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Generalization of Uniqueness of Meromorphic Functions Sharing Fixed Point

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Abstract

In this paper, we study the uniqueness problems of entire and meromorphic functions concerning differential polynomials sharing fixed point and obtain some results which generalize the results due to Subhas S. Bhoosnurmath and Veena L. Pujari [1].

Keywords

Entire Functions, Uniqueness, Meromorphic Functions, Fixed Point, Differential Polynomials

1. Introduction and Main Results

Let $f(z)$ be a non constant meromorphic function in the whole complex plane $\mathbb{C}$. We will use the following standard notations of value distribution theory: $T(r,f), m(r,f), N(r,f), \bar{N}(r,f), \cdots$ (see [2] [3]). We denote by $S(r,f)$ any function satisfying $S(r,f) = o\{T(r,f)\}$, as $r \to +\infty$, possibly outside of a set with finite linear measure.

Let a be a finite complex number and k a positive integer. We denote by $N_{k)}\left(r, \frac{1}{(f-a)}\right)$ the counting function for zeros of $f(z)-a$ in $|z| \le r$ with multiplicity $\le k$ and by $\bar{N}_{k)}\left(r, \frac{1}{(f-a)}\right)$ the corresponding one for which multiplicity is not counted. Let $N_{(k}\left(r, \frac{1}{(f-a)}\right)$ be the counting function for zeros of $f(z)-a$

in $|z| \le r$ with multiplicity $\ge k$ and $\bar{N}_{(k}\left(r, \frac{1}{(f-a)}\right)$ the corresponding one for which multiplicity is not counted. Set

$$N_k\left(r,\frac{1}{f-a}\right)=\bar{N}\left(r,\frac{1}{f-a}\right)+\bar{N}_{(2}\left(r,\frac{1}{f-a}\right)+\cdots+\bar{N}_{(k}\left(r,\frac{1}{f-a}\right).$$

Let $g(z)$ be a non constant meromorphic function. We denote by $\bar{N}_L\left(r,\frac{1}{f-a}\right)$ the counting function for a-points of both $f(z)$ and $g(z)$ about which $f(z)$ has larger multiplicity than $g(z)$, where multiplicity is not counted. Similarly, we have notation $\bar{N}_L\left(r,\frac{1}{g-a}\right)$.

We say that *f* and *g* share *a* CM (counting multiplicity) if $(f-a)$ and $(g-a)$ have same zeros with the same multiplicities. Similarly, we say that *f* and *g* share *a* IM (ignoring multiplicity) if $(f-a)$ and $(g-a)$ have same zeros with ignoring multiplicities.

In 2004, Lin and Yi [4] obtained the following results.

Theorem A. Let *f* and *g* be two transcendental meromorphic functions, $n \ge 12$ an integer. If $f^n(f-1)f'$ and $g^n(g-1)g'$ share *z* CM, then either $f(z)\equiv g(z)$ or

$$g=\frac{(n+2)\left(1-h^{n+1}\right)}{(n+1)\left(1-h^{n+2}\right)},\quad f=\frac{(n+2)h\left(1-h^{n+1}\right)}{(n+1)\left(1-h^{n+2}\right)}$$

where *h* is a non constant meromorphic function.

Theorem B. Let *f* and *g* be two transcendental meromorphic functions, $n \ge 13$ an integer. If $f^n(f-1)^2 f'$ and $g^n(g-1)^2 g'$ share *z* CM, then $f(z)\equiv g(z)$.

In 2013, Subhas S. Bhoosnurmath and Veena L. Pujari [1] extended the above theorems A and B with respect to differential polynomials sharing fixed points. They proved the following results.

Theorem C. Let *f* and *g* be two non constant meromorphic functions, $n \ge 11$ a positive integer. If $f^n(f-1)f'$ and $g^n(g-1)g'$ share *z* CM, *f* and *g* share ∞ IM, then either $f(z)\equiv g(z)$ or

$$g=\frac{(n+2)\left(1-h^{n+1}\right)}{(n+1)\left(1-h^{n+2}\right)},\quad f=\frac{(n+2)h\left(1-h^{n+1}\right)}{(n+1)\left(1-h^{n+2}\right)}$$

where *h* is a non constant meromorphic function.

Theorem D. Let *f* and *g* be two non constant meromorphic functions, $n \ge 12$ a positive integer. If $f^n(f-1)^2 f'$ and $g^n(g-1)^2 g'$ share *z* CM, *f* and *g* share ∞ IM, then $f(z)\equiv g(z)$.

Theorem E. Let *f* and *g* be two non constant entire functions, $n \ge 7$ an integer. If $f^n(f-1)f'$ and $g^n(g-1)g'$ share *z* CM, then $f(z)\equiv g(z)$.

In this paper, we generalize theorems C, D, E and obtain the following results.

Theorem 1. Let *f* and *g* be two non constant meromorphic functions, $n \ge m+10$ an integer. If $f^n(f-1)^m f'$ and $g^n(g-1)^m g'$ share *z* CM, *f* and *g* share ∞ IM, then $f(z)\equiv g(z)$.

For $m=1, n\ge 11$, we get Theorem C.

For $m=2, n\ge 12$, we get Theorem D.

Theorem 2. Let *f* and *g* be two non constant entire functions, $n \ge m+6$ an integer. If $f^n(f-1)^m f'$ and $g^n(g-1)^m g'$ share *z* CM, then $f(z)\equiv g(z)$.

2. Some Lemmas

Lemma 2.1 (see [5]). Let f_1, f_2 and f_3 be non constant meromorphic functions such that $f_1+f_2+f_3=1$. If f_1, f_2 and f_3 are linearly independent, then

$$T(r,f_1)<\sum_{i=1}^{3}N_2\left(r,\frac{1}{f_i}\right)+\sum_{i=1}^{3}\bar{N}(r,f_i)+o(T(r)),$$

where $T(r)=\max_{1\le i\le 3}\{T(r,f_i)\}$ and $r\notin E$.

Lemma 2.2 (see [2]). Let f_1 and f_2 be two non constant meromorphic functions. If $c_1f_1+c_2f_2=c_3$, where c_1,c_2 and c_3 are non-zero constants, then

$$T(r,f_1)\le \bar{N}(r,f_1)+\bar{N}\left(r,\frac{1}{f_1}\right)+\bar{N}\left(r,\frac{1}{f_2}\right)+S(r,f_1)$$

Lemma 2.3 (see [2]). Let *f* be a non constant meromorphic function and let *k* be a non-negative integer, then

$$N\left(r,\frac{1}{f^{(k)}}\right)\le N\left(r,\frac{1}{f}\right)+k\bar{N}(r,f)+S(r,f)$$

Lemma 2.4 (see [6]). Suppose that $f(z)$ is a meromorphic function in the complex plane and $P(f)=a_0f^n+a_1f^{n-1}+\cdots+a_n$, where $a_0(\ne 0),a_1,\cdots,a_n$ are small meromorphic functions of $f(z)$. Then

$$T(r,P(f))=nT(r,f)+S(r,f)$$

Lemma 2.5 (see [7]). Let f_1,f_2 and f_3 be three meromorphic functions satisfying $\sum_{j=1}^{3}f_j=1$, let $g_1=-f_3/f_2, g_2=1/f_2$ and $g_3=-f_1/f_2$. If f_1,f_2 and f_3 are linearly independent then g_1,g_2 and g_3 are linearly independent.

Lemma 2.6 (see [8]). Let $Q(w)=(n-1)^2(w^n-1)(w^{n-2}-1)-n(n-2)(w^{n-1}-1)^2$, then $Q(w)=(w-1)^4(w-\beta_1)(w-\beta_2)\cdots(w-\beta_{2n-6})$ where $\beta_j\in C\setminus\{0,1\}(j=1,2,3,\cdots,2n-6)$ which are distinct respectively.

The following lemmas play a cardinal role in proving our results.

Lemma 2.7 Let *f* and *g* be two non constant meromorphic functions. If $f^n(f-1)^m f'$ and $g^n(g-1)^m g'$ share *z* CM and $n>m+5$, then

$$T(r,g)\le\left(\frac{n+m+2}{n+m-7}\right)T(r,f)+\log r+S(r,g)$$

Proof. Applying Nevanlinna's second fundamental theorem (see [3]) to $g^n(g-1)^m g'$, we have

$$\begin{aligned}T\left(r,g^n(g-1)^m g'\right)&\le \bar{N}\left(r,g^n(g-1)^m g'\right)+\bar{N}\left(r,\frac{1}{g^n(g-1)^m g'}\right)+\bar{N}\left(r,\frac{1}{g^n(g-1)^m g'-z}\right)+S(r,g)\\&\le \bar{N}(r,g)+\bar{N}\left(r,\frac{1}{g}\right)+\bar{N}\left(r,\frac{1}{g-1}\right)+\bar{N}\left(r,\frac{1}{g'}\right)+\bar{N}\left(r,\frac{1}{f^n(f-1)^m f'-z}\right)+S(r,g)\end{aligned}\tag{1}$$

By first fundamental theorem (see [3]) and (1), we have

$$\begin{aligned}(n+m)T(r,g)&\le T\left(r,g^n(g-1)^m\right)+S(r,g)\\&\le T\left(r,g^n(g-1)^m g'\right)+T\left(r,\frac{1}{g'}\right)+S(r,g)\\&\le \bar{N}(r,g)+\bar{N}\left(r,\frac{1}{g}\right)+\bar{N}\left(r,\frac{1}{g-1}\right)+\bar{N}\left(r,\frac{1}{g'}\right)\\&\quad+\bar{N}\left(r,\frac{1}{f^n(f-1)^m f'-z}\right)+T(r,g')+S(r,g)\end{aligned}\tag{2}$$

We know that,

$$\overline{N}\left(r,\frac{1}{f^n(f-1)^m f'-z}\right)\le T\left(r,\frac{1}{f^n(f-1)^m f'-z}\right)$$

$$=T\left(r,f^n(f-1)^m f'-z\right)+O(1) \tag{3}$$

$$\le nT(r,f)+mT(r,(f-1))+T(r,f')+\log r+O(1)$$

$$\le (n+m+2)T(r,f)+\log r+O(1)$$

Therefore, using Lemma 2.3, (2) becomes

$$(n+m)T(r,g)\le \overline{N}(r,g)+\overline{N}\left(r,\frac{1}{g}\right)+\overline{N}\left(r,\frac{1}{g-1}\right)+\overline{N}\left(r,\frac{1}{g'}\right)$$
$$+(n+m+2)T(r,f)+\log r+T(r,g')+S(r,g)$$

Using $\overline{N}\left(r,\frac{1}{g'}\right)\le 2T(r,g), T(r,g')\le 2T(r,g)$, we get

$$(n+m)T(r,g)\le 7T(r,g)+(n+m+2)T(r,f)+\log r+S(r,g)$$

$$(n+m-7)T(r,g)\le (n+m+2)T(r,f)+\log r+S(r,g) \tag{4}$$

since $n>m+5$, we have

$$T(r,g)\le\left(\frac{n+m+2}{n+m-7}\right)T(r,f)+\log r+S(r,g)$$

This completes the proof of Lemma 2.7.

Lemma 2.8 Let *f* and *g* be two non constant entire functions. If $f^n(f-1)^m f'$ and $g^n(g-1)^m g'$ share *z* CM and $n>m+2$, then

$$T(r,g)\le\left(\frac{n+m+1}{n+m-4}\right)T(r,f)+\log r+S(r,g) \tag{5}$$

Proof. Since f and g are entire functions, we have $\overline{N}(r,g)=0$. Proceeding as in the proof of Lemma 2.7, we can easily prove Lemma 2.8.

3. Proof of Theorems

Proof of Theorem 1. By assumption, $f^n(f-1)^m f'$ and $g^n(g-1)^m g'$ share *z* CM, *f* and *g* share ∞ IM. Let

$$H=\frac{f^n(f-1)^m f'-z}{g^n(g-1)^m g'-z} \tag{6}$$

Then, *H* is a meromorphic function satisfying

$$T(r,H)=T\left(r,\frac{f^n(f-1)^m f'-z}{g^n(g-1)^m g'-z}\right)$$
$$\le T\left(r,f^n(f-1)^m f'-z\right)+T\left(r,g^n(g-1)^m g'-z)\right)+O(1)$$

By (3), we get

$$T(r,H)\le(n+m+2)\left[T(r,f)+T(r,g)\right]+O(\log r)$$

Therefore,

$$T(r,H)=O\left[T(r,f)+T(r,g)\right] \tag{7}$$

From (6), we easily see that the zeros and poles of H are multiple and satisy

$$\overline{N}(r,H)\le \overline{N_L}(r,f),\qquad \overline{N}\left(r,\frac{1}{H}\right)\le \overline{N_L}(r,g) \tag{8}$$

Let

$$f_1=\frac{f^n(f-1)^m f'}{z},\quad f_2=H,\quad f_3=-\frac{Hg^n(g-1)^m g'}{z} \tag{9}$$

Then, $f_1+f_2+f_3=1$ and $T(r)$ denote the maximum of $T(r,f_j)$, $j=1,2,3$.

We have, $T(r,f_1)=O(T(r,f))$, (10)

$$T(r,f_2)=O(T(r,f)+T(r,g)),$$

$$T(r,f_3)=O(T(r,f)+T(r,g)). \tag{11}$$

Therefore, $T(r)=O(T(r,f)+T(r,g))$,
and thus

$$S(r,f)+S(r,g)=o(T(r)) \tag{12}$$

Now, we discuss the following three cases.

Case 1. Suppose that neither f_2 nor f_3 is a constant. If f_1, f_2 and f_3 are linearly independent, then by Lemma 2.1 and 2.4, we have

$$\begin{aligned}
T(r,f_1)&<\sum_{i=1}^{3}N_2\left(r,\frac{1}{f_i}\right)+\sum_{i=1}^{3}\overline{N}(r,f_i)+o(T(r))\\
&\le N_2\left(r,\frac{1}{f_1}\right)+N_2\left(r,\frac{1}{f_2}\right)+N_2\left(r,\frac{1}{f_3}\right)+\overline{N}(r,f_1)+\overline{N}(r,f_2)+\overline{N}(r,f_3)+o(T(r))\\
&\le N_2\left(r,\frac{z}{f^n(f-1)^m f'}\right)+N_2\left(r,\frac{1}{H}\right)+N_2\left(r,\frac{z}{Hg^n(g-1)^m g'}\right)\\
&\quad+\overline{N}\left(r,\frac{f^n(f-1)^m f'}{z}\right)+\overline{N}(r,H)+\overline{N}\left(r,\frac{Hg^n(g-1)^m g'}{z}\right)+o(T(r))\\
&=N_2\left(r,\frac{1}{f^n(f-1)^m f'}\right)+N_2\left(r,\frac{1}{H}\right)+N_2\left(r,\frac{1}{Hg^n(g-1)^m g'}\right)\\
&\quad+\overline{N}\left(r,f^n(f-1)^m f'\right)+\overline{N}(r,H)+\overline{N}\left(r,Hg^n(g-1)^m g'\right)+2\log r+o(T(r))
\end{aligned} \tag{13}$$

Using (8), we note that

$$\begin{aligned}
N_2\left(r,\frac{1}{Hg^n(g-1)^m g'}\right)&\le N_2\left(r,\frac{1}{H}\right)+N_2\left(r,\frac{1}{g^n(g-1)^m g'}\right)\\
&\le 2\overline{N_L}(r,g)+N_2\left(r,\frac{1}{g^n(g-1)^m g'}\right)
\end{aligned}$$

since, $\overline{N_L}(r,g)=0$, We obtain that,

$$N_2\left(r,\frac{1}{Hg^n(g-1)^m g'}\right)\le N_2\left(r,\frac{1}{g^n(g-1)^m g'}\right) \tag{14}$$

$$\bar{N}\left(r, Hg^n(g-1)^m g'\right) \le \bar{N}(r,H) + \bar{N}\left(r, g^n(g-1)^m g'\right) \le \overline{N_L}(r,f) + \bar{N}(r,g)$$

But $\overline{N_L}(r,f)=0$, so we get

$$\bar{N}\left(r, Hg^n(g-1)^m g'\right) \le \bar{N}(r,g) \tag{15}$$

Using (14) and (15) in (13), we get

$$T(r,f_1) \le N_2\left(r, \frac{1}{f^n(f-1)^m f'}\right) + N_2\left(r, \frac{1}{H}\right) + N_2\left(r, \frac{1}{g^n(g-1)^m g'}\right)$$
$$+\bar{N}(r,H) + \bar{N}(r,f) + \bar{N}(r,g) + 2\log r + o\left(T(r)\right)$$

Since f and g share ∞ IM, we have $\bar{N}(r,f) = \bar{N}(r,g)$.
Using this with (8), we get

$$T(r,f_1) \le N_2\left(r, \frac{1}{f^n(f-1)^m f'}\right) + 2\overline{N_L}(r,g) + N_2\left(r, \frac{1}{g^n(g-1)^m g'}\right)$$
$$+2\bar{N}(r,f) + \overline{N_L}(r,f) + 2\log r + o\left(T(r)\right)$$
$$\le N\left(r, \frac{1}{f^n(f-1)^m f'}\right) - \left[N_{(3}\left(r, \frac{1}{f^n(f-1)^m f'}\right) - 2\bar{N}_{(3}\left(r, \frac{1}{f^n(f-1)^m f'}\right)\right] \tag{16}$$
$$+N\left(r, \frac{1}{g^n(g-1)^m g'}\right) - \left[N_{(3}\left(r, \frac{1}{g^n(g-1)^m g'}\right) - 2\bar{N}_{(3}\left(r, \frac{1}{g^n(g-1)^m g'}\right)\right]$$
$$+2\overline{N_L}(r,g) + \overline{N_L}(r,f) + 2\bar{N}(r,f) + 2\log r + o\left(T(r)\right)$$

If z_0 is a zero of f with multiplicity p, then z_0 is a zero of $f^n(f-1)^m f'$ with multiplicity $np+p-1 \ge 3$, we have

$$\left[N_{(3}\left(r, \frac{1}{f^n(f-1)^m f'}\right) - 2\bar{N}_{(3}\left(r, \frac{1}{f^n(f-1)^m f'}\right)\right] \ge (n-2)N\left(r, \frac{1}{f}\right) \tag{17}$$

Similarly,

$$\left[N_{(3}\left(r, \frac{1}{g^n(g-1)^m g'}\right) - 2\bar{N}_{(3}\left(r, \frac{1}{g^n(g-1)^m g'}\right)\right] \ge (n-2)N\left(r, \frac{1}{g}\right) \tag{18}$$

Let

$$f_1^* = \frac{f^{n+m+1}}{n+m+1} - \frac{mC_1}{n+m} f^{n+m} + \frac{mC_2}{n+m-1} f^{n+m-1} + \cdots + (-1)^p \frac{1}{n+1} f^{n+1} \tag{19}$$

By Lemma 2.6, we have

$$T\left(r, f_1^*\right) = (n+m+1)T(r,f) + S(r,f)$$

Since $\left(f_1^*\right)' = zf_1$, we have

$$m\left(r, \frac{1}{f_1^*}\right) \le m\left(r, \frac{1}{zf_1}\right) + m\left(r, \frac{\left(f_1^*\right)'}{f_1^*}\right) \le m\left(r, \frac{1}{f_1}\right) + \log r + S(r,f)$$

By the first fundamental theorem, we have

$$T\left(r, f_1^*\right) \le T(r,f_1) + N\left(r, \frac{1}{f_1^*}\right) - N\left(r, \frac{1}{f_1}\right) + \log r + S(r,f) \tag{20}$$

we have

$$N\left(r,\frac{1}{f_1^*}\right)=(n+1)N\left(r,\frac{1}{f}\right)+N\left(r,\frac{1}{f-a_1}\right)+N\left(r,\frac{1}{f-a_2}\right)+\cdots+N\left(r,\frac{1}{f-a_m}\right) \tag{21}$$

where $a_1,a_2,\cdots,a_m$ are distinct roots of algebraic equation,

$$\frac{mC_0}{n+m+1}z^m-\frac{mC_1}{n+m}z^{m-1}+\frac{mC_2}{n+m-1}z^{m-2}+\cdots+(-1)^p\frac{1}{n+1}=0$$

From (16)-(21), we get

$$\begin{aligned}T(r,f_1^*)\le{}& N\left(r,\frac{1}{f^n(f-1)^m f'}\right)-(n-2)N\left(r,\frac{1}{f}\right)+N\left(r,\frac{1}{g^n(g-1)^m g'}\right)\\&-(n-2)N\left(r,\frac{1}{g}\right)+2\overline{N_L}(r,g)+\overline{N_L}(r,f)+2\overline{N}(r,f)+(n+1)N\left(r,\frac{1}{f}\right)\\&+N\left(r,\frac{1}{f-a_1}\right)+N\left(r,\frac{1}{f-a_2}\right)+\cdots+N\left(r,\frac{1}{f-a_m}\right)\\&-N\left(r,\frac{1}{f^n(f-1)^m f'}\right)+3\log r+o(T(r))\end{aligned}$$

Using Lemma 2.3, we get

$$\begin{aligned}&(n+m+1)T(r,f)\\&\le 3N\left(r,\frac{1}{f}\right)+3N\left(r,\frac{1}{g}\right)+mN\left(r,\frac{1}{g-1}\right)+\overline{N}(r,g)+2\overline{N_L}(r,g)+\overline{N_L}(r,f)\\&\quad+2\overline{N}(r,f)+N\left(r,\frac{1}{f-a_1}\right)+N\left(r,\frac{1}{f-a_2}\right)+\cdots+N\left(r,\frac{1}{f-a_m}\right)\\&\quad+3\log r+o(T(r))\end{aligned} \tag{22}$$

Let

$$g_1=-\frac{f_3}{f_2}=\frac{g^n(g-1)^m g'}{z},\quad g_2=\frac{1}{f_2}=\frac{1}{H},\quad g_3=\frac{-f_1}{f_2}=\frac{-f^n(f-1)^m f'}{zH}$$

Then $g_1+g_2+g_3=1$. By Lemma 2.5, g_1,g_2 and g_3 are linearly independent. In the same manner as above, we get expression for $(n+m+1)T(r,g)$.

Note that $\overline{N_L}(r,f)+\overline{N_L}(r,g)\le\overline{N}(r,f)=\overline{N}(r,g)$. We have,

$$\begin{aligned}&(n+m+1)\left[T(r,f)+T(r,g)\right]\\&\le 6\left[N\left(r,\frac{1}{f}\right)+N\left(r,\frac{1}{g}\right)\right]+3\left[\overline{N}(r,f)+\overline{N}(r,g)\right]+3\left[\overline{N_L}(r,f)+\overline{N_L}(r,g)\right]\\&\quad+N\left(r,\frac{1}{f-a_1}\right)+N\left(r,\frac{1}{f-a_2}\right)+\cdots+N\left(r,\frac{1}{f-a_m}\right)\\&\quad+N\left(r,\frac{1}{g-a_1}\right)+N\left(r,\frac{1}{g-a_2}\right)+\cdots+N\left(r,\frac{1}{g-a_m}\right)\\&\quad+m\left[N\left(r,\frac{1}{f-1}\right)+N\left(r,\frac{1}{g-1}\right)\right]+6\log r+o(T(r))\end{aligned}$$

Simplifying, we get

$$(n-m-5)\left[T(r,f)+T(r,g)\right] \le 3\bar{N}(r,f)+6\bar{N}(r,g)+6\log r+o(T(r)) \tag{23}$$

or

$$(n-m-5)\left[T(r,f)+T(r,g)\right] \le 3\bar{N}(r,g)+6\bar{N}(r,f)+6\log r+o(T(r)) \tag{24}$$

Combining (23) and (24), we get

$$\left(n-m-\frac{19}{2}\right)\left[T(r,f)+T(r,g)\right] \le 6\log r+o(T(r)) \tag{25}$$

By $n \ge m+10$ and (12), we get a contradiction. Thus f_1, f_2 and f_3 are linearly dependent. Then, there exists three constants $(c_1,c_2,c_3) \ne (0,0,0)$ such that

$$c_1 f_1 + c_2 f_2 + c_3 f_3 = 0 \tag{26}$$

If $c_1 = 0$, from (26) $c_2 \ne 0$, $c_3 \ne 0$ and $f_3 = -\frac{c_2}{c_3} f_2$

$$\Rightarrow g^n (g-1)^m g' = \frac{c_2}{c_3} z$$

On integrating, we get

$$\frac{g^{n+m+1}}{n+m+1} - mC_1 \frac{g^{n+m}}{n+m} + \cdots + (-1)^p \frac{g^{n+1}}{n+1} = \frac{c_2}{c_3}\frac{z^2}{2} + k,\ k \text{ is constant} \tag{27}$$

$$\Rightarrow T\left(r, \frac{g^{n+m+1}}{n+m+1} - mC_1 \frac{g^{n+m}}{n+m} + \cdots + (-1)^p \frac{g^{n+1}}{n+1}\right) \le T\left(r,z^2\right) + O(1)$$

$$\Rightarrow (n+m+1)T(r,g) \le 2\log r + O(1)$$

Since $n \ge m+10$, we get a contradiction. Thus, $c_1 \ne 0$ and by (26), we have

$$c_1 f_1 = -c_2 f_2 - c_3 f_3$$

$$\Rightarrow f_1 = \frac{-c_2}{c_1} f_2 - \frac{c_3}{c_1} f_3$$

Substituting this in $f_1 + f_2 + f_3 = 1$, we get

$$\frac{-c_2}{c_1} f_2 - \frac{c_3}{c_1} f_3 + f_2 + f_3 = 1$$

that is, $\left(1-\frac{c_2}{c_1}\right) f_2 + \left(1-\frac{c_3}{c_1}\right) f_3 = 1$, where $c_1 \ne c_3, c_2 \ne c_3$

From (9), we obtain

$$\left(1-\frac{c_3}{c_1}\right)\frac{g^n (g-1)^m g'}{z} + \frac{1}{H} = \left(1-\frac{c_2}{c_1}\right) \tag{28}$$

Applying Lemma 2.2, to the above equation, we get

$$T\left(r, \frac{g^n (g-1)^m g'}{z}\right) \le \bar{N}\left(r, \frac{g^n (g-1)^m g'}{z}\right) + \bar{N}\left(r, \frac{z}{g^n (g-1)^m g'}\right) + \bar{N}(r,H) + S(r,g) \tag{29}$$

Note that,

$$T\left(r, g^n (g-1)^m g'\right) \le T\left(r, \frac{g^n (g-1)^m g'}{z}\right) + T(r,z) \le T\left(r, \frac{g^n (g-1)^m g'}{z}\right) + \log r$$

Using (29), we get

$$T\left(r, g^n(g-1)^m g'\right) \le \bar{N}\left(r, \frac{1}{g^n(g-1)^m g'}\right) + 2\bar{N}(r,g) + 2\log r + S(r,g) \tag{30}$$

By, Lemmas 2.3, 2.4 and (30), we have

$$(n+m)T(r,g) = T\left(r, g^n(g-1)^m\right) + S(r,g)$$

$$\le T\left(r, g^n(g-1)^m g'\right) + T\left(r, \frac{1}{g'}\right) + S(r,g)$$

$$\Rightarrow (n-m-8)T(r,g) \le 2\log r + S(r,g)$$

We obtain $n \le 8-m$, which contradicts $n \ge m+10$.

Case 2. Suppose that $f_2 = c(\ne 0)$, where c is constant If $c \ne 1$, then, we have

$$f_1 + f_2 + f_3 = 1 \tag{31}$$

$$\Rightarrow \frac{f^n(f-1)^m f'}{z} - c\frac{g^n(g-1)^m g'}{z} = 1-c$$

Applying Lemma 2.2 to the above equation, we have

$$T\left(r, \frac{f^n(f-1)^m f'}{z}\right) \le \bar{N}(r,f) + \bar{N}\left(r, \frac{1}{g^n(g-1)^m g'}\right) + \bar{N}\left(r, \frac{1}{f^n(f-1)^m f'}\right) + \log r + S(r,f)$$

$$\Rightarrow T\left(r, f^n(f-1)^m f'\right) \le \bar{N}(r,f) + \bar{N}\left(r, \frac{1}{g^n(g-1)^m g'}\right) + \bar{N}\left(r, \frac{1}{f^n(f-1)^m f'}\right) + 2\log r + S(r,f) \tag{32}$$

By Lemmas 2.3, 2.4 and (32), we have

$$(n+m)T(r,f) = T\left(r, f^n(f-1)^m\right) + S(r,f)$$

$$\le T\left(r, f^n(f-1)^m f'\right) + T\left(r, \frac{1}{f'}\right) + S(r,f)$$

$$\Rightarrow (n+m-7)T(r,f) \le 4T(r,g) + 2\log r + S(r,f)$$

Using Lemma 2.7, we get

$$(n+m-7)T(r,f) \le 4\left(\frac{n+m+2}{n+m-7}\right)T(r,f) + 2\log r + S(r,f) \tag{33}$$

Since $n \ge m+10$, we get contradiction

Therefore, $c=1$ and by (6), (8), we have

$$f^n(f-1)^m f' = g^n(g-1)^m g' \tag{34}$$

On integrating, we get

$$\begin{aligned} &\frac{f^{n+m+1}}{n+m+1} - mC_1\frac{f^{n+m}}{n+m} + \cdots + (-1)^p\frac{f^{n+1}}{n+1} \\ &= \frac{g^{n+m+1}}{n+m+1} - mC_1\frac{g^{n+m}}{n+m} + \cdots + (-1)^p\frac{g^{n+1}}{n+1} + k \end{aligned} \tag{35}$$

$$F^* = G^* + k, \text{ where } k \text{ is a constant}$$

We claim that $k=0$. Suppose that $k \ne 0$, then

$$\Theta(0,F^*)+\Theta(k,F^*)+\Theta(\infty,F^*)=\Theta(0,F^*)+\Theta(0,G^*)+\Theta(\infty,F^*) \tag{36}$$

We have,

$$\overline{N}\left(r,\frac{1}{F^*}\right)=\overline{N}\left(r,\frac{1}{f}\right)+\overline{N}\left(r,\frac{1}{f-a_1}\right)+\cdots+\overline{N}\left(r,\frac{1}{f-a_m}\right)\le(m+1)T(r,f)$$

similarly, $\overline{N}\left(r,\frac{1}{G^*}\right)\le(m+1)T(r,g)$

$$\overline{N}(r,F^*)=\overline{N}(r,f)\le T(r,f)$$

Using Lemma 2.4, we have

$$T(r,F^*)=(n+m+1)T(r,f)+S(r,f)$$

$$T(r,G^*)=(n+m+1)T(r,g)+S(r,g) \tag{37}$$

Thus,

$$\Theta(0,F^*)=1-\overline{\lim_{r\to\infty}}\frac{\overline{N}(r,1/F^*)}{T(r,F^*)}\ge1-\frac{(m+1)T(r,f)}{(n+m+1)T(r,f)}\ge1-\frac{(m+1)}{(n+m+1)} \tag{38}$$

similarly, $\Theta(0,G^*)\ge1-\frac{(m+1)}{(n+m+1)}$

$$\Theta(\infty,F^*)=\overline{\lim_{r\to\infty}}\frac{\overline{N}(r,F^*)}{T(r,F^*)}\ge1-\frac{1}{(n+m+1)}$$

Therefore, (36) becomes,

$$\Theta(0,F^*)+\Theta(k,F^*)+\Theta(\infty,F^*)\ge2\left(1-\frac{(m+1)}{(n+m+1)}\right)+1-\frac{1}{(n+m+1)}$$

$$=\frac{3n+m}{n+m+1}>2\quad\text{for } n\ge m+10$$

which contradicts $\sum_{a\in\overline{\mathscr{C}}}\Theta(a,f)\le2$. Thus we have

$$\frac{f^{n+m+1}}{n+m+1}-mC_1\frac{f^{n+m}}{n+m}+\cdots+(-1)^p\frac{f^{n+1}}{n+1}=\frac{g^{n+m+1}}{n+m+1}-mC_1\frac{g^{n+m}}{n+m}+\cdots+(-1)^p\frac{g^{n+1}}{n+1} \tag{39}$$

Let $h=f/g$, substituting $f=hg$ in the above equation, we can easily get

$$\begin{aligned}&(n+m)(n+m-1)\cdots(n+1)g^m\left(h^{n+m-1}-1\right)\\&-mC_1(n+m+1)(n+m-1)\cdots(n+1)g^{m-1}\left(h^{n+m}-1\right)\\&+\cdots+(-1)^p(n+m+1)(n+m)\cdots(n)\left(h^{n+1}-1\right)=0\end{aligned} \tag{40}$$

If h is not a constant, then with simple calculations we get contradiction (refer [9]). Therefore h is a constant. We have from (40) that $h^{n+m}-1=0, h^{n+1}-1=0$, which imply $h=1$. Hence $f\equiv g$.

Case 3. Suppose that $f_3=c(\ne0)$, where c is a constant. If $c\ne1$, then

$$f_1+f_2+f_3=1$$

$$\frac{f^n(f-1)^m f'}{z}-\frac{cz}{g^n(g-1)^m g'}=1-c \tag{41}$$

Applying Lemma 2.2 to above equation, we have

$$T\left(r,\frac{f^n(f-1)^m f'}{z}\right)$$
$$\le \bar{N}\left(r,\frac{f^n(f-1)^m f'}{z}\right)+\bar{N}\left(r,\frac{z}{f^n(f-1)^m f'}\right)+\bar{N}\left(r,\frac{g^n(g-1)^m g'}{z}\right)+S(r,f)$$
$$\le \bar{N}(r,f)+\bar{N}\left(r,\frac{1}{f^n(f-1)^m f'}\right)+\bar{N}(r,g)+2\log r+S(r,f)$$

$$T\left(r,f^n(f-1)^m f'\right)\le \bar{N}(r,f)+\bar{N}\left(r,\frac{1}{f^n(f-1)^m f'}\right)+\bar{N}(r,g)+3\log r+S(r,f) \tag{42}$$

Using Lemmas 2.4, 2.3 and (42), we have

$$(n+m)T(r,f)\le T\left(r,f^n(f-1)^m\right)+S(r,f)$$
$$\le T\left(r,f^n(f-1)^m f'\right)+T\left(r,\frac{1}{f'}\right)+S(r,f)$$
$$\Rightarrow (n+m-7)T(r,f)\le T(r,g)+3\log r+S(r,f)$$

Using Lemma 2.7, we get

$$(n+m-7)T(r,f)\le \frac{n+m+2}{n+m-7}T(r,f)+3\log r+S(r,f) \tag{43}$$

Since $n\ge m+10$, we get contradiction.
Therefore $c=1$
Hence,

$$\frac{f^n(f-1)^m f'}{z}-\frac{z}{g^n(g-1)^m g'}=0 \tag{44}$$

$$f^n(f-1)^m f'\cdot g^n(g-1)^m g'=z^2$$

Let z_0 be a zero of *f* of order *p*. From (44) we know that z_0 is a pole of *g*. Suppose z_0 is a pole of *g* of order *q*, from (44), we obtain

$$np+p-1=nq+mq+q+1$$
$$(n+1)(p-q)=mq+2\ge n+1.$$

Hence,

$$p\ge\frac{n+m+1}{m} \tag{45}$$

Let z_1 be a zero of $(f-1)$ of order p_1. From (44) we know that z_1 is a pole of *g*. (say order q_1). From (44), we obtain

$$p_1+p_1-1=nq_1+mq_1+q_1+1$$
$$p_1\ge\frac{(n+m+3)}{2} \tag{46}$$

Let z_2 be a zero of f' of order p_2, that is not zero of $f(f-1)$, then from (44), z_2 is a pole of *g* of order q_2. From (44), we have

$$p_2 = nq_2 + mq_2 + q_2 + 1$$

$$p_2 \geq n + m + 2 \tag{47}$$

In the same manner as above, we have similar results for zeros of $g^n(g-1)^m g'$. From (44)-(47), we have

$$\bar{N}\left(r, f^n(f-1)^m f'\right) = \bar{N}\left(r, \frac{z^2}{g^n(g-1)^m g'}\right) \tag{48}$$

$$\begin{aligned}\Rightarrow \bar{N}(r,f) &\leq \bar{N}\left(r,\frac{1}{g}\right) + \bar{N}\left(r,\frac{1}{g-1}\right) + N\left(r,\frac{1}{g'}\right)\\ &\leq \left(\frac{m}{n+m-1}\right)T(r,g) + \left(\frac{2}{n+m+3}\right)T(r,g) + \left(\frac{2}{n+m+2}\right)T(r,g)\end{aligned} \tag{49}$$

By Nevanlinna's second fundamental theorem, we have from (45), (46) and (49) that,

$$\begin{aligned}T(r,f) &\leq \bar{N}\left(r,\frac{1}{f}\right) + \bar{N}\left(r,\frac{1}{f-1}\right) + \bar{N}(r,f) + S(r,f)\\ &\leq \left[\left(\frac{m}{n+m-1}\right) + \left(\frac{2}{n+m+3}\right)\right]T(r,f)\\ &\quad + \left[\left(\frac{m}{n+m-1}\right) + \left(\frac{2}{n+m+3}\right) + \left(\frac{2}{n+m+2}\right)\right]T(r,g) + S(r,f) + S(r,g)\end{aligned} \tag{50}$$

Similarly,

$$\begin{aligned}T(r,g) &\leq \left[\left(\frac{m}{n+m-1}\right) + \left(\frac{2}{n+m+3}\right)\right]T(r,g)\\ &\quad + \left[\left(\frac{m}{n+m-1}\right) + \left(\frac{2}{n+m+3}\right) + \left(\frac{2}{n+m+2}\right)\right]T(r,f) + S(r,f) + S(r,g)\end{aligned} \tag{51}$$

From (50) and (51), we get

$$\begin{aligned}&T(r,f) + T(r,g)\\ &\leq \left[\left(\frac{m}{n+m-1}\right) + \left(\frac{2}{n+m+3}\right)\right]\left(T(r,f) + T(r,g)\right)\\ &\quad + \left[\left(\frac{m}{n+m-1}\right) + \left(\frac{2}{n+m+3}\right) + \left(\frac{2}{n+m+2}\right)\right]\left(T(r,f) + T(r,g)\right) + S(r,f) + S(r,g)\\ &\left[1 - \left(\frac{2m}{n+m-1}\right) - \left(\frac{4}{n+m+3}\right) - \left(\frac{2}{n+m+2}\right)\right]\left(T(r,f) + T(r,g)\right) \leq S(r,f) + S(r,g)\end{aligned}$$

since $n \geq m + 10$, we get a contradiction.

This completes the proof of Theorem 1.

Proof of Theorem 2. By the assumption of the theorems, we know that either both f and g are two transcendental entire functions or both f and g are polynomials. If f and g are transcendental entire functions, using $\bar{N}(r,f) = 0, \bar{N}(r,g) = 0$ and similar arguments as in the proof of Theorem 1, we can easily obtain Theorem 2. If f and g are polynomials, $f^n(f-1)^m f'$ and $g^n(g-1)^m g'$ share z CM, we get

$$f^n(f-1)^m f' - z = k\left(g^n(g-1)^m g' - z\right) \tag{52}$$

where k is a non-zero constant. Suppose that $k \neq 1$, (52) can be written as,

$$\frac{f^n(f-1)^m f'}{z} - k\frac{g^n(g-1)^m g'}{z} = 1 - k \tag{53}$$

Apply Lemma 2.2 to above equation, we have

$$T\left(r,\frac{f^n(f-1)^m f'}{z}\right)\le \bar{N}\left(r,\frac{f^n(f-1)^m f'}{z}\right)+\bar{N}\left(r,\frac{z}{g^n(g-1)^m g'}\right)+\bar{N}\left(r,\frac{z}{f^n(f-1)^m f'}\right)+S(r,f)$$

Since f is a polynomial, it does not have any poles. Thus, we have

$$T\left(r,\frac{f^n(f-1)^m f'}{z}\right)\le \bar{N}\left(r,\frac{1}{f^n(f-1)^m f'}\right)+\bar{N}\left(r,\frac{1}{g^n(g-1)^m g'}\right)+2\log r+S(r,f)$$

Therefore,

$$T\left(r,f^n(f-1)^m f'\right)\le \bar{N}\left(r,\frac{1}{f^n(f-1)^m f'}\right)+\bar{N}\left(r,\frac{1}{g^n(g-1)^m g'}\right)+3\log r+S(r,f) \tag{54}$$

Using Lemmas 2.4, 2.3 and (54), we have

$$(n+m)T(r,f)=T\left(r,f^n(f-1)^m\right)+S(r,f)$$
$$\le 4T(r,f)+3T(r,g)+3\log r+S(r,f)$$
$$(n+m-4)T(r,f)\le 3T(r,g)+3\log r+S(r,f)$$

Using Lemma 2.8, we get

$$(n+m-4)T(r,f)\le 3\left(\frac{n+m+1}{n+m-4}\right)T(r,f)+3\log r+S(r,f) \tag{55}$$

since $n\ge m+6$, we get a contradiction. Therefore, $k=1$. So, (52) becomes

$$f^n(f-1)^m f'=g^n(g-1)^m g' \tag{56}$$

On Integrating, we get

$$\frac{f^{n+m+1}}{n+m+1}-mC_1\frac{f^{n+m}}{n+m}+\cdots+(-1)^p\frac{f^{n+1}}{n+1}=\frac{g^{n+m+1}}{n+m+1}-mC_1\frac{g^{n+m}}{n+m}+\cdots+(-1)^p\frac{g^{n+1}}{n+1}+c$$

$$\Rightarrow F^*=G^*+c,\ \text{where } c \text{ is a constant} \tag{57}$$

We claim that $c=0$. Suppose that $c\ne 0$, then

$$\Theta\left(0,F^*\right)+\Theta\left(c,F^*\right)=\Theta\left(0,F^*\right)+\Theta\left(0,G^*\right) \tag{58}$$

Proceeding as in Theorem 1,
we get $f\equiv g$.

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