Applied Mathematics, 2016, 7, 1215-1224
Published Online July 2016 in SciRes. http://www.scirp.org/journal/am
http://dx.doi.org/10.4236/am.2016.711107

Numerical Solutions of a Generalized *N*th Order Boundary Value Problems Using Power Series Approximation Method

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Received 6 May 2016; accepted 8 July 2016; published 11 July 2016

Abstract

In this paper, a new approach called Power Series Approximation Method (PSAM) is developed for the numerical solution of a generalized linear and non-linear higher order Boundary Value Problems (BVPs). The proposed method is efficient and effective on the experimentation on some selected thirteen-order, twelve-order and ten-order boundary value problems as compared with the analytic solutions and other existing methods such as the Homotopy Perturbation Method (HPM) and Variational Iteration Method (VIM) available in the literature. A convergence analysis of PSAM is also provided.

Keywords

Power Series, Linear and Nonlinear Problems, Boundary Value Problem (BVP), Numerical Simulation

1. Introduction

Higher order boundary value problems in linear and non-linear form have been a major concern in recent years. This is due to its applicability in many areas of Mathematical Physics and other sciences in its precise analysis of nonlinear phenomena such as computation of radiowave attenuation in the atmosphere, interface conditions determination in electromagnetic field, potential theory and determination of wave nodes in wave propagation. Most conventional analytic methods for higher order boundary value problems are prone to rounding-off and computation errors. As a result, the analytics methods are less dependent in seeking the solution of higher order boundary values problems in most cases, especially the non-linear type. Thus, numerical methods have gained

How to cite this paper: Njoseh, I.N. and Mamadu, E.J. (2016) Numerical Solutions of a Generalized *N*th Order Boundary Value Problems Using Power Series Approximation Method. *Applied Mathematics*, **7**, 1215-1224.
http://dx.doi.org/10.4236/am.2016.711107

momentum in seeking the solution of higher order boundary value problems.

Over the years, several numerical techniques have been developed, such as the Variational Iteration Method (VIM) [1], Homotopy Perturbation Method (HPM) [2], Spline-Collocation Approximations Method (SCAM) [3], Spline Method [4], etc. that possess an elaborate procedure and structurally complex, which nevertheless yields efficient results. Siddiqi and Iftikhar [5] worked on a numerical solution of higher order boundary value problems. Also, Siddiqi and Iftikhar [6] adopted the technique of variation of parameter methods for the solution of seventh order boundary value problems. Iftikhar *et al*. [7] solved the thirteenth order value problems by Differential transform method. Akram and Rehman [8] presented a numerical solution of eighth order boundary value problems in reproducing kernel space. Wu *et al*. [9] presented a precise and rigorous work on nonlinear functional analysis of boundary value problems: novel theory, methods and applications. Mamadu and Njoseh [10] have proposed a method which efficiently finds exact solutions and is used to solve linear Volterra integral equations.

In this present work, the Power Series Approximation Method (PSAM) is a new approach developed for the numerical solution of a generalized *N*th order boundary value problems. The proposed method is structurally simple with well posed Mathematical formulae. It involves transforming the given boundary value problems into system of ODEs together with the boundary conditions prescribed. Thereafter, the coefficients of the power series solution are uniquely obtained with a well posed recurrence relation along the boundary ξ_0, which leads to the solution. The unknown parameters in the solution are determined at the other boundary ξ_1. This finally leads to a system of algebraic equations, which on solving yields the required approximate series solution. The method is accurate and efficient in obtaining the approximate solutions of linear and non-linear boundary value problems. The method requires no discretization and linearization or perturbation. Also, computational and rounding-off errors are avoided. The method has an excellent rate of convergence as compared with existing methods in [1] [2] and the exact solutions available in the literature.

The rest of this paper will be organized as follows: Section 2 of this work give detailed Mathematical formulation of *N*th order BVPs using PSAM. Section 3 presents the error analysis and convergence theorem of the method. Section 4 offers numerical stimulation of the method on some selected thirteen-order, twelve-order and ten-order boundary value problems. Finally, the conclusion is presented in Section 5.

2. Power Series Approximation Method (PSAM)

We consider the *N*th order BVP of the form

$$y^{(N)}(x)+f(x)y(x)=g(x),\ \ \xi_0<x<\xi_1 \tag{1}$$

with the boundary conditions

$$y^{(2m)}(\xi_0)=\lambda_{2m},\ \ m=0,1,2,3,\cdots,(n-1), \tag{2}$$

$$y^{(2m)}(\xi_1)=\beta_{2m},\ \ m=0,1,2,3,\cdots,(n-1) \tag{3}$$

where $f(x), g(x)$ and $y(x)$ are assumed real and continuous on $\xi_0\le x\le\xi_1$, λ_{2m} and β_{2m}, $m=0,1,2,3,\cdots,(n-1)$ are finite real constants.

The given *n*th order BVP (1), (2) and (3) are transformed to systems of ODEs such that we have

$$\begin{aligned}
&\frac{\mathrm{d}y}{\mathrm{d}x}=y_1,\\
&\frac{\mathrm{d}y_1}{\mathrm{d}x}=y_2,\\
&\frac{\mathrm{d}y_2}{\mathrm{d}x}=y_3,\\
&\quad\vdots\\
&\frac{\mathrm{d}y}{\mathrm{d}x}=g(x)-f(x)y(x),
\end{aligned} \tag{4}$$

with the boundary conditions

$$y_1(\xi_0)=\lambda_0, y_2(\xi_0)=\lambda_1, y_3(\xi_0)=\lambda_2,\cdots, y_{2n}(\xi_0)=\lambda_{2n-1} \tag{5}$$

and

$$y_1(\xi_1)=\beta_0,, y_2(\xi_1)=\beta_1, y_3(\xi_1)=\beta_2,\cdots, y_{2n}(\xi_1)=\beta_{2n-1}. \tag{6}$$

Let the series approximation of (1), (2) and (3) be given as

$$y_N(x)=\sum_{i=0}^{N} a_i x^i,\ \ N<\infty, \tag{7}$$

where $a_i, i=0(1)N$ are unknown constants to be determined and $x\in[\xi_0,\xi_1]$.

Now, we estimate the unknown constants $a_i, i=0(1)N$, at $x=\xi_0$ by substituting (7) in (4) successively, which is as follows:

We consider the first derivative of y_N wrt to *x* as y_1, *i.e.*,

$$\frac{\mathrm{d}y_N}{\mathrm{d}x}=y_1 \Rightarrow i\sum_{i=1}^{N} a_i x^{i-1}=y_1 \Rightarrow a_1+i\sum_{i=2}^{N} a_i x^{i-1}=y_1. \tag{8}$$

At $y_1(\xi_0)=\lambda_0$, we have,

$$a_1+i\sum_{i=2}^{N} a_i \xi_0^{i-1}=\lambda_0 \Rightarrow a_1=\lambda_0-i\sum_{i=2}^{N} a_i \xi_0^{i-1}. \tag{9}$$

Thus (8) becomes

$$y_1=\lambda_0-i\sum_{i=2}^{N} a_i \xi_0^{i-1}+i\sum_{i=2}^{N} a_i x^{i-1} \tag{10}$$

Next: $\dfrac{\mathrm{d}y_1}{\mathrm{d}x}=y_2.$

$$\frac{\mathrm{d}y_1}{\mathrm{d}x}=y_2 \Rightarrow i(i-1)\sum_{i=2}^{N} a_i x^{i-2}=y_2 \Rightarrow 2a_2+i(i-1)\sum_{i=3}^{N} a_i x^{i-2}=y_2 \tag{11}$$

$y_2(\xi_0)=\lambda_1$, we obtain,

$$2a_2+i(i-1)\sum_{i=3}^{N} a_i \xi_0^{i-2}=\lambda_1 \Rightarrow a_2=\frac{1}{2}\left[\lambda_1-i(i-1)\sum_{i=3}^{N} a_i \xi_0^{i-2}\right] \tag{12}$$

Thus (11) becomes

$$y_2=\lambda_1-i(i-1)\sum_{i=3}^{N} a_i \xi_0^{i-2}+i(i-1)\sum_{i=3}^{N} a_i x^{i-2} \tag{13}$$

Carrying on the above sequential approach to the n^{th} order we obtain the following recursive formulae at $x=\xi_0$,

$$a_n=\frac{1}{n!}\left[\lambda_n-n!\sum_{i=n+1}^{N} a_i \xi_0^{i-n}\right], n\geq 0 \tag{14}$$

$$y_n=\lambda_n-n!\sum_{i=n+1}^{N} a_i \xi_0^{i-n}+n!\sum_{i=n+1}^{N} a_i x^{i-n}, n\geq 0, \tag{15}$$

Here, the choice of *N* is equivalent to the order of the BVP considered.

3. Error Analysis and Convergence Theorem

An error estimate for the approximate solution (7) of (1), (2) and (3) is obtained here.

Let

$$e_n=y(x)-y_N(x)$$

as the error function of $y_N(x)$ to $y(x)$; where $y(x)$ is the exact solution of (1), (2) and (3).

Hence, $y_N(x)$ satisfies the following problems:

$$y_N^{(N)}(x) = g(x) - f(x)y_N(x) + H_N(x),\ x \in [\xi_0, \xi_1], \tag{16}$$

$$y_N^{(2m)}(\xi_0) = \lambda_{2m},\ m = 0,1,2,\cdots,(n-1) \tag{17}$$

$$y_N^{(2m)}(\xi_1) = \beta_{2m},\ m = 0,1,2,\cdots,(n-1) \tag{18}$$

The perturbation term $H_N(x)$ can be obtained by substituting the computed solution $y_N(x)$ to obtain

$$H_N(x) = y_N^{(N)}(x) - g(x) + f(x)y_N^{(N)}(x) \tag{19}$$

We then transform (16), (17) and (18) into systems of ordinary differential equations and proceed to find an approximate $e_{N,n}(x)$ to the error function $e_n(x)$ in the same way as we did before for the solution of the problem (1), (2) and (3).

Thus, the error function satisfies the problem

$$e_n^{(N)}(x) - g(x) + f(x)e_n(x) = -H_N(x),\ x \in [\xi_0, \xi_1], \tag{20}$$

with the homogeneous conditions

$$y_N^{(2m)}(\xi_0) = 0,\ m = 0,1,2,\cdots,N \tag{21}$$

$$y_N^{(2m)}(\xi_1) = 0,\ m = 0,1,2,\cdots,N \tag{22}$$

3.1. Convergence Theorem

We now prove that if the solution series by PSAM is convergent, it must be an exact solution by increasing the order of approximation.

Theorem 1:

If the solution series $y_N(x) = \sum_{i=0}^{N} a_i x^i$ converges it must be an exact solution by increasing the order of approximation.

Proof:

Let the series $\sum_{i=0}^{N} a_i x^i$ be convergent. Then

$$y(x) = \sum_{i=0}^{N} a_i x^i \tag{23}$$

$$\lim_{i\to\infty} y_i(x) = 0 \tag{24}$$

We have

$$\sum_{i=0}^{N}\left[a_i x^i - x_i a_{i-1} x^{i-1}\right] \tag{25}$$

Using Equation (23),

$$\sum_{i=0}^{N}\left[a_i x^i - x_i a_{i-1} x^{i-1}\right] = \lim_{i\to\infty} y_i(x) = 0 \tag{26}$$

Using Equation (14),

$$\sum_{i=0}^{N}\left[a_i x^i - x_i a_{i-1} x^{i-1}\right] = \sum_{i=0}^{N}\left[a_i x^{i-1}\right] \tag{27}$$

Since $a_i \neq 0$ in Equation (27), we have

$$\sum_{i=0}^{N} a_i \tilde{x}^{i-1} = 0 \tag{28}$$

If the value of N is so large or approaches infinity as in (14) and (15),

$$\sum_{i=0}^{N} a_i \tilde{x}^{i-1} = \sum_{i=0}^{N} \left[a_i + x^{i-1} + y_i \right] = 0$$

and this completes the proof.

4. Numerical Examples

To implement the method developed, three examples are considered.

Example 1

Consider the following thirteenth-order problem [1]

$$y^{(13)}(x) = \cos x - \sin x\,, \tag{29}$$

$$y^{(0)}(0) = 1,$$

$$y^{(1)}(0) = 1,$$

$$y^{(2)}(0) = -1,$$

$$y^{(3)}(0) = -1,$$

$$y^{(4)}(0) = 1,$$

$$y^{(5)}(0) = 1,$$

$$y^{(6)}(0) = -1,$$

$$y^{(0)}(1) = 1,$$

$$y^{(1)}(1) = -1,$$

$$y^{(2)}(1) = -1,$$

$$y^{(3)}(1) = 1,$$

$$y^{(4)}(1) = 1,$$

$$y^{(5)}(1) = -1,$$

The exact solution is

$$y(x) = \sin x + \cos x.$$

The given 13th order BVP (29) are transformed to systems of ODEs such that we have

$$\begin{aligned} \frac{\mathrm{d}y}{\mathrm{d}x} &= y_1, \\ \frac{\mathrm{d}y_1}{\mathrm{d}x} &= y_2, \\ \frac{\mathrm{d}y_2}{\mathrm{d}x} &= y_3, \\ &\vdots \\ \frac{\mathrm{d}y}{\mathrm{d}x} &= \cos x - \sin x, \end{aligned}$$

with the boundary conditions at $x = \xi_0 = 0$

$$y_1(0)=1, y_2(0)=1, y_3(0)=-1, y_4(0)=-1, y_5(0)=1, y_6(0)=1,$$

$$y_7(0)=-1, y_8(0)=a, y_9(0)=b, y_{11}(0)=d, y_{12}(0)=e, y_{13}(0)=f.$$

The series approximation of (29) is given as Equation (7)
where the unknown constants $a_i, i=0(1)N$ are uniquely determined by Equation (14).

Since, $\xi_0=0$, we have Equation (14) as

$$a_n=\frac{\lambda_n}{n!}, n\geq 0 \tag{30}$$

Using Equation (30) for $n=0(1)11$, we have the following:

$$\begin{aligned} &a_0=1, a_1=1, a_2=-\frac{1}{2}, a_3=-\frac{1}{6}, a_4=\frac{1}{24}, a_5=\frac{1}{120}, a_6=-\frac{1}{720}, \\ &a_7=\frac{a}{5040}, a_8=\frac{b}{40320}, a_9=\frac{c}{362880}, a_{10}=\frac{d}{3628800}, a_{11}=\frac{e}{39916800}. \end{aligned} \tag{31}$$

Substituting (31) into Equation (7) for $N = 0$ (1) 11 we obtain

$$\begin{aligned} y(x)=&\frac{1}{3628800}x^{10}d+\frac{1}{362880}x^9c+\frac{1}{40320}x^8b+\frac{1}{5040}x^7a-\frac{1}{720}x^6 \\ &+\frac{1}{120}x^5+\frac{1}{24}x^4-\frac{1}{6}x^3-\frac{1}{2}x^2+x+1+\frac{1}{39916800}x^{11}e \end{aligned} \tag{32}$$

Using boundary condition at $x=\xi_1=1$ in Equation (32) we obtain the values of *a*, *b*, *c*, *d* and *e*, as $a=1$, $b=1$, $c=-1$, $d=0.999997$ and $e=-1$.

The above values of a,b,c,d and *e* coincide with the results in [1], where Variational Iteration Method is used for the same problem considered.

Thus, the final approximation solution of BVP (29) can be written as

$$\begin{aligned} y(x)=&2.755723655E10^{-7}x^{10}-\frac{1}{362880}x^9+\frac{1}{40320}x^8+\frac{1}{5040}x^7 \\ &-\frac{1}{720}x^6+\frac{1}{120}x^5+\frac{1}{24}x^4-\frac{1}{6}x^3-\frac{1}{2}x^2+x+1 \end{aligned}$$

The comparison of the approximate solution of example 1 obtained with the help of PSAM and the approximate solution using VIM obtained in [1] is given in **Table 1**. From the numerical results, it is clear that the PSAM is more efficient and accurate. By increasing the order of approximation more accuracy can be obtained.

Example 2

Consider the following linear tenth-order problem [2]

$$y^{(10)}(x)=\mathrm{e}^{-x}y^2(x), a\leq x\leq b. \tag{33}$$

with the following boundary conditions

$$y^{2(k)}(0)=(1), k=0,1,2,3,4. \tag{34}$$

$$y^{(2k)}(0)=(1), k=0,1,2,3,4. \tag{35}$$

The exact solution is

$$y(x)=\mathrm{e}^x.$$

The given 10th order BVP (33) is transformed to systems of ODEs such that we have

$$\frac{\mathrm{d}y}{\mathrm{d}x} = y_1,$$
$$\frac{\mathrm{d}y_1}{\mathrm{d}x} = y_2,$$
$$\frac{\mathrm{d}y_2}{\mathrm{d}x} = y_3,$$
$$\vdots$$
$$\frac{\mathrm{d}y}{\mathrm{d}x} = \mathrm{e}^{-x} y^2(x)$$

with the boundary conditions at $x = \xi_0 = 0$

$$y_1(0) = 1, y_2(0) = 1, y_3(0) = 1, y_4(0) = 1, y_5(0) = 1,$$
$$y_6(0) = a, y_7(0) = b, y_8(0) = c, y_9(0) = d, y_{10}(0) = e.$$

Since, $\xi_0 = 0$, we have Equation (14) as

$$a_n = \frac{\lambda_n}{n!}, n \ge 0.$$

Hence for $n = 0(1)9$ we have

$$a_0 = 1, a_1 = 1, a_2 = \frac{1}{2}, a_3 = \frac{1}{6}, a_4 = \frac{1}{24}, a_5 = \frac{a}{120},$$
$$a_6 = \frac{b}{720}, a_7 = \frac{c}{5040}, a_8 = \frac{d}{40320}, a_9 = \frac{e}{362880}.$$

Hence, substituting the above values of $a_n, n = 0(1)9$ in (7), we obtain

$$y(x) = 1 + x + \frac{1}{2}x^2 + \frac{1}{6}x^3 + \frac{1}{24}x^4 + \frac{1}{120}ax^5 + \frac{1}{720}bx^6 + \frac{1}{5040}cx^7 + \frac{1}{40320}dx^8 + \frac{1}{362880}ex^9 \tag{36}$$

Using boundary condition at $x = \xi_1 = 1$ on equation (36) we obtain the values of *a*, *b*, *c*, *d* and *e*, as $a = 1.000029332$, $b = 0.9997112299$, $c = 1.002812535$, $d = 0.9735681663$, and $e = 1.218281800$

Thus, the final approximation solution of the BVP (33) can be written as

$$y(x) = 1 + x + \frac{1}{2}x^2 + \frac{1}{6}x^3 + \frac{1}{24}x^4 + 0.008333577767x^5 + 0.001388487819x^6$$
$$+ 0.0001989707411x^7 + 0.00002414603587x^8 + 3.357258047E10^{-7}x^9$$

The comparison of the approximate solution of Example 2 obtained with the help of PSAM and the approximate solution using HPM [2] is given in **Table 2**. From the numerical results, it is clear that the PSAM is more efficient and accurate. By increasing the order of approximation more accuracy can be obtained.

Example 3

Consider the following twelve-order problem

$$y^{(12)}(x) = 2\mathrm{e}^{x} y^{(2)}(x) + y^{(3)}(x), a \le x \le b. \tag{37}$$

with the following boundary conditions

$$y^{(2k)}(0) = 1, \ k = 0,1,2,3,4,5 \tag{38}$$

$$y^{(2k)}(1) = \left(\frac{1}{\mathrm{e}}\right), k = 0,1,2,3,4,5 \tag{39}$$

The exact solution is

$$y(x) = \mathrm{e}^{-x}.$$

The given 12th order BVP (37) is transformed to systems of ODEs such that we have

$$\frac{\mathrm{d}y}{\mathrm{d}x}=y_1,$$
$$\frac{\mathrm{d}y_1}{\mathrm{d}x}=y_2,$$
$$\frac{\mathrm{d}y_2}{\mathrm{d}x}=y_3,$$
$$\vdots$$
$$\frac{\mathrm{d}y}{\mathrm{d}x}=2\mathrm{e}^x y^2(x)+y^{(3)}(x),$$

with the boundary conditions (at $x=\xi_0=0$)

$$y_1(0)=1, y_2(0)=a, y_3(0)=1, y_4(0)=b, y_5(0)=1, y_6(0)=c, y_7(0)=1,$$
$$y_8(0)=d, y_9(0)=1, y_{10}(0)=e, y_{11}(0)=1 \text{ and } y_{12}(0)=f.$$

Since, $\xi_0=0$, we have Equation (14) as

$$a_n=\frac{\lambda_n}{n!},\quad n\geq 0.$$

Hence for $n=0(1)11$ we obtain the following

$$a_0=1, a_1=a, a_2=\frac{1}{2}, a_3=\frac{b}{6}, a_4=\frac{1}{24}, a_5=\frac{c}{120}, a_6=\frac{1}{720}, a_7=\frac{d}{5040},$$
$$a_8=\frac{1}{40320}, a_9=\frac{e}{362880}, a_{10}=\frac{1}{3628800}, a_{11}=\frac{f}{39916800}.$$

Hence, substituting the above values of $a_n, n=0(1)11$ in (7), we obtain

$$\begin{aligned}y(x)=1+ax+\frac{1}{2}x^2+\frac{1}{6}bx^3+\frac{1}{24}x^4+\frac{1}{120}cx^5+\frac{1}{720}x^6+\frac{1}{5040}dx^7\\+\frac{1}{40320}x^8+\frac{1}{362880}ex^9+\frac{1}{3628800}x^{10}+\frac{1}{39916800}fx^{11}\end{aligned}\tag{40}$$

Using boundary condition at $x=\xi_1=1$ in Equation (40) we obtain the values of *a*, *b*, *c*, *d*, *e* and *f*, as $a=-0.9999940293$, $b=-1.000058885$, $c=-0.9994190942$, $d=-1.005725028$, $e=-0.9434337955$ and $f=-1.632120555$.

Thus, substituting the values *a*, *b*, *c*, *d*, *e* and *f* in (40), the final approximation solution of BVP (37) can be written as

$$\begin{aligned}y(x)=1-0.9999940293x+\frac{1}{2}x^2-0.1666764808x^3+\frac{1}{24}x^4\\-0.008328492452x^5+\frac{1}{720}x^6-0.0001995486167x^7+\frac{1}{40320}x^8\\-0.000002599850627x^9+\frac{1}{3628800}x^{10}-4.088806104E10^{-8}x^{11}\end{aligned}$$

The comparison of the approximate solution of Example 3 obtained with the help of PSAM and the approximate solution using HPM [2] is given in **Table 3**. From the numerical results, it is clear that the PSAM is more efficient and accurate. By increasing the order of approximation more accuracy can be obtained.

5. Conclusion

In this paper, the Power Series Approximation Method has been applied to obtain the numerical solution of linear and nonlinear generalized N^{th} order boundary value problems. The PSAM requires no discretization, linea-rization or perturbation. By increasing the order of approximation more accuracy can be obtained. Comparison of the results obtained with existing techniques [1] [2] shows that the PSAM is more efficient and accurate. Hence, it is easier and more economical to apply PSAM in solving BVPs.

Table 1. Comparison of results of PSAM with Variational Iteration Method (VIM).

X	Exact Solution	PSAM	VIM
0.00	1.0000000	1.0000000	1.000000
0.10	1.0948376	1.0948376	0.994054
0.20	1.1787359	1.1787359	0.931864
0.30	1.2508567	1.2508568	0.769356
0.40	1.3104793	1.3104800	0.784691
0.50	1.3570081	1.3570112	0.659287
0.60	1.3899781	1.3899892	0.537115
0.70	1.4090599	1.4090924	0.381117
0.80	1.4140628	1.4141457	0.240714
0.90	1.4049369	1.4051257	0.129106
1.00	1.3817733	1.3821676	0.000000

Table 2. Comparison of results of PSAM with HPM.

X	Exact Solution	PSAM	HPM
0.0	0.100000000	0.100000000	0.1000000000
0.2	0.122140276	0.122140276	0.1221408246
0.4	0.149182470	0.149182470	0.1491833581
0.6	0.182211880	0.182211878	0.1822127686
0.8	0.222554093	0.222554055	0.2225546413
1.0	0.271828183	0.271827885	0.2718281799

Table 3. Comparison of results of PSAM with HPM.

X	Exact Solution	PSAM	HPM
0.0	10.000000000	10.000000000	10.000000000
0.2	8.187307531	8.187318703	8.187308703
0.4	6.703200460	6.703218540	6.703208540
0.6	5.488116361	5.488134449	5.488114451
0.8	4.493289641	4.493300834	4.493289646
1.0	3.678794412	3.678794408	3.678794453

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