(S(n) = k_1) = \binom{n}{k_1} p^{k_1} q^{n-k_1} \tag{3}$$

Existence of a Finite Search Plan

Assuming that λ, ζ be positive integers such that: $\zeta > 1, \lambda = k\theta$, where $k = 1, 2, \dots$ and θ are positive integer numbers greater than one and $V = 1$. We will shall define the following sequences $\{G_i\}_{i \geq 1}$ and $\{H_{ji}\}_{i \geq 1, j = 1, 2}$, for all the searchers $S_r, r = 1, 2, 3, 4$ on the line L_j , to obtain the distances which the searcher should do them as the functions of λ and ζ . In **Figure 2** we can define

$$G_i = 2^{\frac{1}{2} \lceil 1 - (-1)^{i+1} \rceil} \lambda \left(\zeta^{\frac{i}{2} + \frac{1}{4}} - \frac{1}{4} (-1)^i - 1 \right), \tag{4}$$

And

$$H_{ji} = G_{2i-1} = \frac{1}{2} G_{2i} \tag{5}$$

Also, we shall define the search paths as follows: for any $t \in I^+$, If $G_i \leq t \leq G_{i+1}$, $i = 1, 2, 3, \dots$, then

$$\phi_j(t) = \left(\frac{1}{2} H_{j \frac{i+1}{2}} \right) + (-1)^{i+1} \left(\frac{1}{2} H_{j \frac{i+1}{2}} \right) + (-1)^i (t - G_i) \tag{6}$$

and $\phi_k(t) = -\phi_{k+1}(t), k = 1, 3$. We define the notations $\varphi_j(t) = S(t) - t$ and $\tilde{\varphi}_j(t) = S(t) + t$ on $L_j, j = 1, 2$, respectively, τ_{ϕ_0} is the first meeting between one of the searcher and the target.

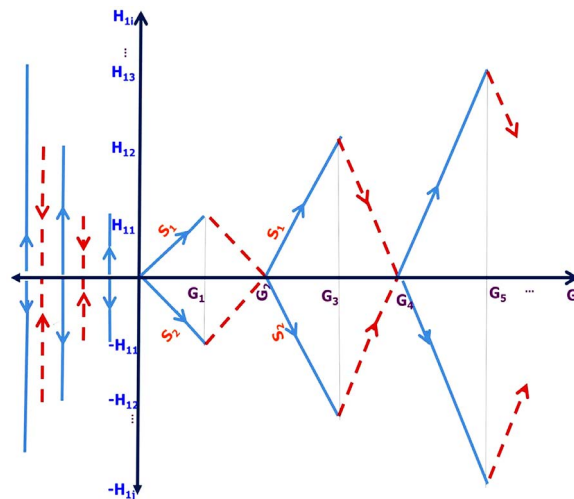


Figure 2. Plots the searched distances H_{ji} and times G_i on L_1 .

Theorem 1. If $\phi_r \in \Phi_0, r = 1, 2, 3, 4$ is a search plan, then the expectation $E(\tau_{\phi_0})$ is finite if:

$$\begin{aligned}
 B_1(z_1) &= \sum_{i=1}^{\infty} (\zeta^i - 1) p(\tilde{\varphi}_1(G_{2i-1}) < -z_1), \\
 B_2(z_1) &= \sum_{i=1}^{\infty} (\zeta^i - 1) p(\varphi_1(G_{2i-1}) > -z_1), \\
 B_3(z_1) &= \sum_{i=1}^{\infty} (\zeta^i (\zeta - 2) + 1) p(\tilde{\varphi}_1(G_{2i}) < -z_1), \\
 B_4(z_1) &= \sum_{i=1}^{\infty} (\zeta^i (\zeta - 2) + 1) p(\varphi_1(G_{2i}) > -z_1), \\
 B_5(z_2) &= \sum_{i=1}^{\infty} (\zeta^i - 1) p(\tilde{\varphi}_2(G_{2i-1}) < -z_2), \\
 B_6(z_2) &= \sum_{i=1}^{\infty} (\zeta^i - 1) p(\varphi_2(G_{2i-1}) > -z_2), \\
 B_7(z_2) &= \sum_{i=1}^{\infty} (\zeta^i (\zeta - 2) + 1) p(\tilde{\varphi}_2(G_{2i}) < -z_2)
 \end{aligned}$$

and

$$B_8(z_2) = \sum_{i=1}^{\infty} (\zeta^i (\zeta - 2) + 1) p(\varphi_2(G_{2i}) > -z_2),$$

are finite.

Proof:

The hypothesis Z_1 and Z_2 are valued in $2I$ (or $2I + 1$) and independent of $S(t), t > 0$, if $Z_1 > 0$ then $Z_1 + S(t)$ is greater than $\phi_1(t)$ until the first meeting between S_1 and the target on L_1 also if $Z_1 < 0$, then $Z_1 + S(t)$ is smaller than $\phi_2(t)$ until the first meeting between S_2 and the target on L_1 . The same thing for the second line by replacing Z_1 by Z_2 in the second line L_2 and $\phi_1(t), \phi_2(t)$ by $\phi_3(t), \phi_4(t)$ respectively.

Hence, for any $i > 0$:

$$p(\tau_{\hat{\phi}_0} > t) \leq \sum_{r=1}^4 p(\tau_{\phi_r} > t)$$

hence,

$$E(\tau_{\hat{\phi}_0}) = \int_0^{\infty} p(\tau_{\hat{\phi}_0} > t) dt$$

see ([5]),

$$\begin{aligned}
 &= \sum_{i=0}^{\infty} \int_{G_i}^{G_{i+1}} p(\tau_{\hat{\phi}_0} > t) dt \\
 &\leq \sum_{i=0}^{\infty} \int_{G_i}^{G_{i+1}} p(\tau_{\hat{\phi}_0} > G_i) dt \\
 &= \sum_{i=0}^{\infty} [(G_{i+1} - G_i) p(\tau_{\hat{\phi}_0} > G_i)] \\
 &= \sum_{i=0}^{\infty} \left\{ 2^{\lfloor \frac{1}{2} [1 - (-1)^{i+2}] \rfloor} \lambda \left(\zeta^{\frac{i+1}{2} + \frac{1}{4} - \frac{1}{4} (-1)^{i+1}} - 1 \right) \right. \\
 &\quad \left. - 2^{\lfloor \frac{1}{2} [1 - (-1)^{i+1}] \rfloor} \lambda \left(\zeta^{\frac{i}{2} + \frac{1}{4} - \frac{1}{4} (-1)^i} - 1 \right) \right\} p(\tau_{\hat{\phi}_0} > G_i)
 \end{aligned}$$

$$\begin{aligned}
 &= \lambda \left\{ ((\zeta - 2) + 1) \left[p(\tau_{\hat{\phi}_0} > 0) + (\zeta - 1) p(\tau_{\hat{\phi}_0} > G_1) \right] \right. \\
 &\quad + (\zeta(\zeta - 2) + 1) p(\tau_{\hat{\phi}_0} > G_2) + (\zeta^2 - 1) p(\tau_{\hat{\phi}_0} > G_3) \\
 &\quad + (\zeta^2(\zeta - 2) + 1) p(\tau_{\hat{\phi}_0} > G_4) + (\zeta^3 - 1) p(\tau_{\hat{\phi}_0} > G_5) \\
 &\quad \left. + (\zeta^3(\zeta - 2) + 1) p(\tau_{\hat{\phi}_0} > G_6) + \dots \right\}
 \end{aligned}$$

To solve this equation we shall find the value of $p(\tau_{\hat{\phi}_0} > G_{2i-1})$ and the value of $p(\tau_{\hat{\phi}_0} > G_{2i}), i \geq 1$ as the following

$$\begin{aligned}
 p(\tau_{\hat{\phi}_0} > G_{2i-1}) &\leq \int_{-\infty}^0 p(Z_1 + S(G_{2i-1}) < -H_i | Z_i = z_1) v_1(dz_1) \\
 &\quad + \int_0^{\infty} p(Z_1 + S(G_{2i-1}) > H_i | Z_i = z_1) v_1(dz_1) \\
 &\quad + \int_{-\infty}^0 p(Z_2 + S(G_{2i-1}) < -H_i | Z_2 = z_2) v_2(dz_2) \\
 &\quad + \int_0^{\infty} p(Z_2 + S(G_{2i-1}) > H_i | Z_2 = z_2) v_2(dz_2)
 \end{aligned}$$

We get

$$\begin{aligned}
 p(\tau_{\hat{\phi}_0} > G_{2i-1}) &\leq \int_{-\infty}^0 p(\tilde{\varphi}_1(G_{2i-1}) < -z_1) v_1(dz_1) + \int_0^{\infty} p(\varphi_1(G_{2i-1}) > -z_1) v_1(dz_1) \\
 &\quad + \int_{-\infty}^0 p(\tilde{\varphi}_2(G_{2i-1}) < -z_2) v_2(dz_2) + \int_0^{\infty} p(\varphi_2(G_{2i-1}) > -z_2) v_2(dz_2)
 \end{aligned}$$

also we get

$$\begin{aligned}
 p(\tau_{\hat{\phi}_0} > G_{2i}) &\leq \int_{-\infty}^0 p(Z_1 + S(G_{2i}) < -2H_i | Z_1 = z_1) v_1(dz_1) \\
 &\quad + \int_0^{\infty} p(Z_1 + S(G_{2i}) > 2H_i | Z_1 = z_1) v_1(dz_1) \\
 &\quad + \int_{-\infty}^0 p(Z_2 + S(G_{2i}) < -2H_i | Z_2 = z_2) v_2(dz_2) \\
 &\quad + \int_0^{\infty} p(Z_2 + S(G_{2i}) > 2H_i | Z_2 = z_2) v_2(dz_2)
 \end{aligned}$$

We get

$$\begin{aligned}
 p(\tau_{\hat{\phi}_0} > G_{2i}) &\leq \int_{-\infty}^0 p(\tilde{\varphi}_1(G_{2i}) < -z_1) v_1(dz_1) + \int_0^{\infty} p(\varphi_1(G_{2i}) > -z_1) v_1(dz_1) \\
 &\quad + \int_{-\infty}^0 p(\tilde{\varphi}_2(G_{2i}) < -z_2) v_2(dz_2) + \int_0^{\infty} p(\varphi_2(G_{2i}) > -z_2) v_2(dz_2)
 \end{aligned}$$

Hence, we can get,

$$\begin{aligned}
 E(\tau_{\hat{\phi}_0}) &\leq \lambda \left\{ ((\zeta - 2) + 1) p(\tau_{\hat{\phi}_0} > 0) + (\zeta - 1) \left[\int_{-\infty}^0 p(\tilde{\varphi}_1(G_1) < -z_1) v_1(dz_1) \right. \right. \\
 &\quad \left. \left. + \int_0^{\infty} p(\varphi_1(G_1) > -z_1) v_1(dz_1) \right] + (\zeta(\zeta - 2) + 1) \left[\int_{-\infty}^0 p(\tilde{\varphi}_1(G_2) < -z_1) v_1(dz_1) \right. \right.
 \end{aligned}$$

$$\begin{aligned}
 & + \int_0^\infty p(\varphi_1(G_2) > -z_1) v_1(dz_1) \Big] + (\zeta^2 - 1) \Big[\int_{-\infty}^0 p(\tilde{\varphi}_1(G_3) < -z_1) v_1(dz_1) \\
 & + \int_0^\infty p(\varphi_1(G_3) > -z_1) v_1(dz_1) \Big] + (\zeta^2(\zeta - 2) + 1) \Big[\int_{-\infty}^0 p(\tilde{\varphi}_1(G_4) < -z_1) v_1(dz_1) \\
 & + \int_0^\infty p(\varphi_1(G_4) > -z_1) v_1(dz_1) \Big] + (\zeta^3 - 1) \Big[\int_{-\infty}^0 p(\tilde{\varphi}_1(G_5) < -z_1) v_1(dz_1) \\
 & + \int_0^\infty p(\varphi_1(G_5) > -z_1) v_1(dz_1) \Big] + (\zeta^3(\zeta - 2) + 1) \Big[\int_{-\infty}^0 p(\tilde{\varphi}_6(G_2) < -z_1) v_1(dz_1) \\
 & + \int_0^\infty p(\varphi_1(G_6) > -z_1) v_1(dz_1) \Big] \Big\} + \lambda \Big\{ ((\zeta - 2) + 1) p(\tau_{\hat{\phi}} > 0) \\
 & + (\zeta - 1) \Big[\int_{-\infty}^0 p(\tilde{\varphi}_2(G_1) < -z_2) v_2(dz_2) + \int_0^\infty p(\varphi_2(G_1) > -z_2) v_2(dz_2) \Big] \\
 & + (\zeta(\zeta - 2) + 1) \Big[\int_{-\infty}^0 p(\tilde{\varphi}_2(G_2) < -z_2) v_2(dz_2) + \int_0^\infty p(\varphi_2(G_2) > -z_2) v_2(dz_2) \Big] \\
 & + (\zeta^2 - 1) \Big[\int_{-\infty}^0 p(\tilde{\varphi}_2(G_3) < -z_2) v_2(dz_2) + \int_0^\infty p(\varphi_3(G_2) > -z_2) v_2(dz_2) \Big] \\
 & + (\zeta^2(\zeta - 2) + 1) \Big[\int_{-\infty}^0 p(\tilde{\varphi}_4(G_2) < -z_2) v_2(dz_2) + \int_0^\infty p(\varphi_2(G_4) > -z_2) v_2(dz_2) \Big] \\
 & + (\zeta^3 - 1) \Big[\int_{-\infty}^0 p(\tilde{\varphi}_3(G_2) < -z_2) v_2(dz_2) + \int_0^\infty p(\varphi_5(G_2) > -z_2) v_2(dz_2) \Big] \\
 & + (\zeta^3(\zeta - 2) + 1) \Big[\int_{-\infty}^0 p(\tilde{\varphi}_2(G_6) < -z_2) v_2(dz_2) \\
 & + \int_0^\infty p(\varphi_6(G_2) > -z_2) v_2(dz_2) \Big] + \dots \Big\}
 \end{aligned}$$

hence,

$$\begin{aligned}
 E(\tau_{\hat{\phi}_0}) & \leq \lambda \Big\{ ((\zeta - 2) + 1) p(\tau_{\hat{\phi}_0} > 0) + \Big[\int_{-\infty}^0 w_1(z_1) v_1(dz_1) + \int_0^\infty w_2(z_1) v_1(dz_1) \Big] \\
 & + \Big[\int_{-\infty}^0 w_3(z_1) v_1(dz_1) + \int_0^\infty w_4(z_1) v_1(dz_1) \Big] \Big\} \\
 & + \lambda \Big\{ ((\zeta - 2) + 1) p(\tau_{\hat{\phi}_0} > 0) + \Big[\int_{-\infty}^0 w_5(z_2) v_2(dz_2) + \int_0^\infty w_6(z_2) v_2(dz_2) \Big] \\
 & + \Big[\int_{-\infty}^0 w_7(z_2) v_2(dz_2) + \int_0^\infty w_8(z_2) v_2(dz_2) \Big] \Big\}
 \end{aligned}$$

where,

$$B_1(z_1) = \sum_{i=1}^\infty (\zeta^i - 1) p(\tilde{\varphi}_1(G_{2i-1}) < -z_1),$$

$$B_2(z_1) = \sum_{i=1}^\infty (\zeta^i - 1) p(\varphi_1(G_{2i-1}) > -z_1),$$

$$B_3(z_1) = \sum_{i=1}^{\infty} (\zeta^i (\zeta - 2) + 1) p(\tilde{\varphi}_1(G_{2i}) < -z_1),$$

$$B_4(z_1) = \sum_{i=1}^{\infty} (\zeta^i (\zeta - 2) + 1) p(\varphi_1(G_{2i}) > -z_1),$$

$$B_5(z_2) = \sum_{i=1}^{\infty} (\zeta^i - 1) p(\tilde{\varphi}_2(G_{2i-1}) < -z_2),$$

$$B_6(z_2) = \sum_{i=1}^{\infty} (\zeta^i - 1) p(\varphi_2(G_{2i-1}) > -z_2),$$

$$B_7(z_2) = \sum_{i=1}^{\infty} (\zeta^i (\zeta - 2) + 1) p(\tilde{\varphi}_2(G_{2i}) < -z_2)$$

and

$$B_8(z_2) = \sum_{i=1}^{\infty} (\zeta^i (\zeta - 2) + 1) p(\varphi_2(G_{2i}) > -z_2),$$

lemma 1.

For any $\zeta \geq 0$, if $a_n \geq 0$ for $n > 0$, and $a_{n+1} \leq a_n$, $\{d_n\}_{n \geq 0}$ be a strictly increasing sequence of integer numbers with $d_0 = 0$, then

$$\sum_{n=k}^{\infty} (d_{n+1} - d_n) a_{d_{n+1}} \leq \sum_{k=d_k}^{\infty} a_k \leq \sum_{n=k}^{\infty} (d_{n+1} - d_n) a_{d_n},$$

where $\sum_{n=k}^{\infty} (d_{n+1} - d_n) a_{d_{n+1}}$, $\sum_{k=d_k}^{\infty} a_k$ and $\sum_{n=k}^{\infty} (d_{n+1} - d_n) a_{d_n}$ are vectors see [32].

Theorem 2. For the two intersected lines the chosen search plan satisfies:

$$B_1(z_1) \leq B_9(|z_1|), \quad B_2(z_1) \leq B_{10}(|z_1|), \quad B_3(z_1) \leq B_{11}(|z_1|),$$

$$B_4(z_1) \leq B_{12}(|z_1|), \quad B_5(z_2) \leq B_{13}(|z_2|), \quad B_6(z_2) \leq B_{14}(|z_2|),$$

$$B_7(z_2) \leq B_{15}(|z_2|) \quad \text{and} \quad B_8(z_2) \leq B_{16}(|z_2|),$$

where $B_9(|z_1|)$, $B_{10}(|z_1|)$, $B_{11}(|z_1|)$, $B_{12}(|z_1|)$, $B_{13}(|z_2|)$, $B_{14}(|z_2|)$, $B_{15}(|z_2|)$ and $B_{16}(|z_2|)$ are linear functions.

Proof:

We shall prove the theorem for $B_2(z_1)$ and $B_6(z_2)$ since the other cases can be proved by similar ways.

$$B_2(z_1) = \sum_{i=1}^{\infty} (\zeta^i - 1) p(\varphi_1(G_{2i-1}) > -z_1)$$

and

$$B_6(z_2) = \sum_{i=1}^{\infty} (\zeta^i - 1) p(\varphi_2(G_{2i-1}) > -z_2)$$

Let us defined the following:

1) For $z_1 \leq 0$, we have $B_2(z_1) \leq B_2(0)$

And for $z_2 \leq 0$, we have $B_6(z_2) \leq B_6(0)$.

2) but for $z_1 > 0$, we have

$$B_2(z_1) = B_2(0) + \sum_{i=1}^{\infty} (\zeta^i - 1) p(-z_1 < \varphi_1(G_{2i-1}) \leq 0),$$

$$B_2(z_1) - B_2(0) = \sum_{i=1}^{\infty} (\zeta^i - 1) p(-z_1 < \varphi_1(G_{2i-1}) \leq 0).$$

and for $z_2 > 0$, we have

$$B_6(z_2) = B_6(0) + \sum_{i=1}^{\infty} (\zeta^i - 1) p(-z_2 < \varphi_2(G_{2i-1}) \leq 0),$$

$$B_6(z_2) - B_2(0) = \sum_{i=1}^{\infty} (\zeta^i - 1) p(-z_2 < \varphi_2(G_{2i-1}) \leq 0).$$

from theorem 2 see [13], we obtain:

$$B_2(0) = \sum_{i=0}^{\infty} (\zeta^i - 1) p(G_{2i-1} > 0) \leq \sum_{i=1}^{\infty} (\zeta^i - 1) \varepsilon^{G_{2i-1}}, 0 < \varepsilon < 1$$

And

$$B_6(0) = \sum_{i=0}^{\infty} (\zeta^i - 1) p(G_{2i-1} > 0) \leq \sum_{i=1}^{\infty} (\zeta^i - 1) \varepsilon^{G_{2i-1}}, 0 < \varepsilon < 1$$

Let us defined the following:

1)

$$V_1(n) = \frac{\varphi_1(n\theta_1)}{2} = \sum_{i=1}^n B_{1(i)}$$

where $\{B_{1i}\}$ is a sequence of independent identically distributed random variable.

$$V_2(n) = \frac{\varphi_2(n\theta_2)}{2} = \sum_{i=1}^n B_{2i}$$

where $\{B_{2i}\}$ is a sequence of independent identically distributed random variable.

2)

$$d_{1(n)} = \frac{G_{2n-1}}{\theta_1} = K(\zeta^n - 1)$$

$$d_{2(n)} = \frac{G_{2n-1}}{\theta_2} = K(\zeta^n - 1)$$

3) m_1 is an integer such that $d_{m_1} = b_1|z_1| + b_2$, m_2 is an integer such that $d_{m_2} = b_1|z_2| + b_2$.

4)

$$a_1(n) = \frac{n}{n+k} p(-z_1/2 < V_1(n) \leq 0) = \sum_{i=0}^{(|z_1|/2)} p[-(i+1) < V_1(n) \leq (-i)]$$

$$a_2(n) = \frac{n}{n+k} p(-z_2/2 < V_2(n) \leq 0) = \sum_{i=0}^{(|z_2|/2)} p[-(i+1) < V_2(n) \leq (-i)]$$

5)

$$\alpha_1 = \frac{\zeta}{(\zeta - 1)K},$$

$$\alpha_2 = \frac{\zeta}{(\zeta - 1)K}$$

and
6)

$$U_1(j, j+1) = \sum_{n=0}^{\infty} p[-(j+1) < V_1(n) \leq (-j)],$$

$$U_2(j, j+1) = \sum_{n=0}^{\infty} p[-(j+1) < V_2(n) \leq (-j)].$$

Then $U_1(j, j+1)$ and $U_2(j, j+1)$ satisfies the condition of the renewal equation see [33]. Thus, from lemma (1) we have $\alpha_1(n)$ and $\alpha_2(n)$ are non increasing if $n > d_{m_1}$ and $n > d_{m_2}$ see [4], consequently,

$$\begin{aligned} B_2(z_1) - B_2(0) &= \sum_{i=1}^{\infty} (\zeta^i - 1) p(-z_1 < \varphi_1(G_{2i-1}) \leq 0) \\ &= \sum_{i=1}^{n_1} (\zeta^i - 1) p(-z_1 < \varphi_1(G_{2i-1}) \leq 0) \\ &\quad + \sum_{i=n_1+1}^{\infty} (\zeta^i - 1) p(-z_1 < \varphi_1(G_{2i-1}) \leq 0) \\ &= \sum_{n=1}^{n_1} \zeta^n a_1(d_{1(n)}) + \sum_{n=n_j+1}^{\infty} \zeta^n a_1(d_{1(n)}) \\ &\leq \sum_{n=1}^{n_1} \zeta^n + \alpha_1 \sum_{n=n_j+1}^{\infty} (d_{1(n)} - d_{1(n-1)}) a_1(d_{1(n)}) \\ &\leq \sum_{n=1}^{n_1} \zeta^n + \alpha_1 \sum_{n=d_{m_1}}^{\infty} a_1(n) \\ &\leq \sum_{n=1}^{n_1} \zeta^n + \alpha_1 \sum_{n=d_{m_1}}^{\infty} \sum_{i=0}^{\lfloor |z_1|/2 \rfloor} p[-(j+1) < V_1(n) \leq (-j)] \\ &\leq \sum_{n=1}^{n_1} \zeta^n + \alpha_1 \sum_{j=0}^{\lfloor |z_1|/2 \rfloor} U_1(j, j+1) \end{aligned}$$

and

$$\begin{aligned} B_6(z_2) - B_6(0) &= \sum_{i=1}^{\infty} (\zeta^i - 1) p(-z_2 < \varphi_2(G_{2i-1}) \leq 0) \\ &= \sum_{i=1}^{n_2} (\zeta^i - 1) p(-z_2 < \varphi_2(G_{2i-1}) \leq 0) \\ &\quad + \sum_{i=n_2+1}^{\infty} (\zeta^i - 1) p(-z_2 < \varphi_2(G_{2i-1}) \leq 0) \\ &= \sum_{n=1}^{n_2} \zeta^n a_2(d_{2(n)}) + \sum_{n=n_2+1}^{\infty} \zeta^n a_2(d_{2(n)}) \\ &\leq \sum_{n=1}^{n_2} \zeta^n + \alpha_2 \sum_{n=n_2+1}^{\infty} (d_{2(n)} - d_{2(n-1)}) a_2(d_{2(n)}) \\ &\leq \sum_{n=1}^{n_2} \zeta^n + \alpha_2 \sum_{n=d_{m_2}}^{\infty} a_2(n) \\ &\leq \sum_{n=1}^{n_2} \zeta^n + \alpha_2 \sum_{n=d_{m_2}}^{\infty} \sum_{i=0}^{\lfloor |z_2|/2 \rfloor} p[-(j+1) < V_2(n) \leq (-j)] \\ &\leq \sum_{n=1}^{n_2} \zeta^n + \alpha_2 \sum_{j=0}^{\lfloor |z_2|/2 \rfloor} U_2(j, j+1) \end{aligned}$$

Such that for any line $U_1(j, j+1)$ and $U_2(j, j+1)$ satisfies the condition of the renewal equation see [26], we have $U_1(j, j+1)$ and $U_2(j, j+1)$ are bounded for all j by a constant so

$$B_2(z_1) \leq B_2(0) + M_1 + M_2 |z_1| = B_{10}(|z_1|),$$

then

$$B_6(z_1) \leq B_6(0) + M_1 + M_2 |z_2| = B_{14}(|z_2|).$$

The necessary and sufficient condition for the existence of a finite search plan is $E|Z_j| < \infty, j=1,2$, that is sufficient from the consequence of Theorems 1,2 and the following Theorem 3.

Theorem. 3.

If there exists a finite search plan $\hat{\phi}_0 \in \hat{\Phi}_0$, then $E|Z_j|$ is finite, where Z_j is a random variable representing the initial position of the target on a line $L_j, j=1,2$.

Proof

For $E(\tau_{\hat{\phi}_0}) < \infty$, we have $p(\tau_{\hat{\phi}_0} \text{ is finite}) = 1$, and so

$$\sum_{r=1}^4 p(\tau_{\hat{\phi}_r} \text{ is finite}) = 1$$

Therefore,

$$p(\tau_{\hat{\phi}_{01}} \text{ is finite}) + p(\tau_{\hat{\phi}_{02}} \text{ is finite}) = 1$$

where $\tau_{\hat{\phi}_{01}}, \tau_{\hat{\phi}_{02}}$ the first meeting time on the L_1 and L_2 , respectively, then,

$$p(\tau_{\hat{\phi}_{01}} \text{ is finite}) = 1.$$

we conclude that:

$$p(\tau_{\hat{\phi}_{01}} \text{ is finite}) = 1 \text{ and } p(\tau_{\hat{\phi}_{02}} \text{ is finite}) = 0$$

$$\text{or } p(\tau_{\hat{\phi}_{01}} \text{ is finite}) = 0 \text{ and } p(\tau_{\hat{\phi}_{02}} \text{ is finite}) = 1$$

If $p(\tau_{\hat{\phi}_{01}} \text{ is finite}) = 1$ on the first line L_1 , then $Z_1 = \phi(\tau_{\hat{\phi}_{01}}) - S(\tau_{\hat{\phi}_{01}})$ with probability one and hence

$$|Z_1| \leq |\phi(\tau_{\hat{\phi}_{01}})| + |S(\tau_{\hat{\phi}_{01}})| \leq \tau_{\hat{\phi}_{01}} + |S(\tau_{\hat{\phi}_{01}})|,$$

that leads to,

$$E|Z_1| \leq E(\tau_{\hat{\phi}_{01}}) + E|S(\tau_{\hat{\phi}_{01}})|.$$

but $|S(\tau_{\hat{\phi}_{01}})| \leq \tau_{\hat{\phi}_{01}}$, then $E|S(\tau_{\hat{\phi}_{01}})| \leq E(\tau_{\hat{\phi}_{01}})$.

If $E(\tau_{\hat{\phi}_{01}}) < \infty$, then $E|S(\tau_{\hat{\phi}_{01}})| < \infty$, and $E|Z_1|$ is finite. On the second line L_2 .

If $p(\tau_{\hat{\phi}_2} \text{ is finite})=1$, then $Z_2 = \phi(\tau_{\hat{\phi}_2}) - S(\tau_{\hat{\phi}_2})$ with probability one, by the same way we can get $E|\tau_{\hat{\phi}_2}| < \infty$, and $E|Z_2|$ is finite.

3. Existence of an Optimal Search Plan

Definition 1.

Let $\hat{\phi}_{ji} \in \hat{\Phi}(t), j=1,2$ be two sequences of search plans, we say that $\hat{\phi}_{ji}$ converges to $\hat{\phi}_j$ as i tends to ∞ if and only if for any $t \in I^+$, converges to $\hat{\phi}_j$ uniformly on every compact subset see [13].

Theorem 1.4.

Let for any $t \in I^+$, and $S(t)$ be a process(one dimensional random walk). The mapping $\hat{\phi} = (\phi_1, \phi_2, \phi_3, \phi_4) \rightarrow E(\tau_{\hat{\phi}}) \in R^+ \rightarrow E(\tau_{\hat{\phi}}) \in R^+$ is lower semi-continuous on $\hat{\Phi}(t)$.

Proof

Let $I(\hat{\phi}_j, t)$ be the indicator function of the set $\{\tau_{\hat{\phi}_j} \geq t\}$, by the fa-tou-lebesque theorem see [5] we get:

$$E(\tau_{\hat{\phi}_j}) = E\left[\sum_{t=1}^{\infty} I(\hat{\phi}_j, t)\right] = E\left[\sum_{t=1}^{\infty} \liminf_{i \rightarrow \infty} I(\hat{\phi}_{ji}, t)\right] \leq \liminf_{i \rightarrow \infty} E(\tau_{\hat{\phi}_{ji}})$$

for any sequence $\hat{\phi}_{ji} \rightarrow \hat{\phi}_j$ in $\hat{\Phi}(t)$, where $\hat{\Phi}(t)$ is sequentially compact see [29]. Thus the mapping $\hat{\phi}_j \rightarrow E(\tau_{\hat{\phi}_j})$ is lower semi continuous on $\hat{\Phi}(t)$, then this mapping attains its minimum.

4. Conclusions and Future Work

We illustrated that the quasi coordinated linear search technique for a random walk target on one of two intersected real lines has been presented, where the target initial position is given by a random variable. We introduced the proof of conditions that make the search plan which is finite in Theorem 1 based on the continuity of the search plan. In Theorems 2 we showed that the search plan is finite if the conditions, where

$$B_1(z_1) \leq B_9(|z_1|), B_2(z_1) \leq B_{10}(|z_1|), B_3(z_1) \leq B_{11}(|z_1|), B_4(z_1) \leq B_{12}(|z_1|), \\ B_5(z_2) \leq B_{13}(|z_2|), B_6(z_2) \leq B_{14}(|z_2|), B_7(z_2) \leq B_{15}(|z_2|) \text{ and } B_8(z_2) \leq B_{16}(|z_2|),$$

where $B_9(|z_1|), B_{10}(|z_1|), B_{11}(|z_1|), B_{12}(|z_1|), B_{13}(|z_2|), B_{14}(|z_2|), B_{15}(|z_2|)$ and $B_{16}(|z_2|)$ are linear functions. We use Theorem 3 to show that if there exists a finite search plan then the expected value of the target initial position $E|Z_j|$ is finite. It will also be interesting to see a direct consequence of Theorems 1, 2, and 3 satisfying the existence of a finite search plan if and only if $E|Z_j|$ is finite. We pointed to the existence of an optimal search plan in Theorem 4. The effectiveness of this model is illustrated using a real life application.

In future research, we have interesting search problems, study the coordinated search problem using multiple searchers, when the searchers star from any point on more lines rather than the origin.

Conflicts of Interest

The authors declare no conflicts of interest regarding the publication of this paper.

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