Characterization of Self Dual Lattices in $\mathbb{R}$, $\mathbb{R}^2$ and $\mathbb{R}^3$

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Received 19 March 2014; revised 19 April 2014; accepted 26 April 2014

Abstract

This paper shows that the only self dual lattices in $\mathbb{R}, \mathbb{R}^2, \mathbb{R}^3$ are rotations of $\mathbb{Z}, \mathbb{Z} \times \mathbb{Z}$ and $\mathbb{Z} \times \mathbb{Z} \times \mathbb{Z}$.

Keywords

Self Dual Lattice

1. Introduction

Let

$$A := [a_1, \cdots, a_n], B := [b_1, \cdots, b_n],$$

be nonsingular $n \times n$ real matrices with column vectors $a_1, \cdots, a_n$ and $b_1, \cdots, b_n$, respectively. Let

$$\mathcal{L}_A := \{ \sum_{i=1}^{n} m_i a_i : m_1, \cdots, m_n \in \mathbb{Z} \},$$

$$\mathcal{L}_B := \{ \sum_{i=1}^{n} m_i b_i : m_1, \cdots, m_n \in \mathbb{Z} \}.$$

be the lattices in $\mathbb{R}^n$ that are generated by the columns of $A, B$. The lattice $\mathcal{L}_A$ will be a subset of the lattice $\mathcal{L}_B$ if and only if the generators $a_1, \cdots, a_n$ of $\mathcal{L}_A$ all lie in $\mathcal{L}_B$, i.e.,

$$a_k = \sum_{i=1}^{n} m_i b_i, k = 1, 2, \cdots, n.$$
for suitably chosen integers $m_{ik}$. Equivalently, 

$$
\begin{bmatrix}
  a_1, \ldots, a_n \\
\end{bmatrix} = 
\begin{bmatrix}
  b_1, \ldots, b_n \\
\end{bmatrix}
\begin{bmatrix}
  m_{11} & m_{12} & \cdots & m_{1n} \\
  m_{21} & m_{22} & \cdots & m_{2n} \\
  \vdots & \vdots & \ddots & \vdots \\
  m_{n1} & m_{n2} & \cdots & m_{nn}
\end{bmatrix}
$$

i.e.,

$$
M := B^{-1}A
$$
is a matrix of integers. Analogously, the lattice $L_B$ is a subset of $L_A$ if $A^{-1}B$ is a matrix of integers. In this way we see that

$$
L_A = L_B
$$

if and only if both $M = B^{-1}A$ and

$$
A^{-1}B = (B^{-1}A)^{-1} = M^{-1}
$$

are matrices with integer elements. When this is the case, $\det M$ and $\det M^{-1}$ are both integers and since

$$
\det M \det M^{-1} = \det MM^{-1} = \det I = 1,
$$

this implies that

$$
\det M = \det M^{-1} = \pm 1.
$$

Such a matrix is said to be unimodular. The above analysis (that can be found in [1]) is summarized as follows.

**Theorem 1** The lattices $L_A, L_B$ are identical if and only if

$$
M := A^{-1}B
$$
is a matrix of integers with

$$
\det M = \pm 1
$$

**Corollary 1** Lattices are preserved under integer column operations.

**Proof 1** Let $A = [a_1, \ldots, a_n]$ generate the lattice $L_A$, and let

$$
K = 
\begin{bmatrix}
  0 & k_{12} & k_{13} & \cdots & k_{1n} \\
  0 & 0 & k_{23} & \cdots & k_{2n} \\
  \vdots & \vdots & \vdots & \ddots & \vdots \\
  0 & 0 & 0 & \cdots & k_{n-1,n} \\
  0 & 0 & 0 & \cdots & 0
\end{bmatrix}
$$

be a strictly upper triangular matrix of integers. Then $I + K$ is an upper triangular matrix of integers with a unit diagonal, and we can write

$$
(I + K)^{-1} = I + L
$$

where

$$
L := -K + K^2 - K^3 + \cdots + (-1)^{n-1} K^{n-1}
$$
is a strictly upper triangular matrix of integers. The columns of

$$
B := A(I + K)
$$
i.e.,

$$
a_1, k_{12}a_1 + a_2, k_{13}a_1 + k_{23}a_2 + a_3, \ldots
$$
generate the same lattice as the columns of $A$. To see this we observe that
is a matrix of integers with unit determinant.

2. Dual Lattices

Definition 1 Two linearly independent sets of real $n$ (column) vectors $a_1, \cdots, a_n$ and $b_1, \cdots, b_n$ are said to be biorthogonal if
\[
\langle a_i, b_l \rangle := a_i^T b_l = \delta_{il}, k, l = 1, 2, \cdots, n
\]
where $\delta_{il}$ is the Kronecker’s delta, $^T$ denotes the transpose and $\langle \rangle$ denotes the usual inner product. When the columns of
\[
A := [a_1, \cdots, a_n]
\]
and
\[
B := [b_1, \cdots, b_n]
\]
are biorthogonal, we find
\[
A^T B = I
\]
so that
\[
B = (A^T)^{-1} = A^{-T}.
\]
This being the case, given linearly independent vectors $a_1, \cdots, a_n$ we can form $A$ and then obtain the biorthogonal vectors $b_1, \cdots, b_n$ as the columns of $A^{-T}$.

The lattice $\mathcal{L}_A$ generated by vectors biorthogonal to $a_1, \cdots, a_n$ is said to be the dual of the lattice $\mathcal{L}_A$. More generally, $\mathcal{L}_B$ is dual to $\mathcal{L}_A$ if and only if $B$ generates the same lattice as $A^{-T}$, i.e.,
\[
(A^{-T})^{-1} B = A^T B
\]
is a matrix of integers with determinant $\pm 1$.

Suppose now that $A_1, A_2$ generate the same lattice, i.e.,
\[
\mathcal{L}_{A_1} = \mathcal{L}_{A_2}.
\]
Let
\[
B_1 = A_1^{-T}, B_2 = A_2^{-T}
\]
be the generators of lattices $\mathcal{L}_{A_1}, \mathcal{L}_{A_2}$ dual to $\mathcal{L}_{A_1}, \mathcal{L}_{A_2}$, respectively. Since
\[
B_2^{-1} B_1 = (A_2^{-T})^{-1} A_1^{-T} = A_2^T A_1^T = (A_1^{-1} A_2)^T
\]
we see that $A_1^{-1} A_2$ will be a matrix of integers with determinant $\pm 1$ if and only if the same is true of $B_2^{-1} B_1$.

Thus $\mathcal{L}_{B_1} = \mathcal{L}_{B_2}$ if and only if $\mathcal{L}_{A_1} = \mathcal{L}_{A_2}$.

We are interested in characterizing those lattices $\mathcal{L}_A$ that are self dual, i.e.,
\[
\mathcal{L}_A = \mathcal{L}_{A^{-T}}.
\]
This will be the case if and only if
\[
(A^{-T})^{-1} A = A^T A
\]
is a matrix of integers with determinant $\pm 1$. Since
\[
\det A^T A = (\det A)^2,
\]
this will be the case only if
In this way we see that a lattice $\mathcal{L}_d$ is self dual if and only if $A^T A$ is a matrix of integers with unit determinant. The parallelopiped in $\mathbb{R}^n$ with vertices $0, a_1, a_2, \cdots, a_n, a_1 + a_2, a_1 + a_3, \cdots, a_1 + a_2 + \cdots + a_n$, i.e., the unit cell of the lattice has the volume

$$V(a_1, a_2, \cdots, a_n) = |\det A|,$$

[2] [3]. Thus a lattice can be self dual only if each of its primitive cells, has unit volume.

Self dual lattices are preserved under orthogonal transformations. Indeed, let $Q$ be an orthogonal transformation on $\mathbb{R}^n$, i.e.,

$$Q^T Q = I,$$

and let $\mathcal{L}_A, \mathcal{L}_B$ be the lattices generated by the columns of a nonsingular $n \times n$ matrix $A$ and $B := A^T$. The matrix

$$A' := QA$$

has columns

$$a'_1 = Qa_1, a'_2 = Qa_2, \cdots, a'_n = Qa_n$$

that generate the lattice $\mathcal{L}_{A'}$. We can use such a matrix $Q$ to rotate $a_1, a_2, \cdots, a_n$, to reflect one or more vectors of the set $a_1, a_2, \cdots, a_n$, to permute $a_1, a_2, \cdots, a_n$, etc. The lattice $\mathcal{L}_{A'}$ which is dual to $\mathcal{L}_B$ is generated by the columns of

$$B' = (A')^T = (QA)^T = Q^T A^T = QB,$$

i.e., by

$$b'_1 = Qb_1, b'_2 = Qb_2, \cdots, b'_n = Qb_n.$$

Thus the generators of the dual lattice $\mathcal{L}_{A'}$ are transformed in the same way as the generators of the lattice $\mathcal{L}_A$. In this way we see that a lattice $\mathcal{L}_A$ is self dual if and only if the lattice $\mathcal{L}_{A'}$ is self dual. Indeed,

$$(A')^T A' = (QA)^T QA = A^T A$$

so $A^T A$ is a matrix of integers with unit determinant if and only if the same is true of $(A')^T A'$. Moreover, since

$$\|Qx\|_E^2 = x^T Q^T Q x = x^T x = \|x\|_E^2$$

we see that the orthogonal transformation $Q$ preserves the Euclidean lengths of a set of generators for the lattice $\mathcal{L}_d$.

3. Main Results

We will now show that the only self dual lattices in $\mathbb{R}, \mathbb{R}^2, \mathbb{R}^3$ are rotations of $\mathbb{Z}, \mathbb{Z} \times \mathbb{Z}$, and $\mathbb{Z} \times \mathbb{Z} \times \mathbb{Z}$, respectively.

The case $n = 1$

Let $A = [a_1]$ be a vector in $\mathbb{R}$ that generates the lattice $\mathcal{L}_d$. We do not change the lattice if we assume that $a_1 > 0$. Let $b_1 = 1/a_1$ be biorthogonal to $A$. The lattice $\mathcal{L}_B$ generated by $B = [b_1]$ will be identical to the lattice $\mathcal{L}_d$ if and only if

$$a_i = \frac{1}{a_1},$$

i.e., if and only if
Thus the only self dual lattice in $\mathbb{R}$ is the lattice
\[ \mathcal{L} = \mathbb{Z}. \]

The case $n = 2$

**Theorem 2** Every self dual lattice in $\mathbb{R}^2$ is some rotation of $\mathbb{Z} \times \mathbb{Z}$.

**Proof 2** Let $A = [a_1 \ a_2]$ where $a_1, a_2$ are linearly independent vectors in $\mathbb{R}^2$ and assume that $\mathcal{L}_A$ is self dual. Fix the origin at some lattice point of $\mathcal{L}_A$ and rotate the axes, if necessary, so that the nearest nonzero lattice point of $\mathcal{L}_A'$ lies on the positive $x$-axis, i.e.

\[
Q_A = A' = \begin{bmatrix} a_1' & a_2' \\ a_1'' & a_2'' \end{bmatrix} = \begin{bmatrix} \alpha & \beta \\ 0 & \gamma \end{bmatrix}
\]

where $\alpha > 0$ and

\[ \alpha^2 \leq \beta^2 + \gamma^2. \quad (1.1) \]

The lattice $\mathcal{L}_A$ does not change if $a_2'$ is replaced by $-a_2'$ so we can and do assume that $\gamma > 0$. Likewise the lattice $\mathcal{L}_A$ does not change if $a_2'$ is replaced by $a_2' - ka_1', k = 0, \pm 1, \pm 2, \cdots$ since this is the result of an integer column operation. Thus we can and do assume that

\[ |\beta| \leq \alpha/2. \quad (1.2) \]

By hypothesis the lattice $\mathcal{L}_A$ is self dual so the same is true of $\mathcal{L}_A'$. This implies that $\alpha \gamma = \det A' = 1$, and

\[
(A')^T = \begin{bmatrix} \gamma & 0 \\ -\beta & \alpha \end{bmatrix}.
\]

Since $\mathcal{L}_A' \subset \mathbb{R}^2$ is self dual, the first column of $A'$ can be expressed as an integral linear combination of the columns of $(A')^T$, i.e.,

\[
\begin{bmatrix} \alpha \\ 0 \end{bmatrix} = n \begin{bmatrix} \gamma \\ -\beta \end{bmatrix} + m \begin{bmatrix} 0 \\ \alpha \end{bmatrix}
\]

where $n, m \in \mathbb{Z}$. In this way we see in turn that

\[ \alpha = n \gamma, \alpha = n/\alpha, \alpha = \sqrt{n}, \quad (1.3) \]

for some $n = 1, 2, \cdots$,

\[ n \beta = m \alpha, \beta = m/\sqrt{n}, \quad (1.4) \]

for some $m = 0, \pm 1, \pm 2, \cdots$, and

\[ \gamma = 1/\alpha = 1/\sqrt{n}. \quad (1.5) \]

Using these expressions with (1.2) we find

\[
\frac{|m|}{\sqrt{n}} \leq \frac{\sqrt{n}}{2}
\]

so

\[ |m| \leq \frac{n}{2}. \]

Using these expressions with (1.1) we find

\[ n = \alpha^2 \leq \beta^2 + \gamma^2 = \frac{m^2}{n} + \frac{1}{n}, \]

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and since

$$|m| \leq \frac{n}{2}.$$  

this implies that

$$n^2 \leq 4/3.$$  

It follows that \( n = 1 \) and \( m = 0 \). In this way we prove that \( A' = I \), i.e., the columns of \( A' \) and thus those of \( A \) are orthonormal. Thus \( \mathcal{L}_A \) is some rotation of \( \mathbb{Z} \times \mathbb{Z} \).

A theorem of Minkowski [1] states that

$$\|a\|_2 \leq \sqrt{N} |\det A|^{\frac{1}{n}},$$

where \( a \) is the shortest nonzero vector in a lattice \( \mathcal{L}_A \) in \( \mathbb{R}^n \). Within the present context, this leads to the bound

$$\sqrt{n} = \alpha \leq \sqrt{2}$$

which implies that \( n = 1, 2 \). Our argument gives \( n^2 \leq 4/3 \) from which we immediately obtain \( n = 1 \).

Another result in [4] states that if \( A \) is a self-dual lattice in \( \mathbb{R}^n \) then

$$\|a\|_2^2 = \min \{ \langle u, u \rangle | u \in \Lambda, u \neq 0 \} \leq \left[ \frac{n}{8} \right] + 1$$

which leads to

$$\alpha \leq \sqrt{5}/4.$$  

The case \( n = 3 \)

Theorem 3 Every self-dual lattice in \( \mathbb{R}^3 \) is some rotation of \( \mathbb{Z} \times \mathbb{Z} \times \mathbb{Z} \).

Proof 3 Let the self-dual lattice \( \mathcal{L}_A \) in \( \mathbb{R}^3 \) be generated by the columns of \( A = [a_1, a_2, a_3] \) chosen so that \( \|a_1\|_2, \|a_2\|_2, \|a_3\|_2 \) are as small as possible subject to the constraint

$$\|a_1\|_2 \leq \|a_2\|_2 \leq \|a_3\|_2.$$  

Following the analysis from the previous section, we set

$$A' = QA,$$

where \( Q \) is an orthogonal matrix chosen so that

$$A' = [a'_1, a'_2, a'_3] = \begin{bmatrix} \alpha & \beta & \delta \\ 0 & \gamma & \epsilon \\ 0 & 0 & \zeta \end{bmatrix}$$

with

$$\alpha > 0, \gamma > 0, \zeta > 0.$$  

By hypothesis the lattice \( \mathcal{L}_A \) is self dual, and since \( Q \) is orthogonal, the same is true of \( \mathcal{L}_{A'} \). This being the case

$$\alpha \gamma \zeta = |\det A'| = |\det A| = 1.$$  

Since the lengths of the generators of the lattice \( \mathcal{L}_A \) are preserved under the orthogonal transformation \( Q \), it follows that

$$\alpha^2 \leq \beta^2 + \gamma^2 \leq \delta^2 + \epsilon^2 + \zeta^2.$$  \hspace{1cm} (1.6)

The columns of \( A \) (and thus the columns of \( A' \)) have been chosen to be as small as possible subject to the above constraints, so we must have

$$|\beta| \leq \alpha/2, |\delta| \leq \alpha/2, |\epsilon| \leq \gamma/2.$$  \hspace{1cm} (1.7)
It can be verified that $A'$ has the inverse

\[
(A')^{-1} = \begin{bmatrix}
\frac{1}{\alpha} & -\beta \gamma & -\delta \gamma \\
0 & \frac{1}{\gamma} & \frac{\epsilon \gamma}{\zeta} \\
0 & 0 & \frac{1}{\zeta}
\end{bmatrix},
\]

and after using $\alpha \gamma \zeta = 1$ to simplify the components we obtain

\[
(A')^\top = \begin{bmatrix}
\frac{1}{\alpha} & 0 & 0 \\
-\beta \gamma & \frac{1}{\gamma} & 0 \\
-\delta \gamma + \beta \epsilon & -\alpha \epsilon & \frac{1}{\zeta}
\end{bmatrix}.
\]

Since $L_x$ is self dual, the columns of $(A')^\top$ generate the same lattice as the columns of $A'$ so we can write

\[
\begin{bmatrix}
\alpha \\
0 \\
0
\end{bmatrix} = n \begin{bmatrix}
\frac{1}{\alpha} \\
-\beta \gamma \\
-\delta \gamma + \beta \epsilon
\end{bmatrix} + m \begin{bmatrix}
0 \\
\frac{1}{\gamma} \\
-\alpha \epsilon
\end{bmatrix} + l \begin{bmatrix}
0 \\
0 \\
\frac{1}{\zeta}
\end{bmatrix}
\]

for suitably chosen $n, m, l, p, q, r \in \mathbb{Z}$. In this way we see in turn that

\[
\alpha^2 = n \text{ so that } \alpha = \sqrt{n}
\]

(1.8)

\[
\frac{1}{\zeta^2} = p \text{ so that } \zeta = \frac{1}{\sqrt{p}}
\]

(1.9)

for some $n = 1, 2, \ldots, p = 1, 2, \ldots$, and

\[
1 = \alpha \gamma \zeta = \sqrt{n} \gamma \frac{1}{\sqrt{p}} \text{ so that } \gamma = \frac{\sqrt{p}}{\sqrt{n}}.
\]

We also have

\[
0 = -n \beta \gamma + m \gamma \text{ so that } \beta = \frac{m}{\sqrt{n}}
\]

(1.11)

\[
0 = p \epsilon + q \gamma \text{ so that } \epsilon = -\frac{q}{\sqrt{p m}},
\]

(1.12)

for some $m = 0, \pm 1, \pm 2, \ldots, q = 0, \pm 1, \pm 2, \ldots$, and

\[
0 = n \left( -\delta \gamma + \beta \epsilon \right) - m \alpha \gamma + \frac{l}{\zeta} = -\delta \sqrt{n p} + l \sqrt{p}
\]

so that

\[
\delta = \frac{l}{\sqrt{n}} \text{ for some } l = 0, \pm 1, \pm 2, \ldots.
\]

(1.13)

Using (1.7) and (1.8)-(1.12) we find

\[
2|m| \leq n, 2|q| \leq p, 2|l| \leq n.
\]

(1.14)

Using (1.6) and (1.7) we see that,
\[ \alpha^2 \leq \beta^2 + \gamma^2 \leq \left( \frac{\alpha}{2} \right)^2 + \gamma^2 \]

which implies that
\[ \gamma \geq \frac{\sqrt{5}}{2} \alpha. \]

Again using (1.6) and (1.7) we see that,
\[ \gamma^2 \leq \beta^2 + \gamma^2 \leq \delta^2 + \varepsilon^2 + \zeta^2 \leq \left( \frac{\alpha}{2} \right)^2 + \left( \frac{\gamma}{2} \right)^2 + \zeta^2 \]

which implies that
\[ \zeta^2 \geq \frac{3}{4} \gamma^2 - 1 \quad \alpha^2 \geq \frac{9}{16} \alpha^2 - 1 \quad \alpha^2 = \frac{5}{16} \alpha^2 \]

so that
\[ \zeta \geq \frac{\sqrt{5}}{4} \alpha. \]

Since \( \alpha \gamma \zeta = 1 \) we must have
\[ 1 = \alpha \gamma \zeta \geq \alpha \left( \frac{\sqrt{5}}{2} \alpha \right) \left( \frac{\sqrt{5}}{4} \alpha \right) = \frac{\sqrt{5}}{8} \alpha^3 \]

or
\[ \sqrt{n} = \alpha \leq \left( \frac{8}{\sqrt{15}} \right)^{\frac{1}{3}} = 1.2735 \ldots \]

In this way we see in turn that \( n = 1 \) and \( m = l = 0 \) so that \( \alpha = 1, \beta = 0, \delta = 0 \). Finally, we again use (1.6) with (1.13), (1.12), (1.9) to write
\[ p = \gamma^2 \leq \delta^2 + \varepsilon^2 + \zeta^2 = \frac{l^2}{n} + \frac{q^2}{n \alpha} + \frac{1}{\alpha} = \frac{q^2}{p} + \frac{1}{\alpha} \leq \frac{b^2}{p} + \frac{1}{p}. \]

It follows that \( p \leq \frac{\sqrt{4/3}}{2} \) so we must have \( p = 1, q = 0 \) and \( \varepsilon = 0, \gamma = \zeta = 1 \). In this way we see that the columns of \( A' \) (and thus those of \( A \)) must be orthonormal. Thus \( L_{A} \) is some rotation of \( \mathbb{R} \times \mathbb{R} \times \mathbb{R} \).

Suppose now that \( a_1, a_2 \) are linearly independent vectors in \( \mathbb{R}^2 \) and that
\[ \text{grid}_{a_1, a_2}(x) := \sum_{n \in \mathbb{Z}} \sum_{m \in \mathbb{Z}} \delta(x - ma_1 - na_2) = \sum_{d \in A} \delta(x - a) \]

where \( A := [a_1 \quad a_2] \). We know that
\[ \text{grid}_{a_1, a_2}(s) = \det[A_1 \quad A_2] \text{grid}_{a_1, a_2}(s) = \det[A_1 \quad A_2] \sum_{d \in L_{A}} \delta(s - a) \]

where the biorthogonal vectors \( A_1, A_2 \) are the columns of \( A^{-1} \). In this way we see that
\[ \text{grid}_{a_1, a_2} = \text{grid}_{a_1, a_2} \]

if and only if \( L_{A} \) is self dual, where \( 4ptA = [a_1 \quad a_2] \). This proves the following.

**Theorem 4** Let \( a_1, a_2 \) be linearly independent vectors in \( \mathbb{R}^2 \). Then
\[ \text{grid}_{a_1, a_2} = \text{grid}_{a_1, a_2} \]

if and only if
\[ \text{grid}_{a_1, a_2} = \text{grid}_{a_1, a_2} \]
for some orthonormal choice of the vectors $a_1', a_2'$.

Analogously, we can prove the following 3-dimensional generalization.

**Theorem 5** Let $a_1, a_2, a_3$ be linearly independent vectors in $\mathbb{R}^3$. Then

$$\text{grid}_{a_1, a_2, a_3} = \text{grid}_{a_1', a_2', a_3}$$

if and only if

$$\text{grid}_{a_1, a_2} = \text{grid}_{a_1', a_2', a_3'}$$

for some orthonormal choice of the vectors $a_1', a_2', a_3'$.

These results correspond to the familiar identity

$$III' = III$$

from univariate Fourier analysis. The possibility of rotations (other than reflections) in $\mathbb{R}^2, \mathbb{R}^3$ slightly complicates the generalization of this result.

**References**


